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# Homology and topological persistence

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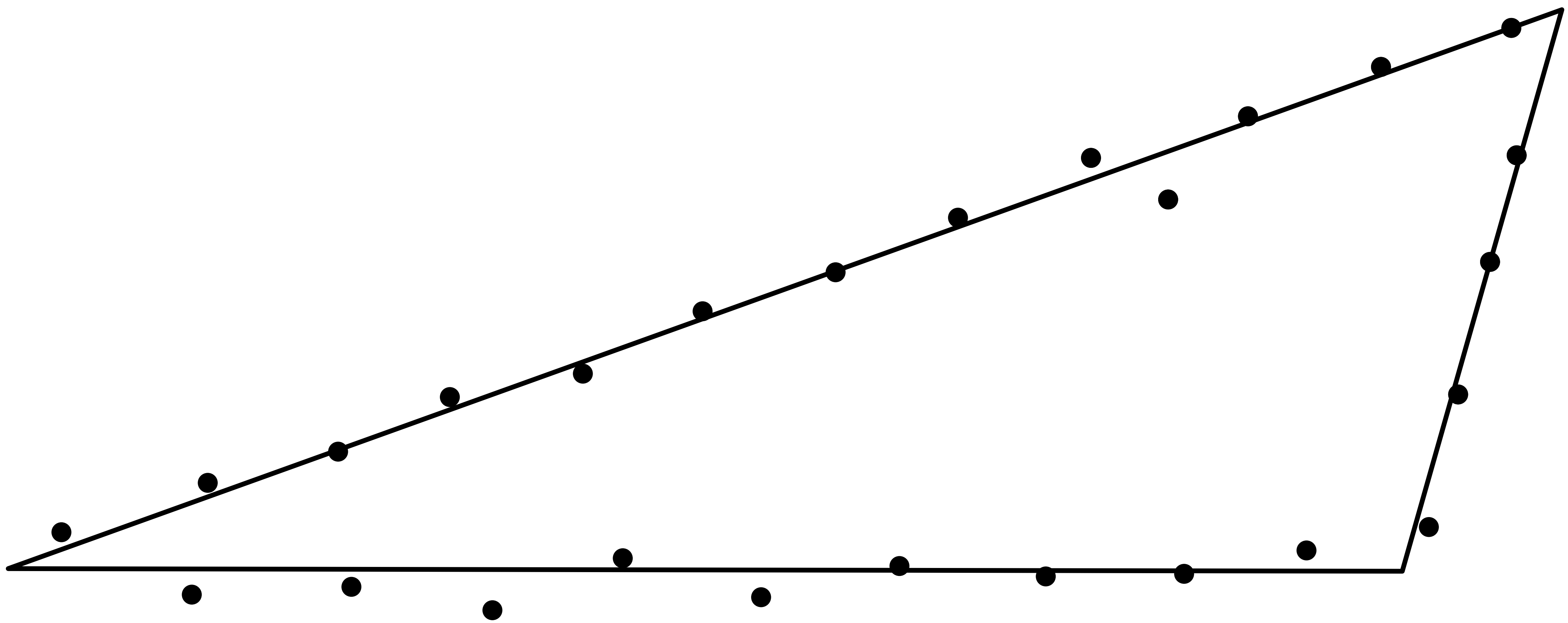
# Motivation: getting topological information without reconsctructing



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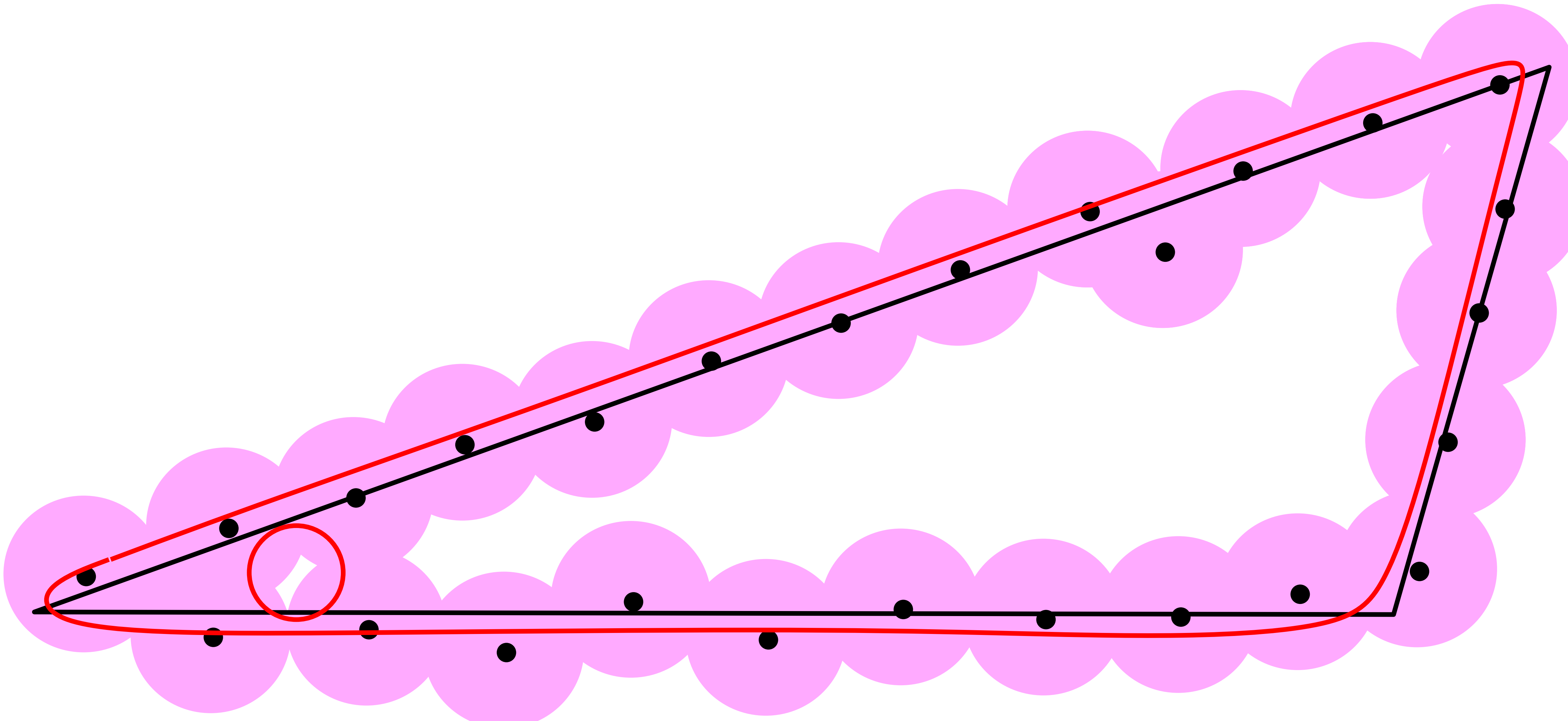

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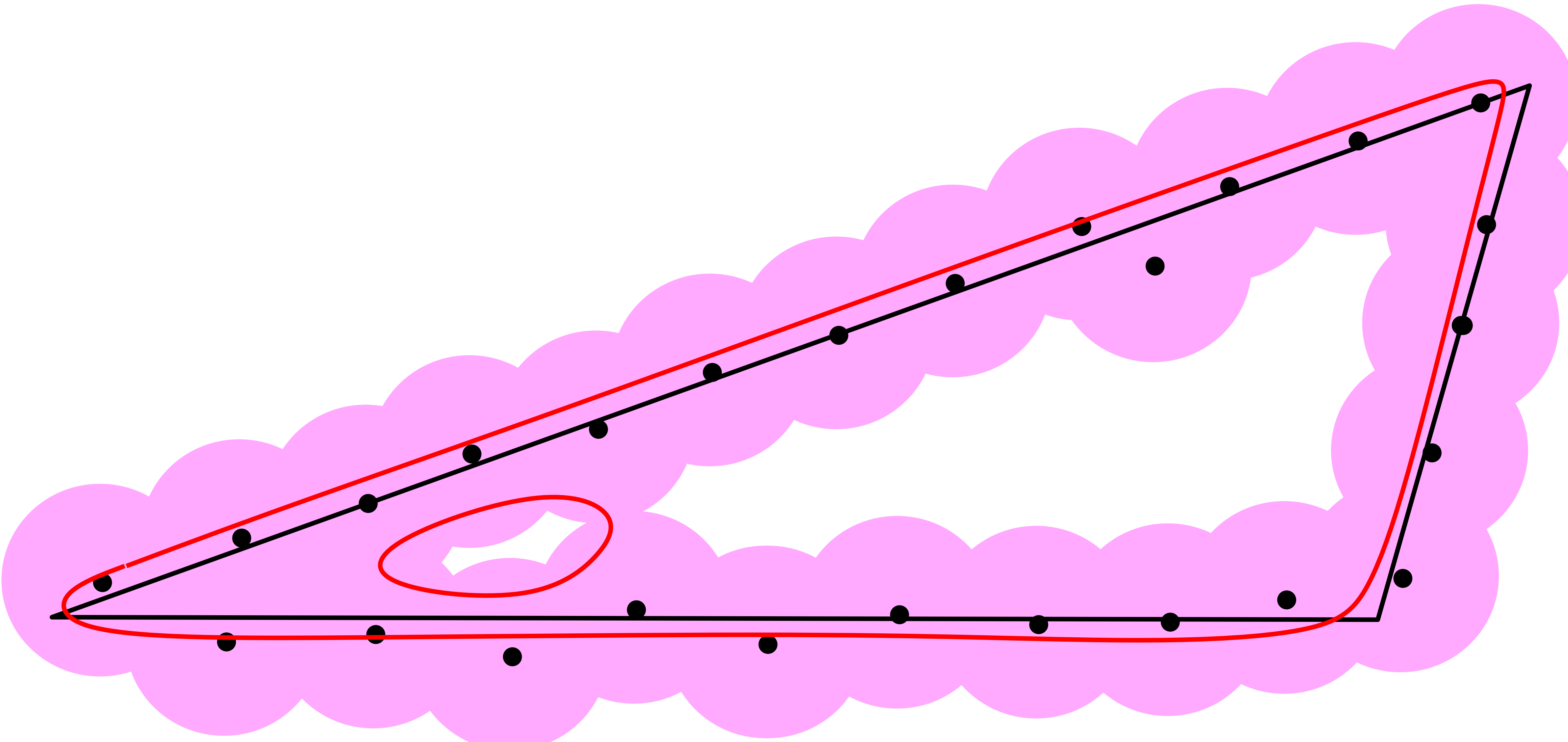

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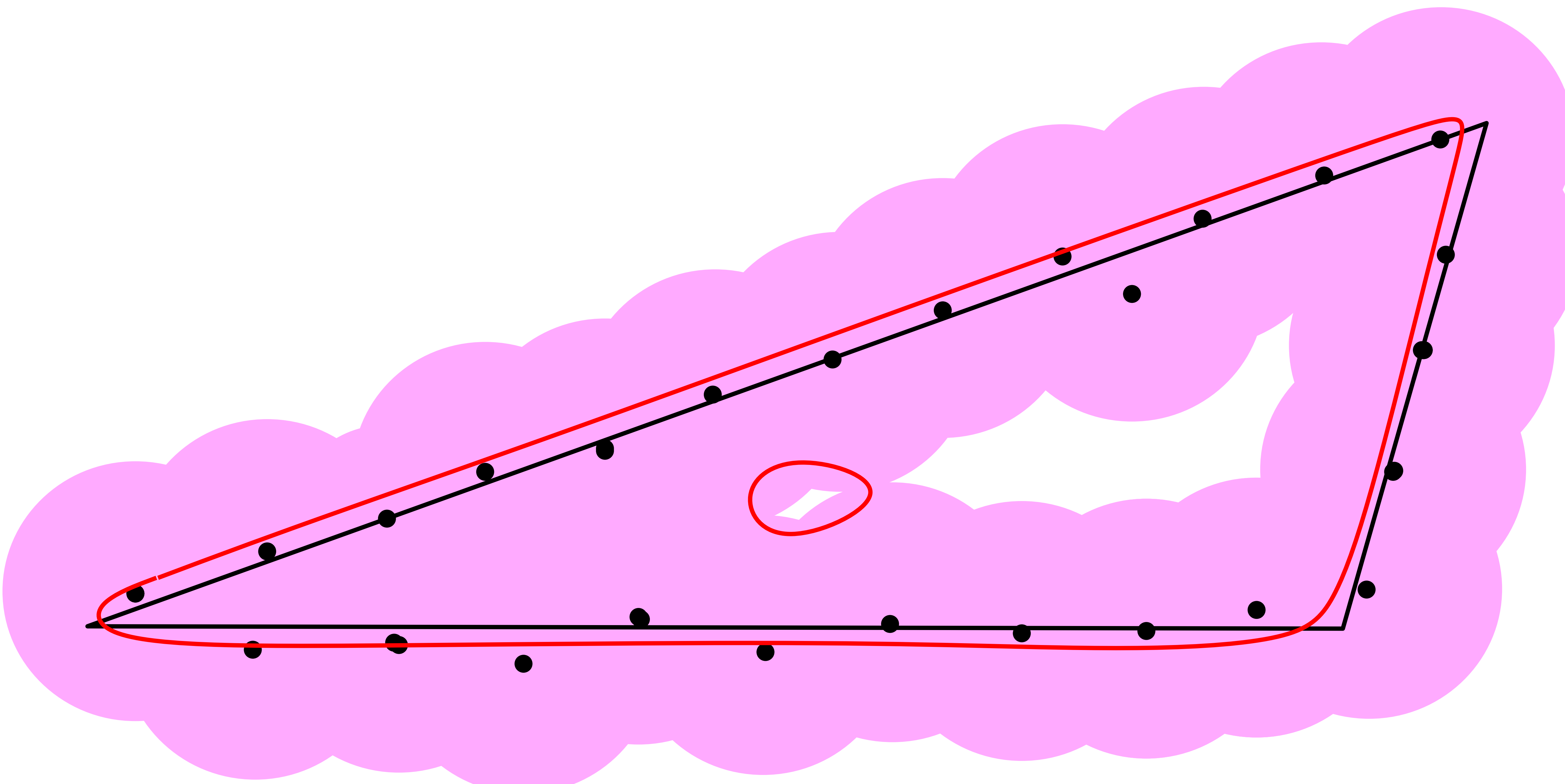

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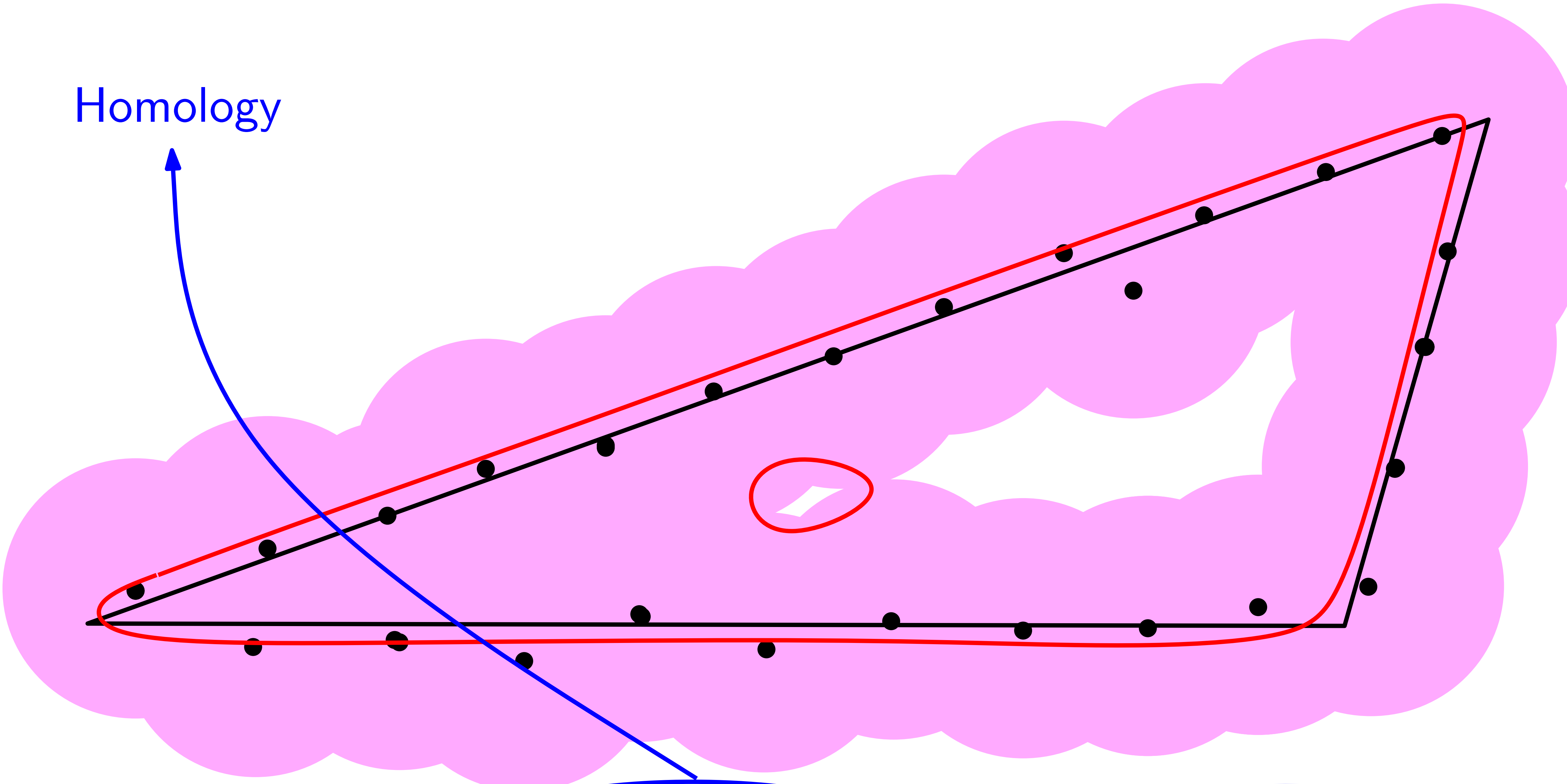
Homology



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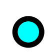
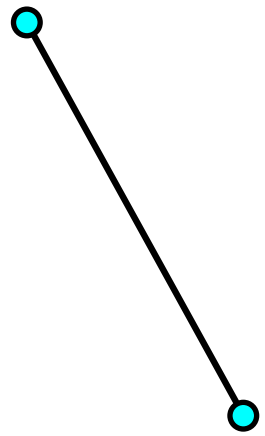
Persistent homology

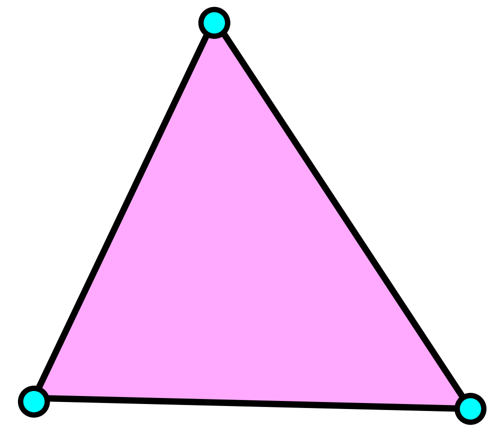
# Simplices



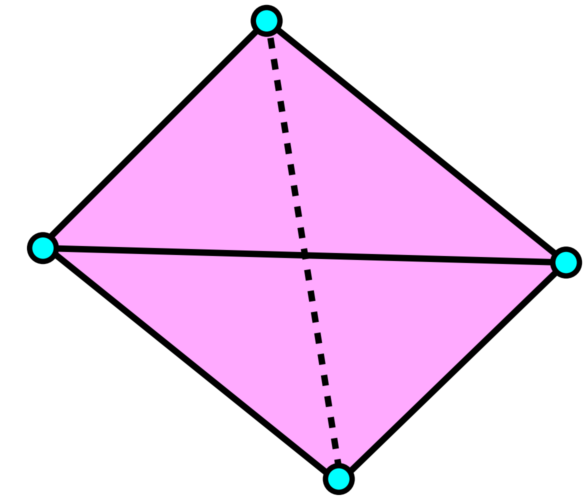
$0$-simplex:
vertex



$1$-simplex:
edge



$2$-simplex:
triangle



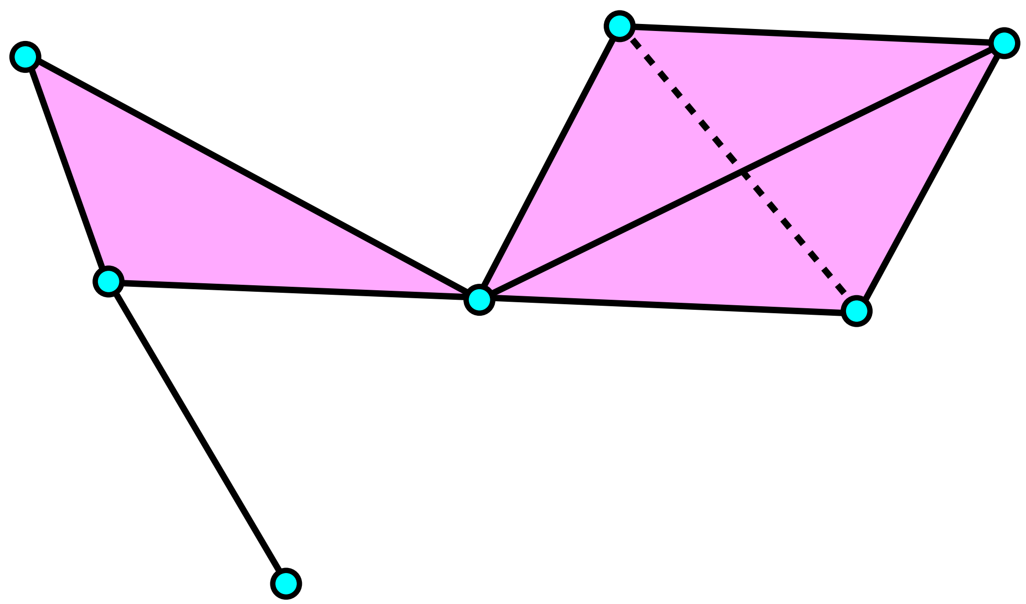
$3$-simplex:
tetraedron

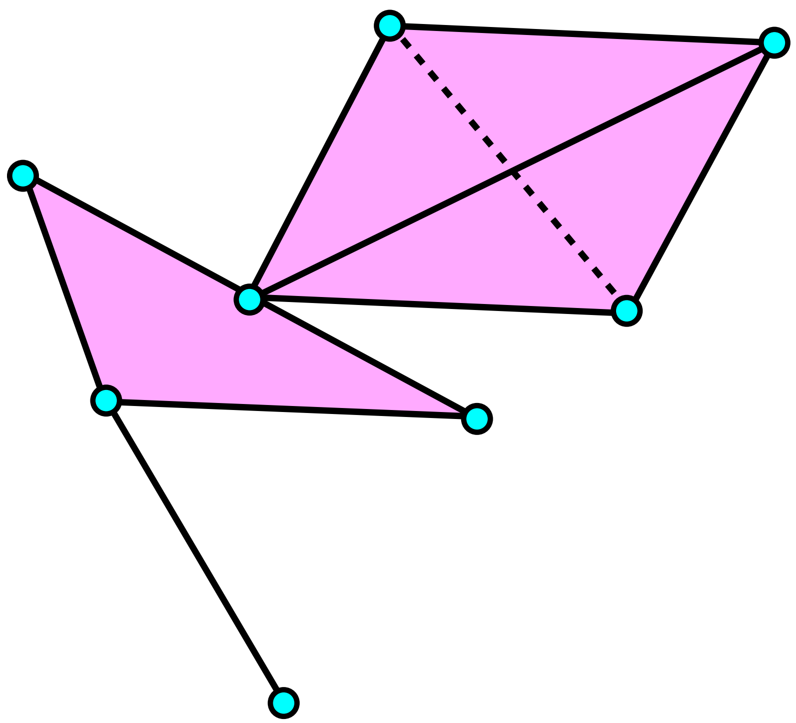
$v_0, v_1, \cdots, v_k \in \mathbb{R}^d$ are affinely independant if

$$\left( \sum_{i=0}^{k} t_i v_i = 0 \text{ and } \sum_{i=0}^{k} t_i = 0 \right) \Rightarrow t_0 = t_1 = \cdots = t_k = 0$$

In this case $\sigma = [v_0, v_1, \cdots, v_k]$ is a simplex of dimension $d$. A simplex generated by a subset of the vertices $v_0, v_1, \cdots, v_k$ of $\sigma$ is a face of $\sigma$.

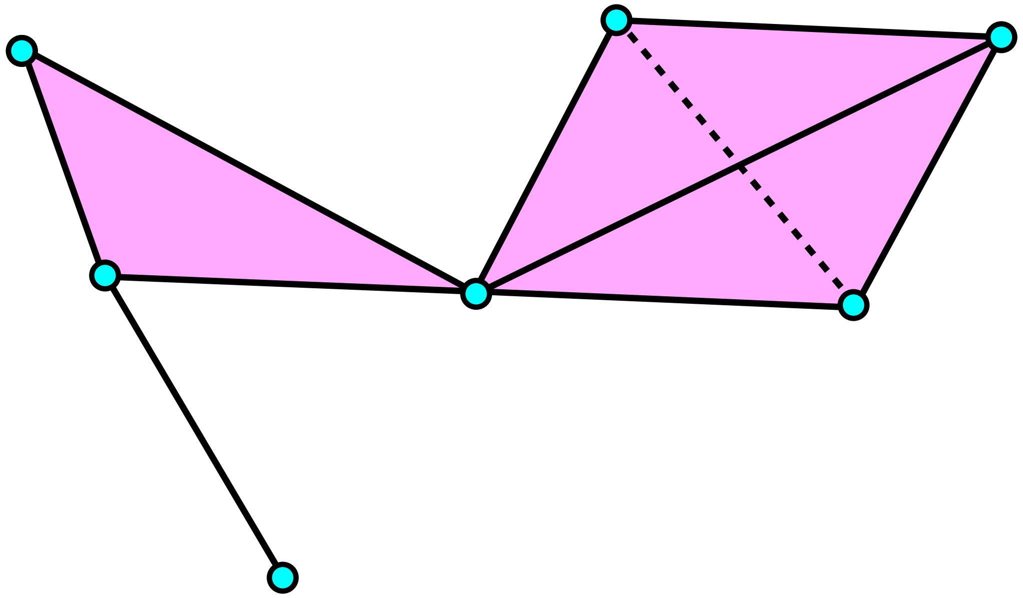
# Simplicial complexes

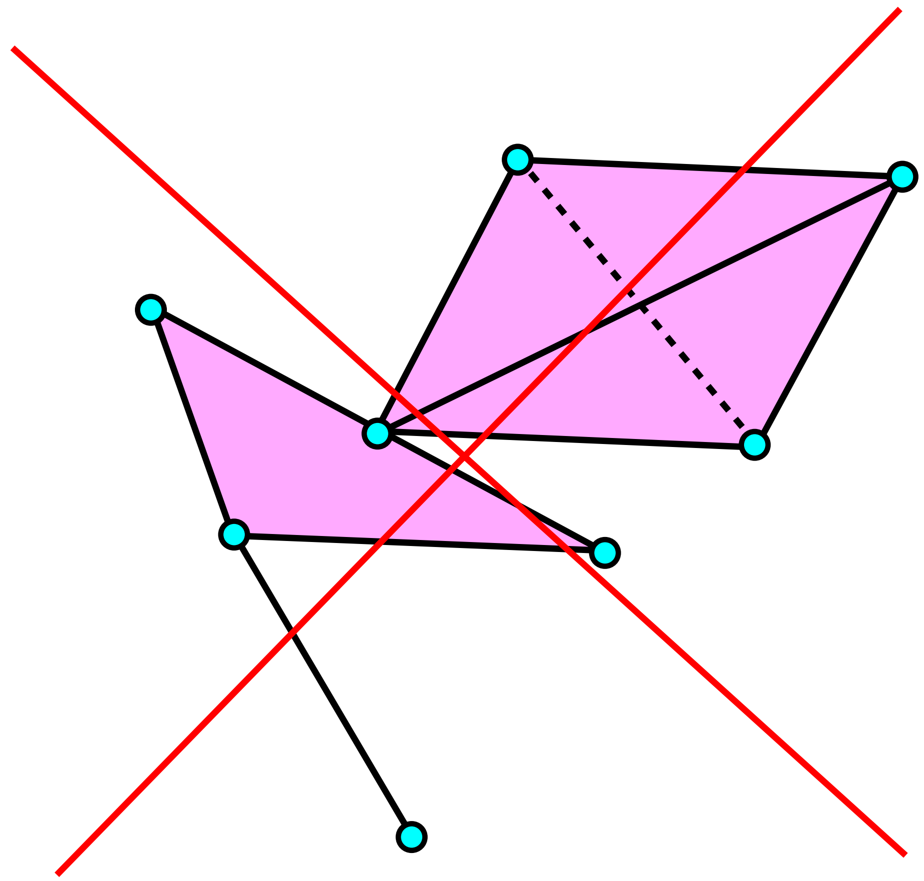




A (finite) simplicial complex $C$ is a (finite) union of simplices s.t.
*i)* for any $\sigma \in C$, all the faces of $\sigma$ are in $C$,
*ii)* the intersection of any two simplices of $C$ is either empty or a simplex which is their common face of highest dimension.

# Simplicial complexes





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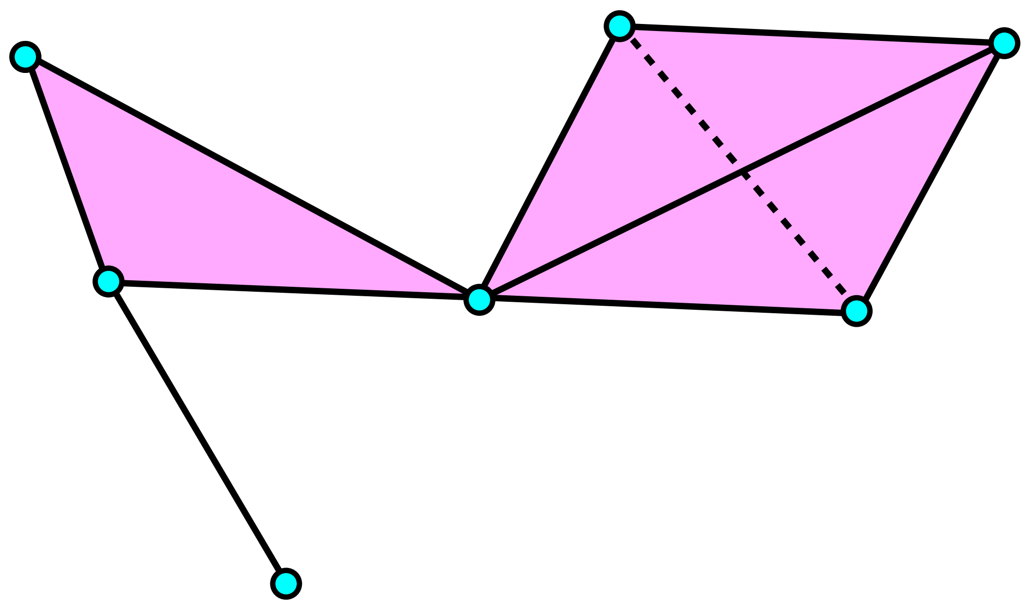

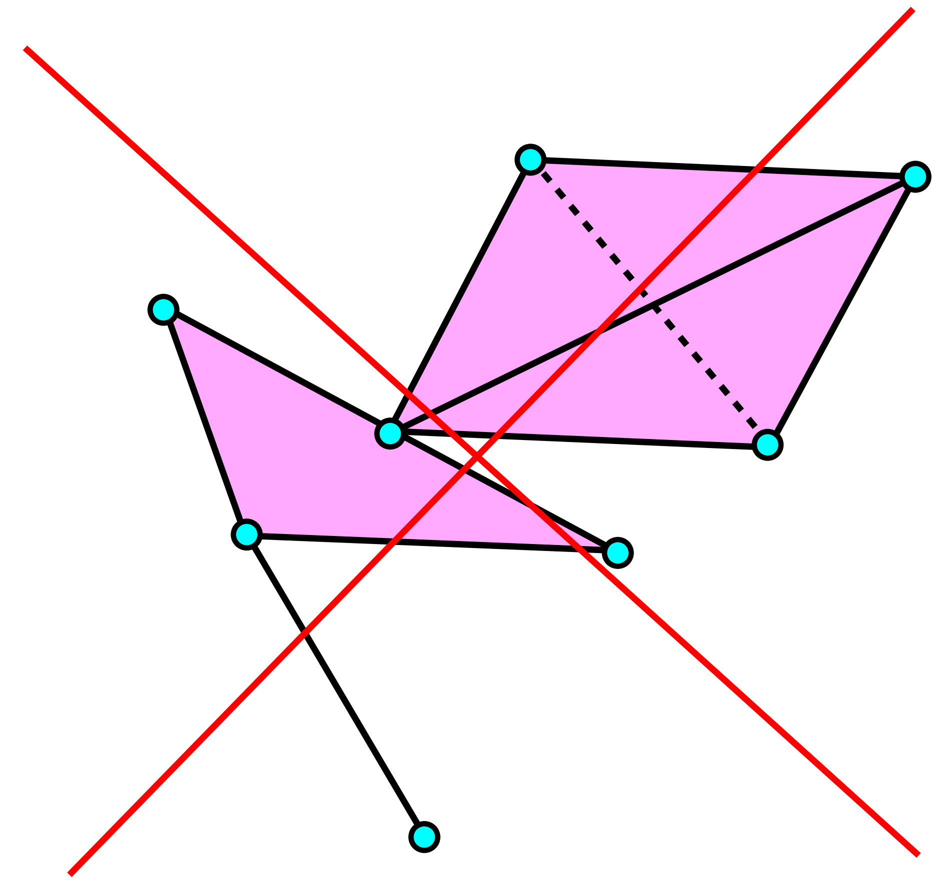

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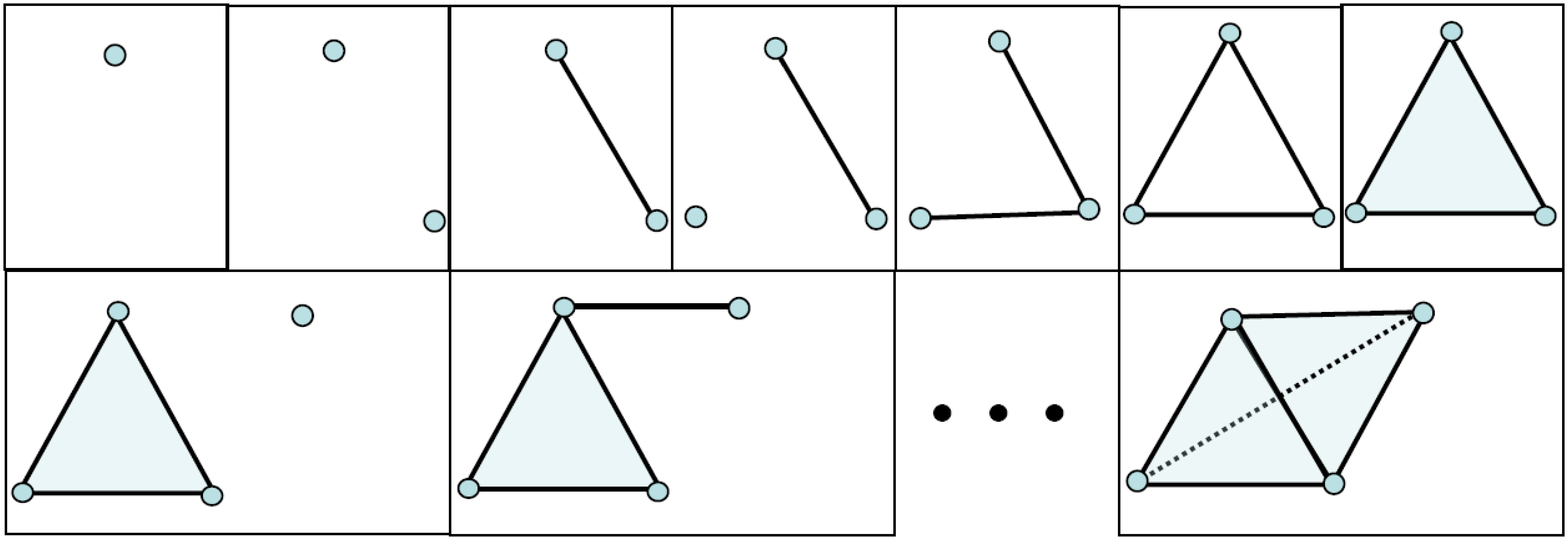
Faces: the simplices of $C$.

$j$-skeleton: the subcomplex made of the simplices of dimension at most $j$.

Dimension of $C$: the maximum of the dimensions of the faces. $C$ is homogenousof dimension $n$ if any of its faces is a face of a $n$-dimensional simplex.

# Filtrations of simplicial complexes



A filtration of a (finite) simplicial complex $K$ is a sequence of subcomplexes such that

*i)* $\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$,

*ii)* $K^{i+1} = K^i \cup \sigma^{i+1}$ where $\sigma^{i+1}$ is a simplex of $K$.

# Example: filtration associated to a function

- $f$ a real valued function defined on the vertices of $K$
- For $\sigma = [v_0, \cdots, v_k] \in K$, $f(\sigma) = \max_{i=0,\cdots,k} f(v_i)$
- The simplices of $K$ are ordered according increasing $f$ values (and dimension in case of equal values on different simplices).

$\Rightarrow$ The sublevel sets filtration.

Exercise: show that this is a filtration.

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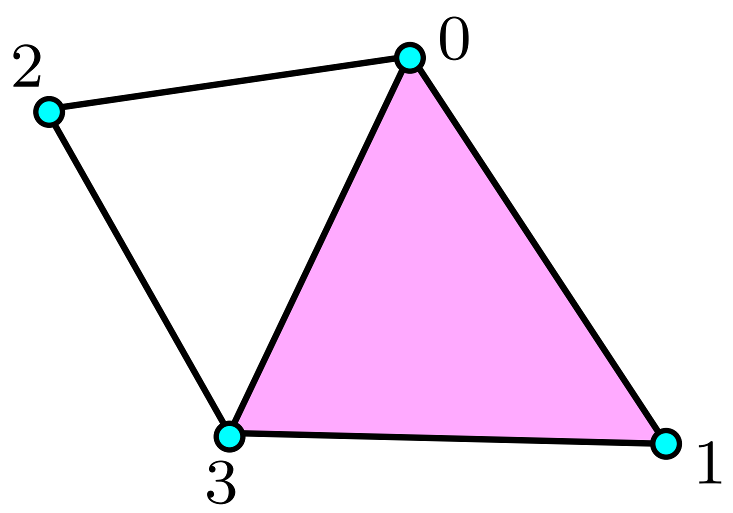


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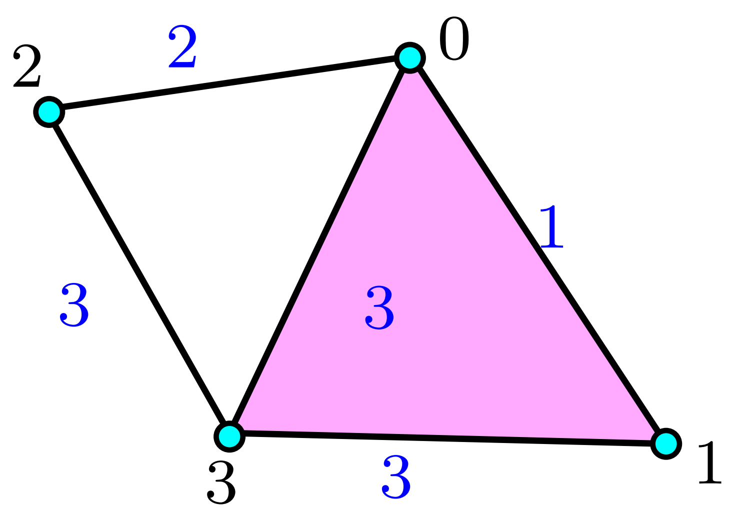


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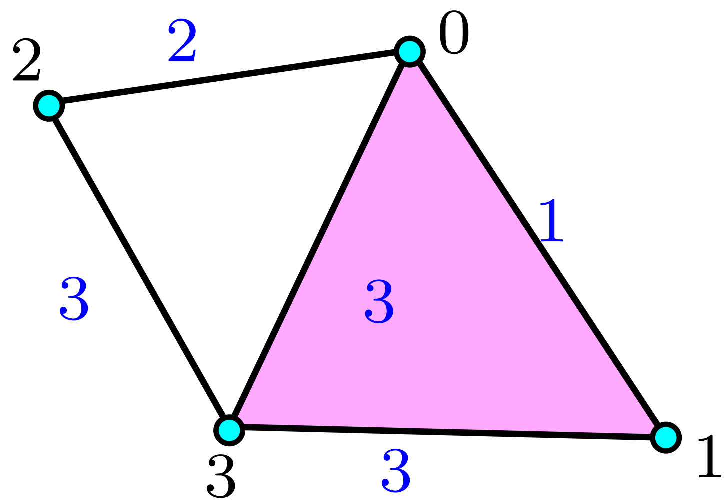


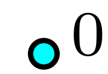


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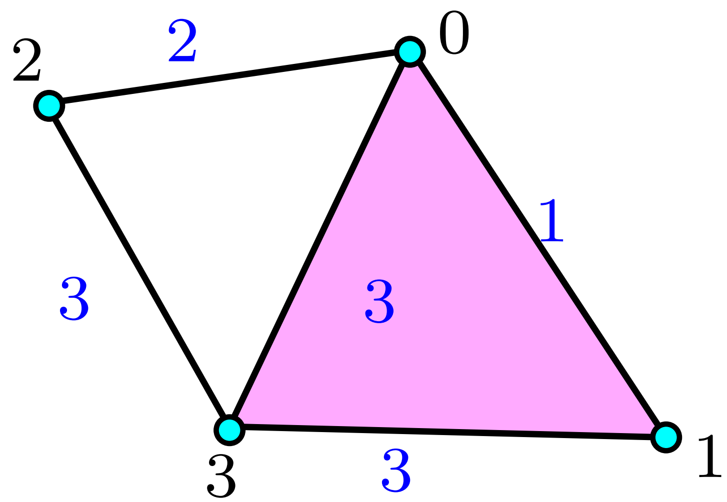



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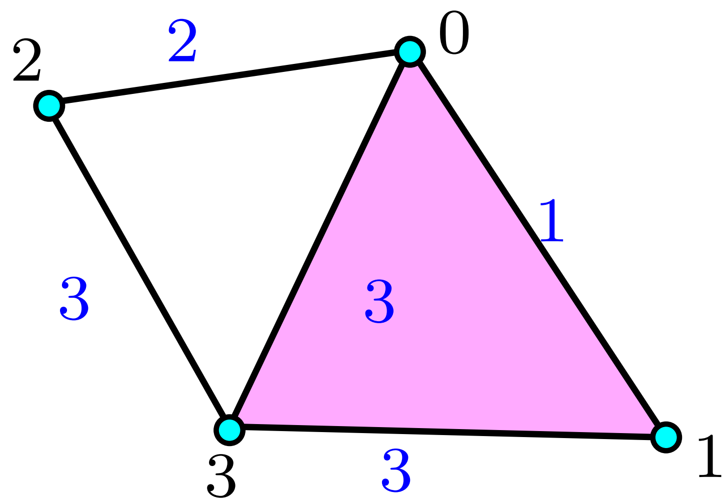


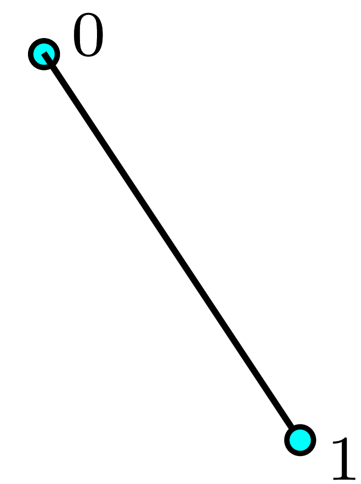


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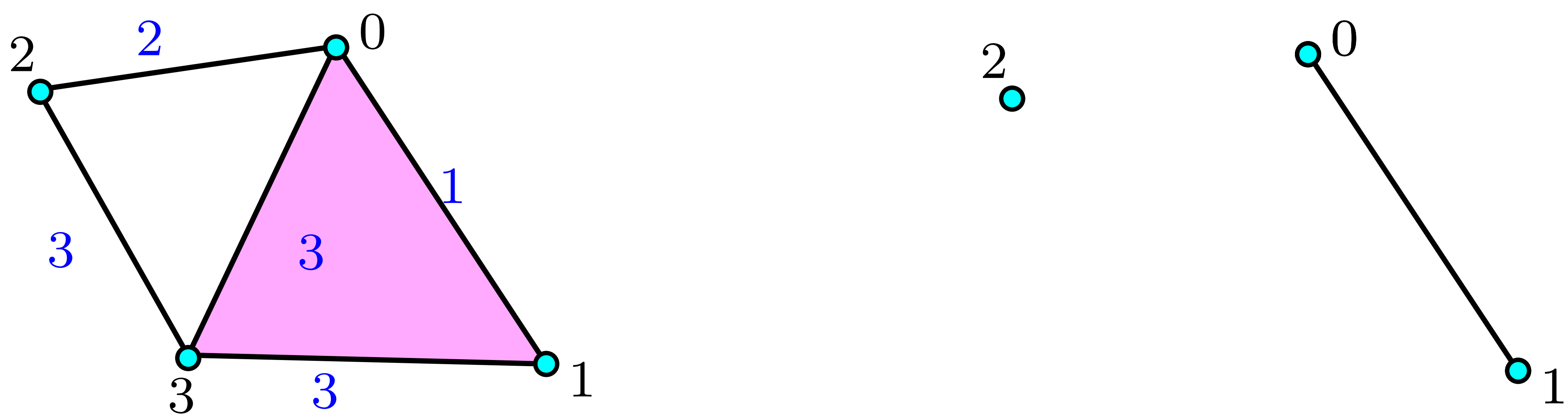


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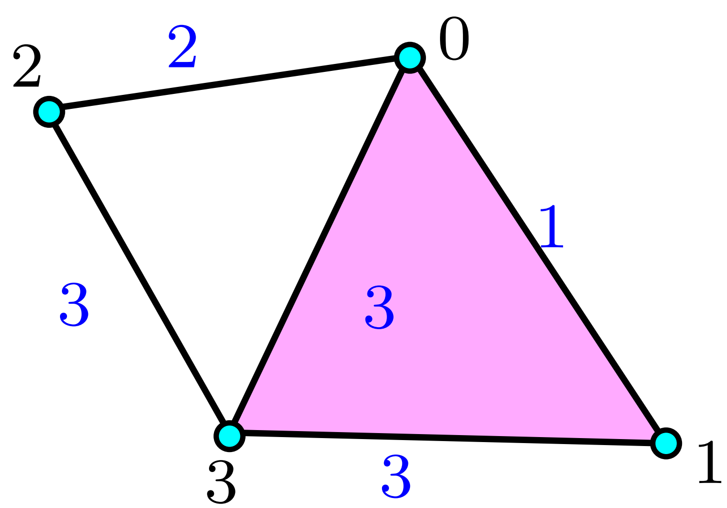


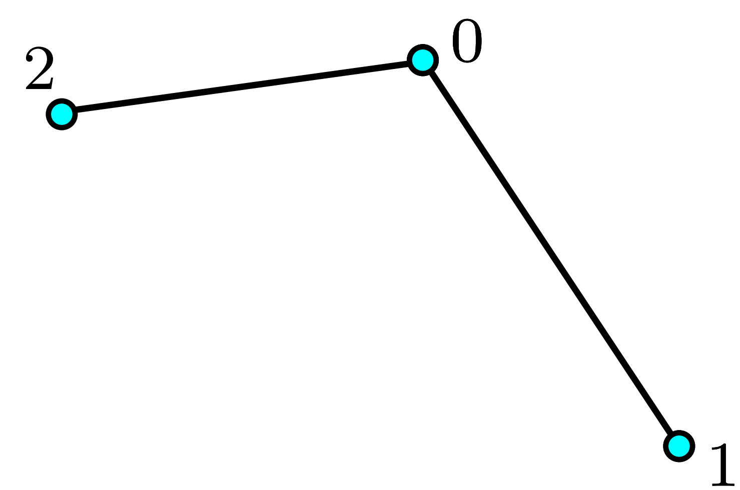


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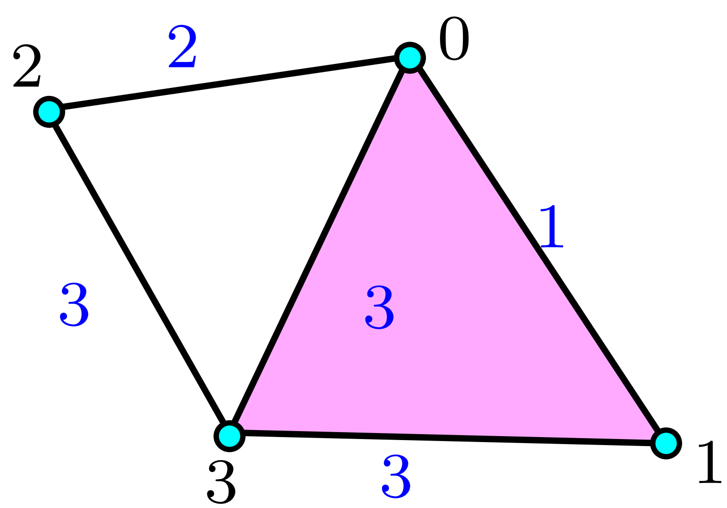


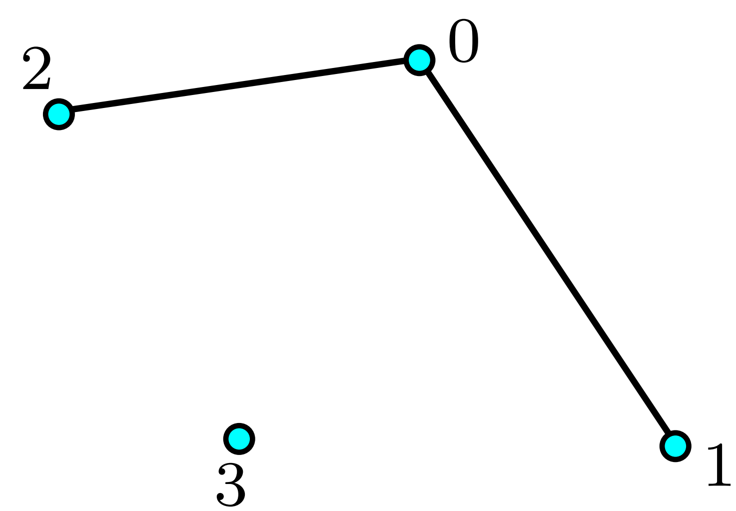


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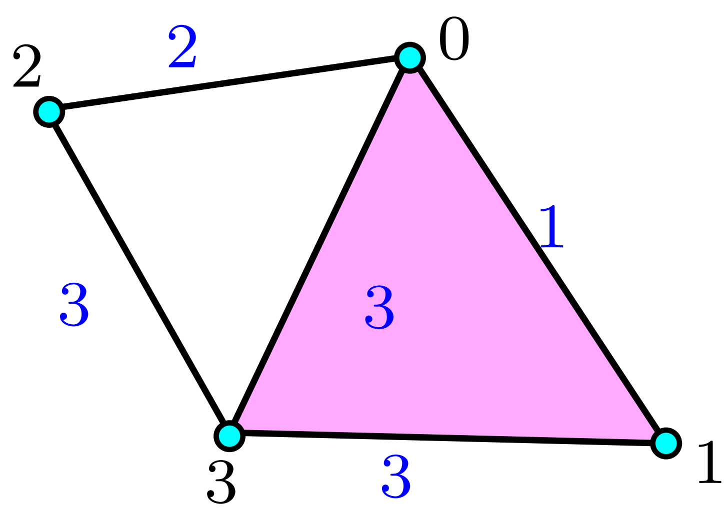


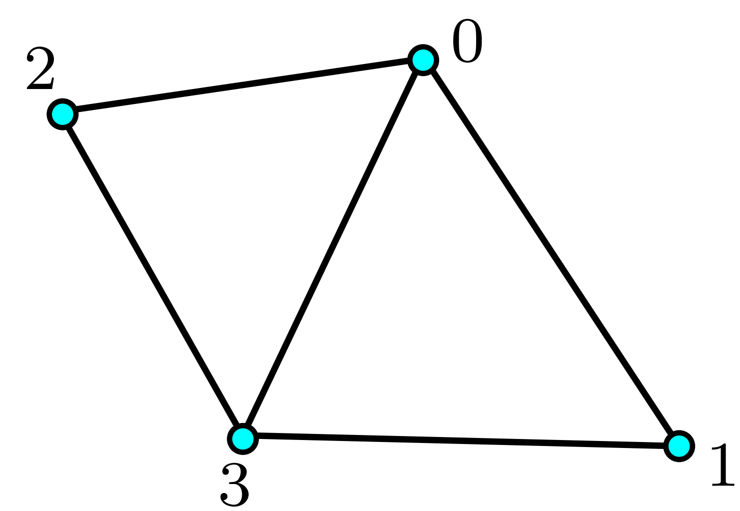


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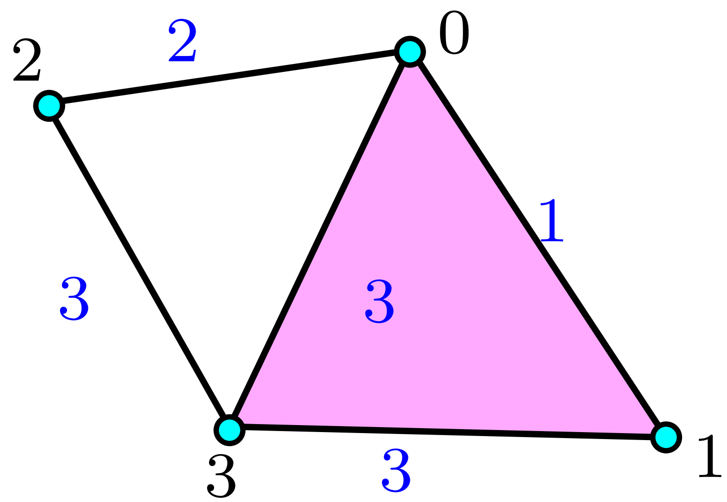


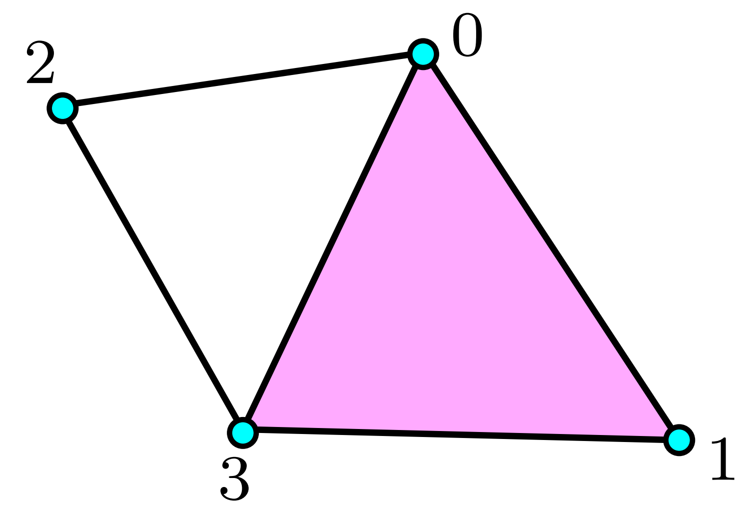


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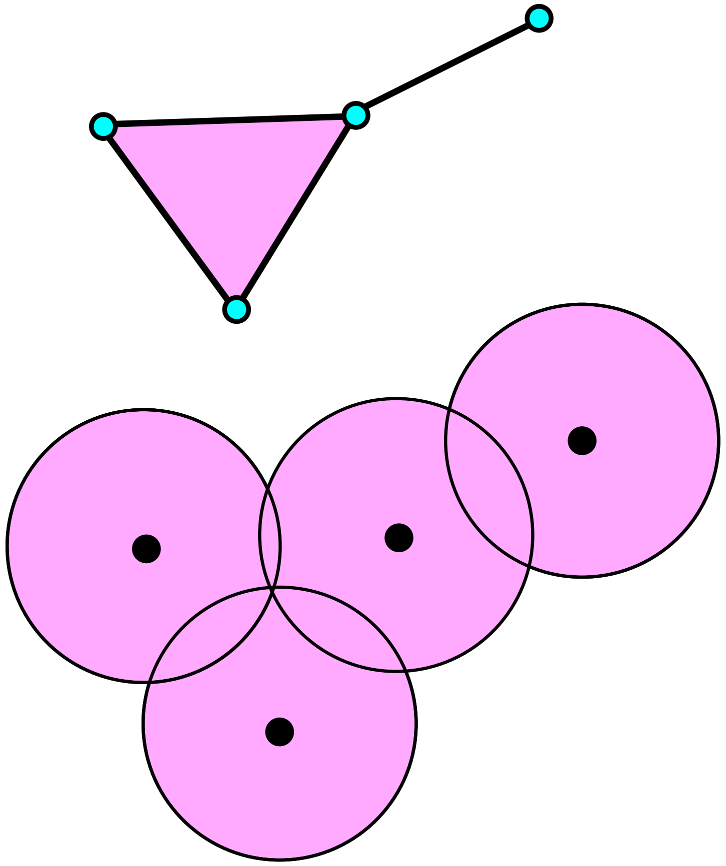
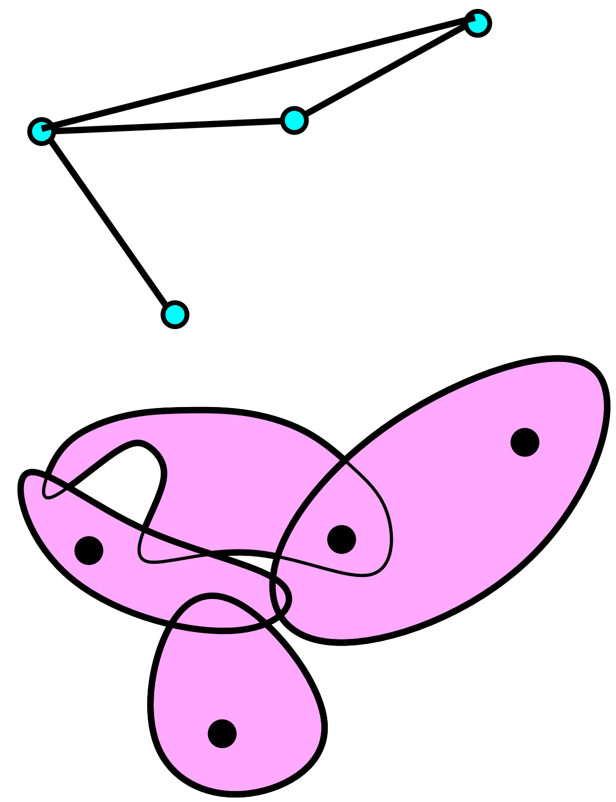
# Example: The Cěch complex





- Let $\mathcal{U} = (U_i)_{i \in I}$ be a covering of a topological space $X$ by open sets: $X = \cup_{i \in I} U_i$.

- The Cěch complex $C(\mathcal{U})$ associated to the covering $\mathcal{U}$ is the simplicial complex defined by:
  - the vertex set of $C(\mathcal{U})$ is the set of the open sets $U_i$
  - $[U_{i_0}, \cdots, U_{i_k}]$ is a $k$-simplex in $C(\mathcal{U})$ iff $\cap_{j=0}^{k} U_{i_j} \neq \emptyset$.

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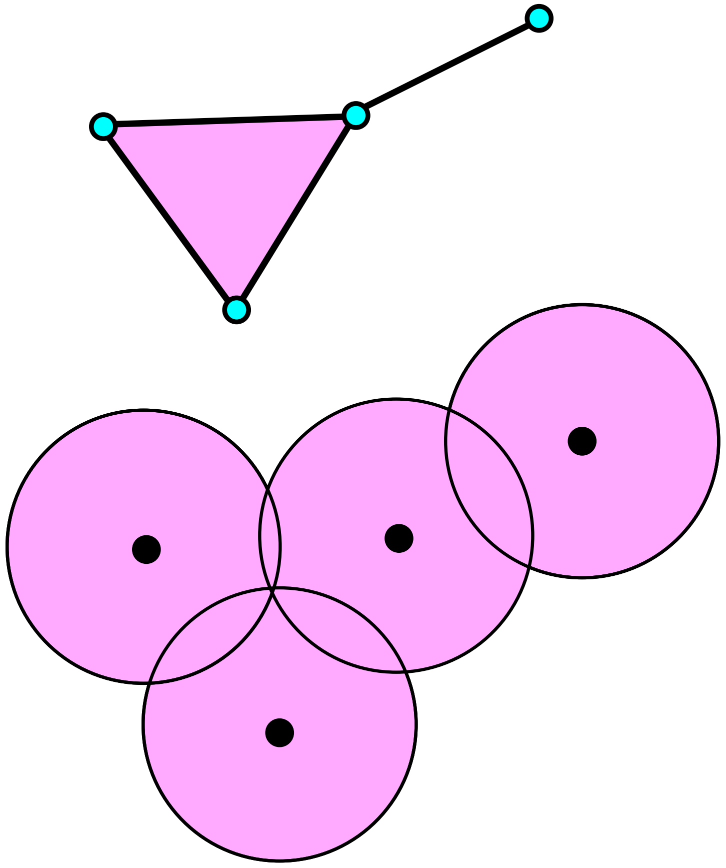

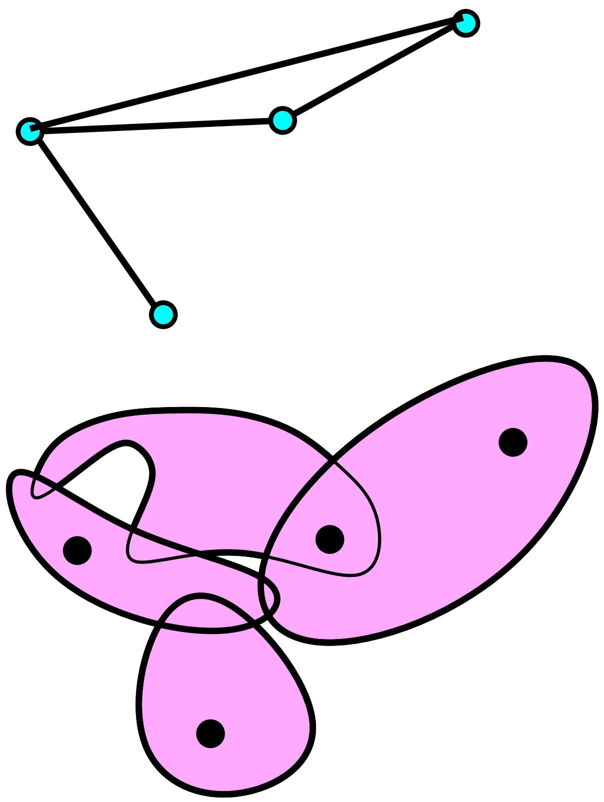

**Nerve theorem (Leray):** If all the intersections between opens in $\mathcal{U}$ are either empty or contractible then $C(\mathcal{U})$ and $X = \cup_{i \in I} U_i$ are homotopy equivalent.

$\Rightarrow$ The combinatorics of the covering (a simplicial complex) carries the topology of the space.

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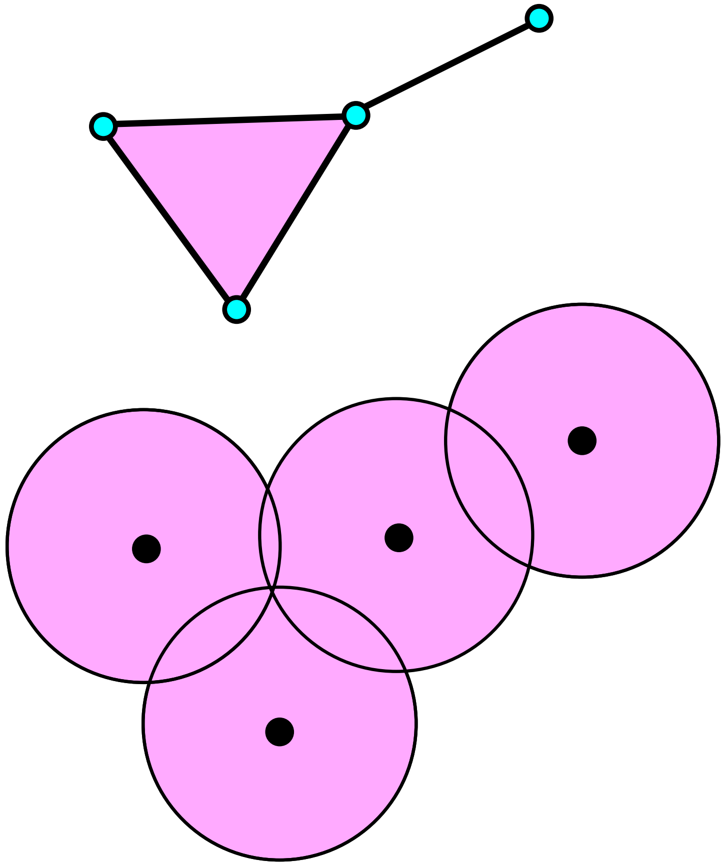

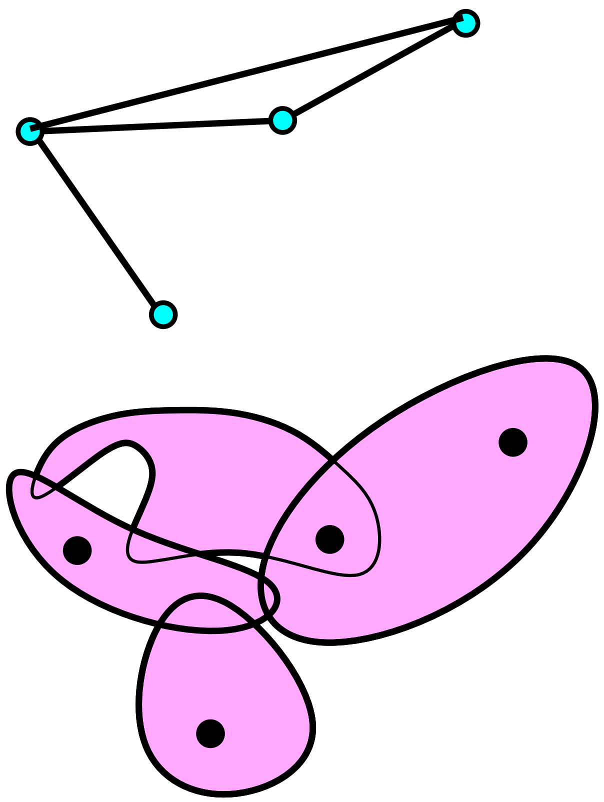

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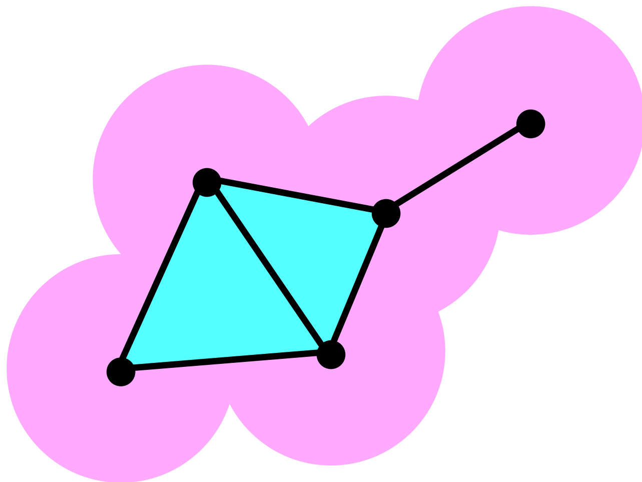
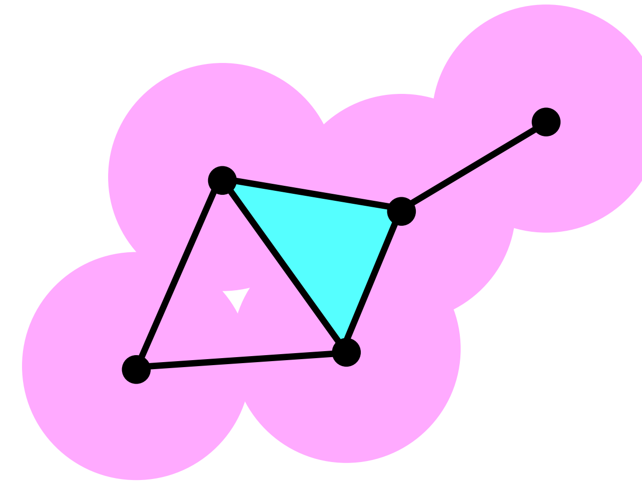
**Warning:** even when the open sets are euclidean balls, the computation of the Cěch complex is a very difficult task!

# Example: the Rips complex



Rips vs Čech



Let $L = \{p_0, \cdots p_n\}$ be a (finite) point cloud (in a metric space).

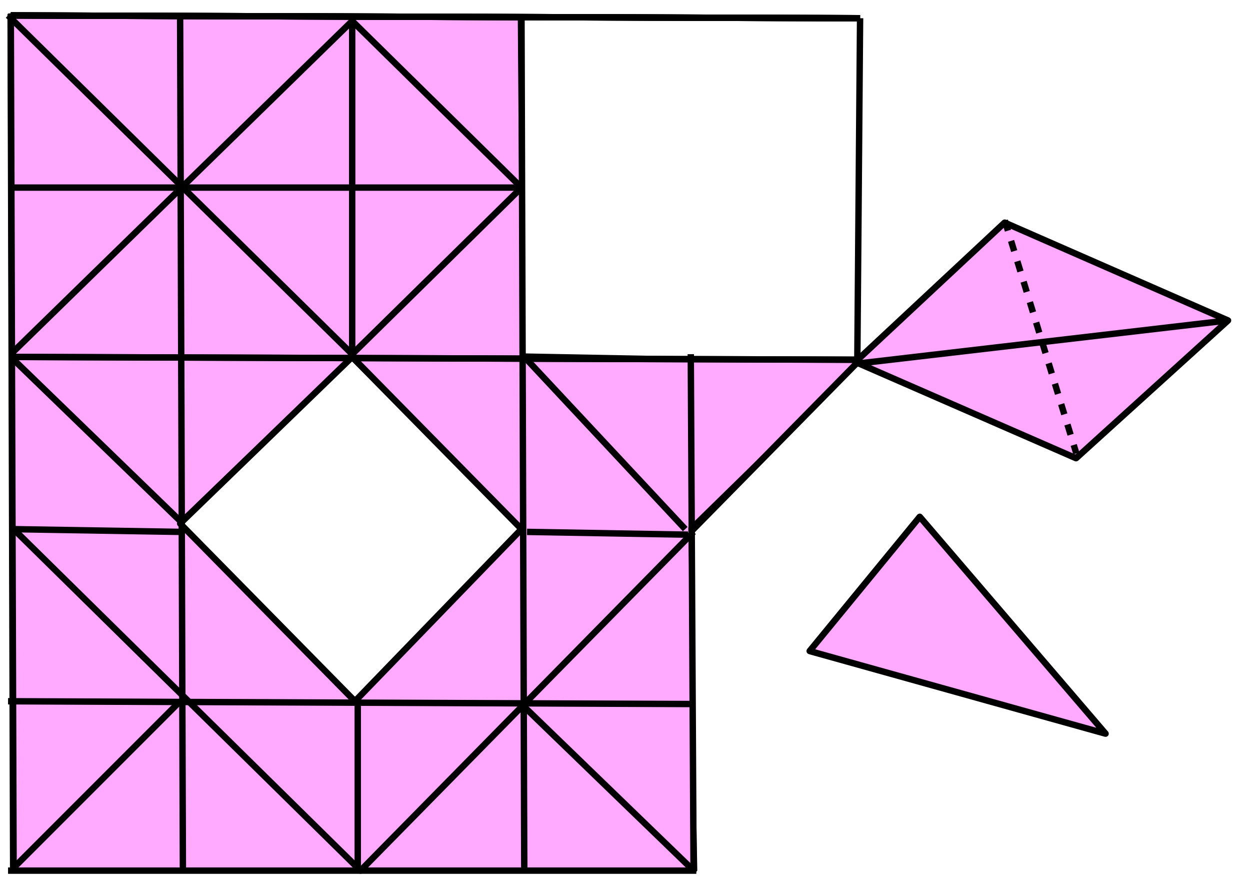
The Rips complex $\mathcal{R}^\alpha(L)$: for $p_0, \cdots p_k \in L$,

$$\sigma = [p_0 p_1 \cdots p_k] \in \mathcal{R}^\alpha(L) \text{ iff } \forall i, j \in \{0, \cdots k\},\ d(p_i, p_j) \leq \alpha$$

- Easy to compute and fully determined by its $1$-skeleton
- Rips-Čech interleaving: for any $\alpha > 0$,

$$\mathcal{C}^{\frac{\alpha}{2}}(L) \subseteq \mathcal{R}^\alpha(L) \subseteq \mathcal{C}^\alpha(L) \subseteq \mathcal{R}^{2\alpha}(L) \subseteq \cdots$$

# Homology of simplicial complexes



- 2 connected components
- Intuitively: 2 cycles

Topological invariants:
- Number of connected components
- Number of cycles: how to define a cycle?
- Number of voids: how to define a void?
- ...

(Simplicial) homology and Betti numbers

In the following: homology with coeeficient in $\mathbb{Z}/2$

Refs: J.R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, 1984.
A. Hatcher, *Algebraic Topology*, Cambridge University Press 2002.

# The space of $k$-chains

Let $K$ be a $d$-dimensional simplicial complex. Let $k \in \{0, 1, \cdots, d\}$ and $\{\sigma_1, \cdots, \sigma_p\}$ be the set of $k$-simplices of $K$.

$k$-chain:

$$c = \sum_{i=1}^{p} \varepsilon_i \sigma_i \quad \text{with} \quad \varepsilon_i \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$$

Sum of $k$-chains:

$$c + c' = \sum_{i=1}^{p} (\varepsilon_i + \varepsilon_i') \sigma_i \quad \text{and} \quad \lambda . c = \sum_{i=1}^{p} (\lambda \varepsilon_i') \sigma_i$$

where the sums $\varepsilon_i + \varepsilon_i'$ and the products $\lambda \varepsilon_i$ are modulo $2$.

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The space $\mathcal{C}_k(K)$ of $k$-chains is a $\mathbb{Z}/2$-vector space

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Geometric interpretation:
$k$-chain $=$ union of $k$-simplices
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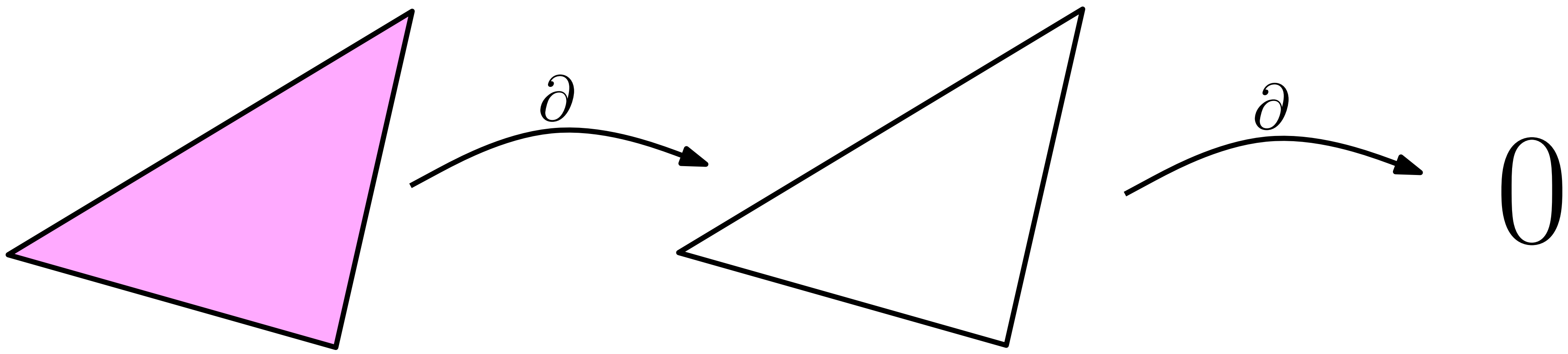
sum $c + c'$ = symmetric difference

# The boundary operator




The boundary $\partial\sigma$ of a $k$-simplex $\sigma$ is the sum of its $(k-1)$-faces. This is a $(k-1)$-chain.

$$If \sigma = [v_0, \cdots, v_k] \ \text{ then } \ \partial\sigma = \sum_{i=0}^{k} [v_0 \cdots \hat{v}_i \cdots v_k]$$

The boundary operator is the linear map defined by

$$\begin{array}{rlll} \partial : & \mathcal{C}_k(K) & \rightarrow & \mathcal{C}_{k-1}(K) \\ & c & \rightarrow & \partial c = \sum_{\sigma \in c} \partial\sigma \end{array}$$

# Fundamental property of the boundary operator

$$\partial\partial := \partial \circ \partial = 0$$

**Proof:** by linearity it is just necessary to prove it for a simplex.

$$\begin{aligned}
\partial\partial\sigma &= \partial\left(\sum_{i=0}^{k}[v_0\cdots\hat{v}_i\cdots v_k]\right) \\
&= \sum_{i=0}^{k}\partial[v_0\cdots\hat{v}_i\cdots v_k] \\
&= \sum_{j<i}[v_0\cdots\hat{v}_j\cdots\hat{v}_i\cdots v_k] + \sum_{j>i}[v_0\cdots\hat{v}_i\cdots\hat{v}_j\cdots v_k] \\
&= 0
\end{aligned}$$

# Cycles and boundaries

The chain complex associated to a complex $K$ of dimension $d$

$$\emptyset \to \mathcal{C}_d(K) \xrightarrow{\partial} \mathcal{C}_{d-1}(K) \xrightarrow{\partial} \cdots \mathcal{C}_{k+1}(K) \xrightarrow{\partial} \mathcal{C}_k(K) \xrightarrow{\partial} \cdots \mathcal{C}_1(K) \xrightarrow{\partial} \mathcal{C}_0(K) \xrightarrow{\partial} \emptyset$$

$k$-cycles:

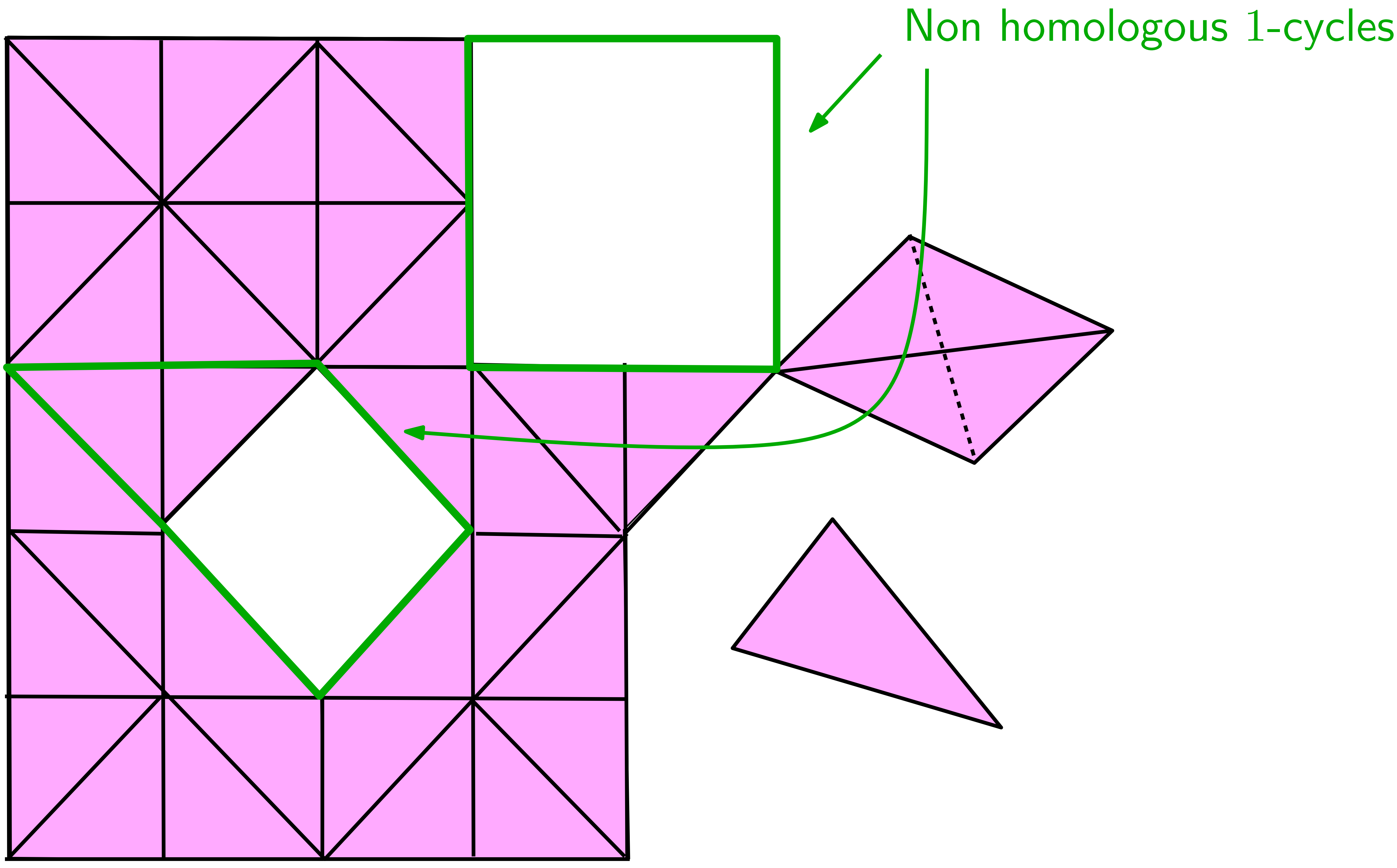
$$Z_k(K) := \ker(\partial : \mathcal{C}_k \to \mathcal{C}_{k-1}) = \{c \in \mathcal{C}_k : \partial c = \emptyset\}$$

$k$-boundaries:

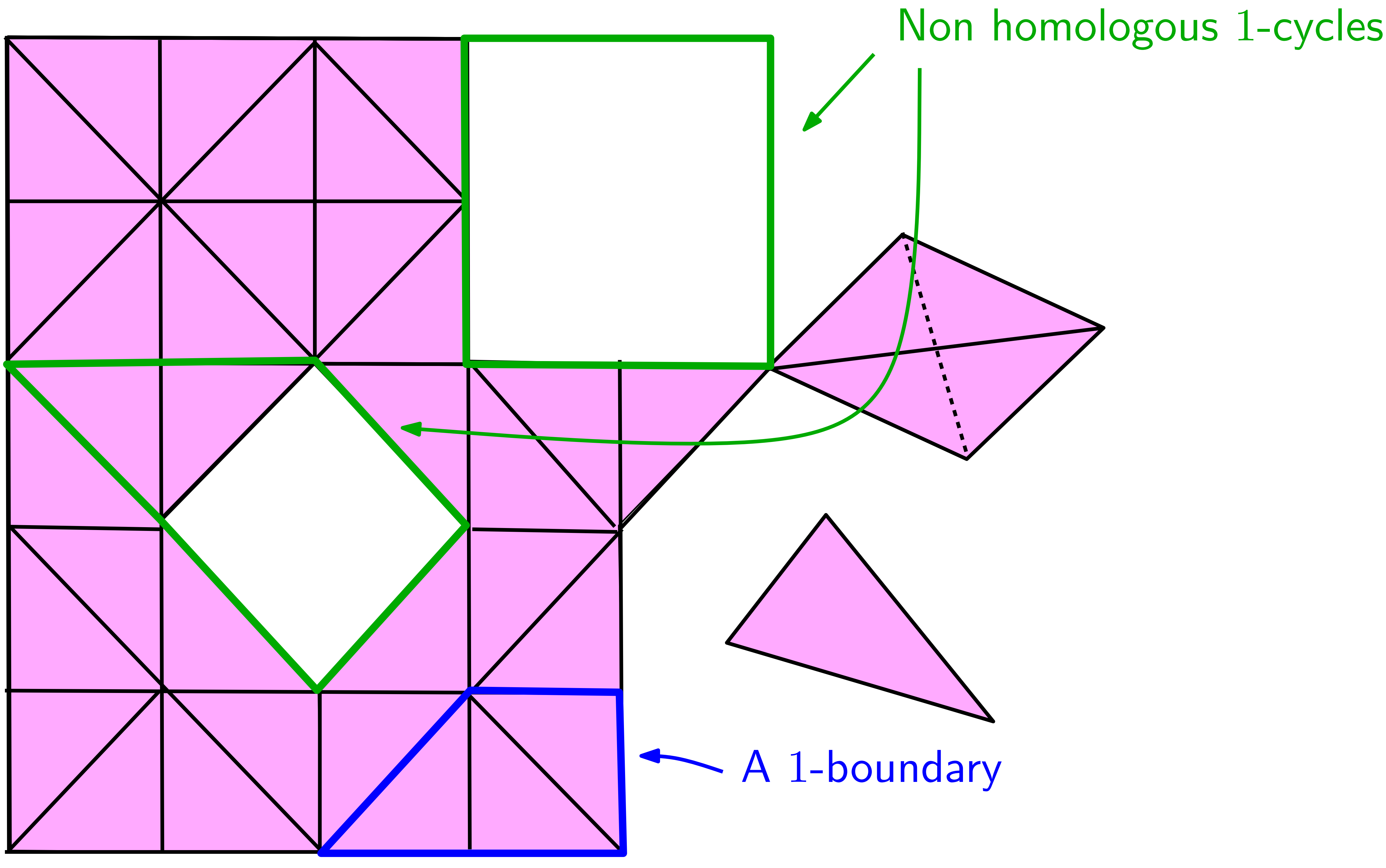
$$B_k(K) := im(\partial : \mathcal{C}_{k+1} \to \mathcal{C}_k) = \{c \in \mathcal{C}_k : \exists c' \in \mathcal{C}_{k+1}, c = \partial c'\}$$

$$B_k(K) \subset Z_k(K) \subset \mathcal{C}_k(K)$$

# Cycles and boundaries



# Cycles and boundaries



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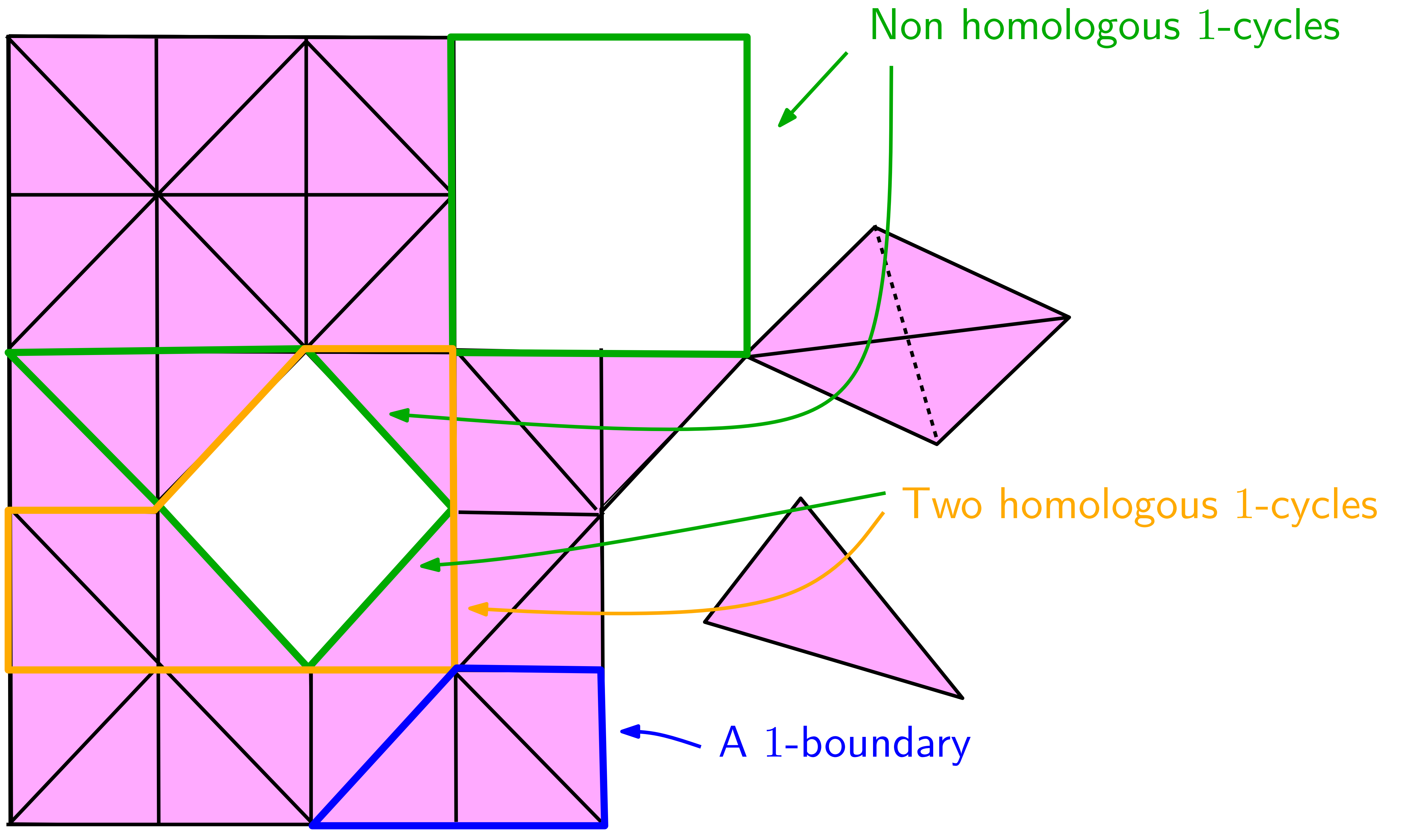
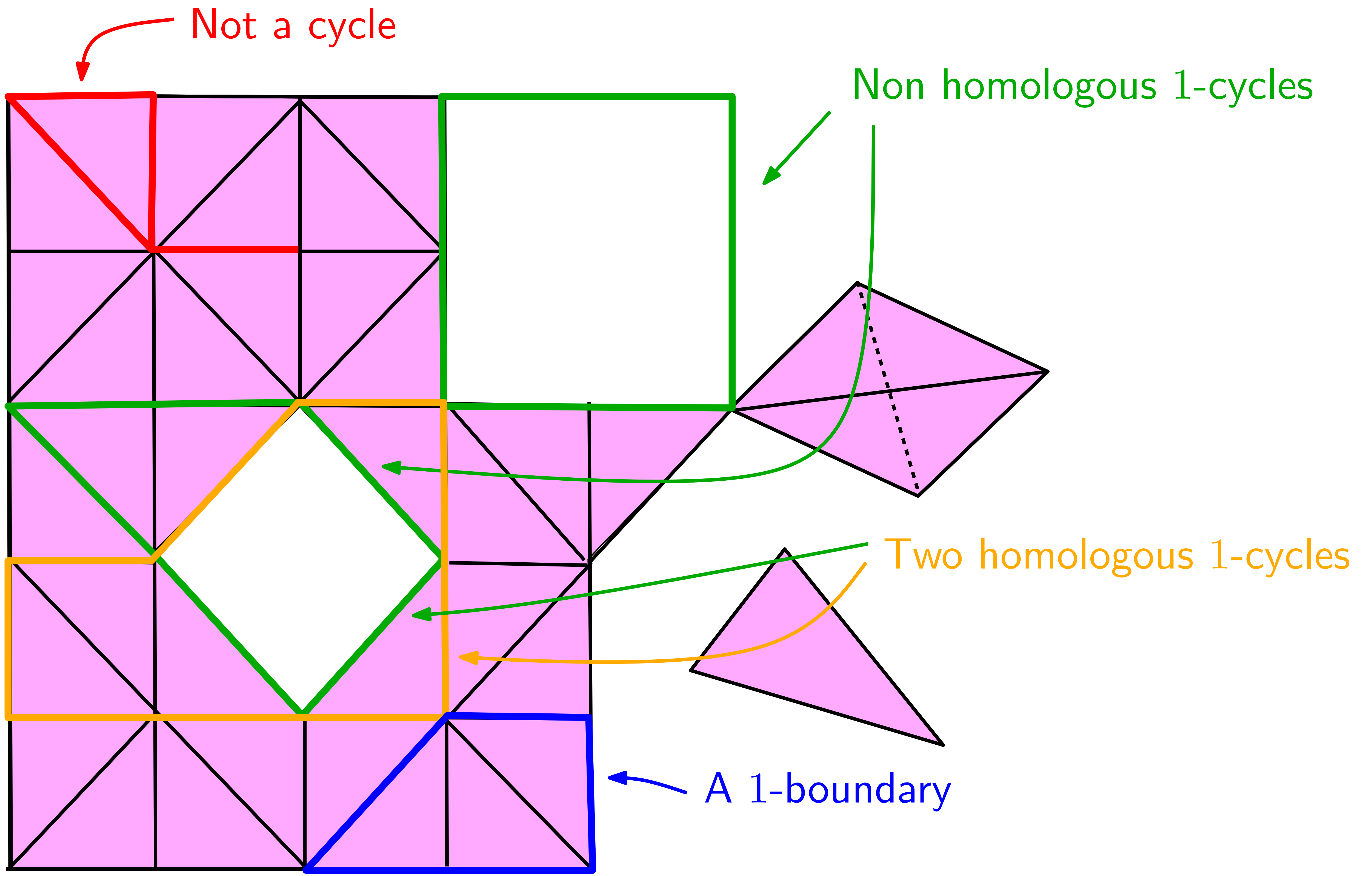


# Cycles and boundaries



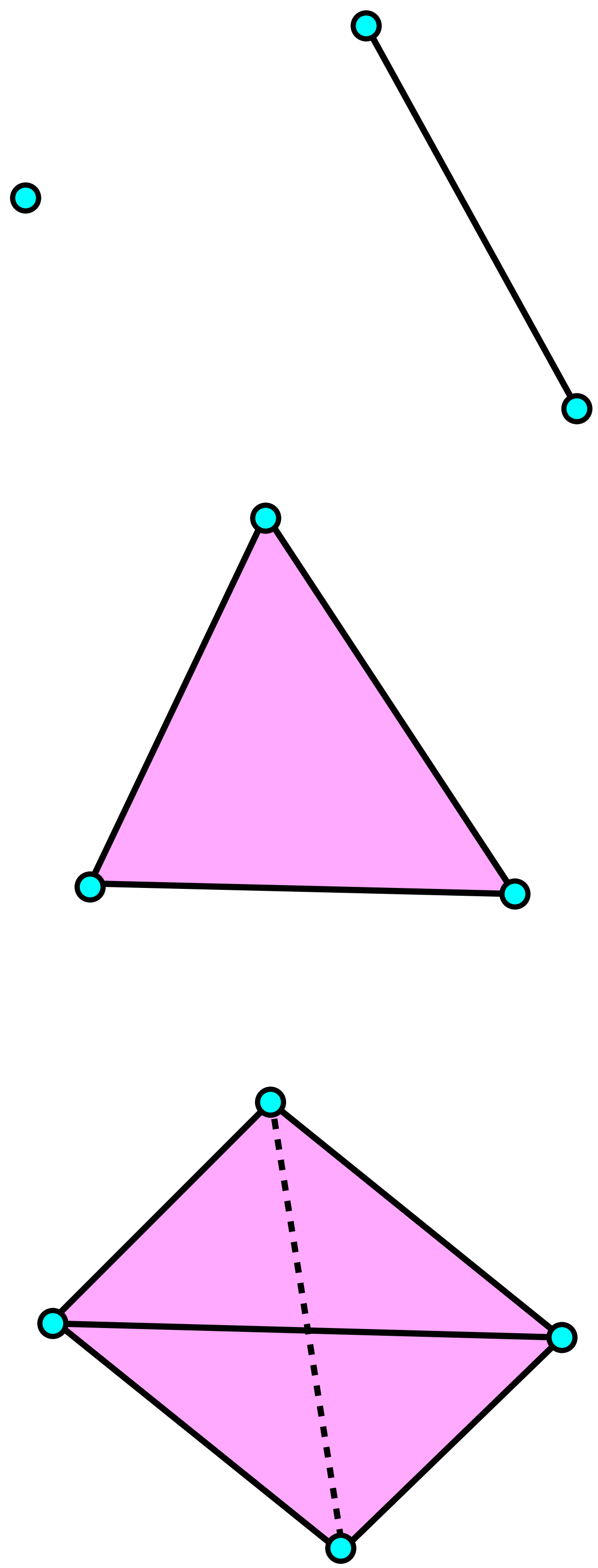
# Homology groups and Betti numbers

$$B_k(K) \subset Z_k(K) \subset \mathcal{C}_k(K)$$

- The $k^{th}$ homology group of $K$: $H_k(K) = Z_k/B_k$
- Tout each cycle $c \in Z_k(K)$ corresponds its homology class $c + B_k(K) = \{c + b : b \in B_k(K)\}$.
- Two cycles $c, c'$ are homologous if they are in the same homology class: $\exists b \in B_k(K)$ s. t. $b = c' - c (= c' + c)$.
- The $k^{th}$ Betti number of $K$: $\beta_k(K) = \dim(H_k(K))$.

# Elementary examples

**Remark:** $\beta_0 =$ number of connected components of $K$



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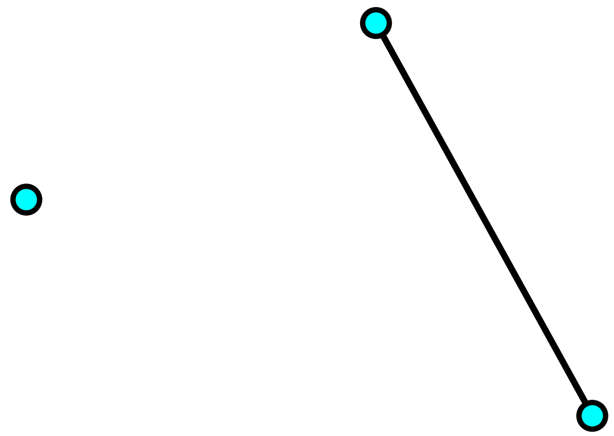
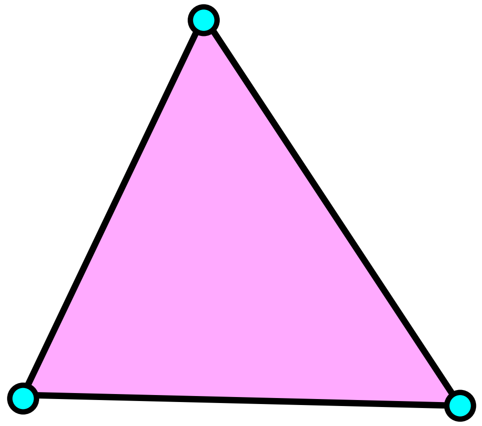
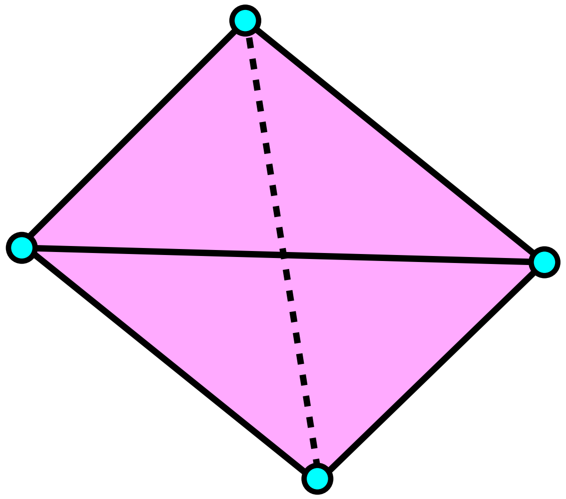


$\beta_0 = 2$
$\beta_1 = 0$
$\beta_2 = 0$





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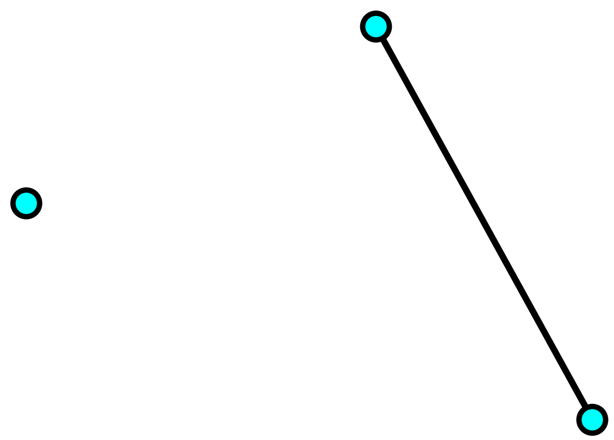

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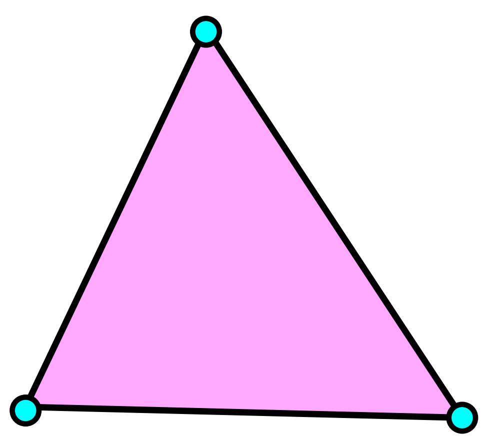
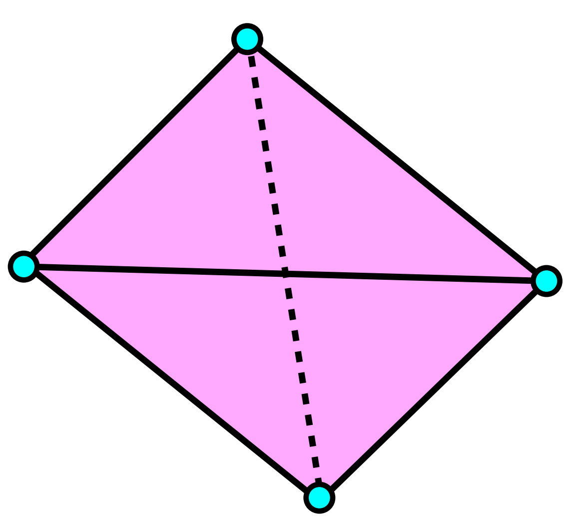


$\beta_0 = 1$
$\beta_1 = 0$
$\beta_2 = 0$



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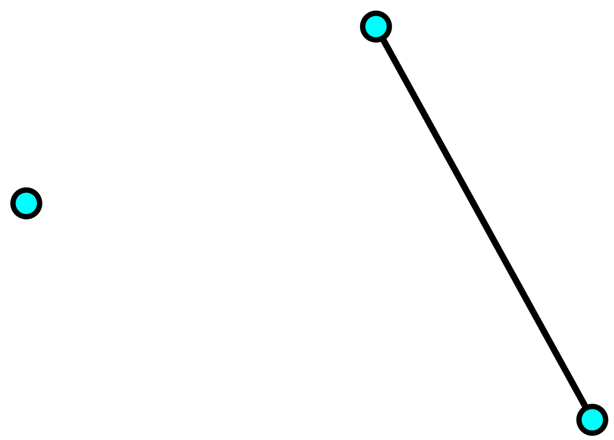

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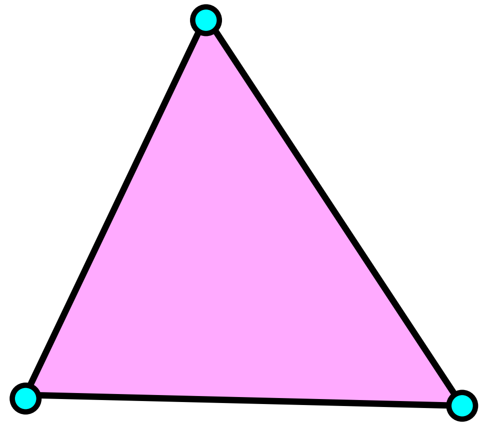

$\beta_0 = 1$
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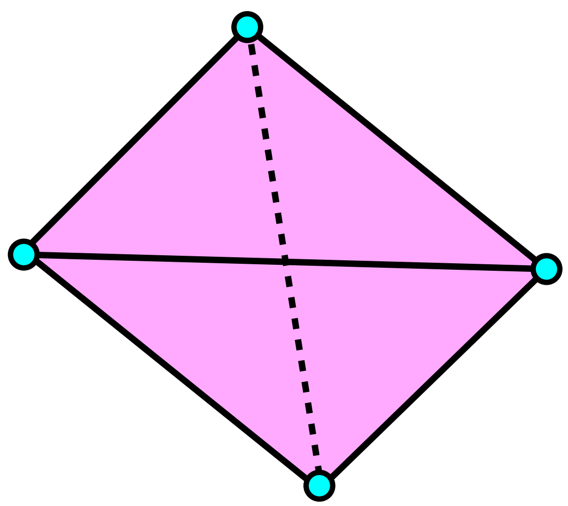
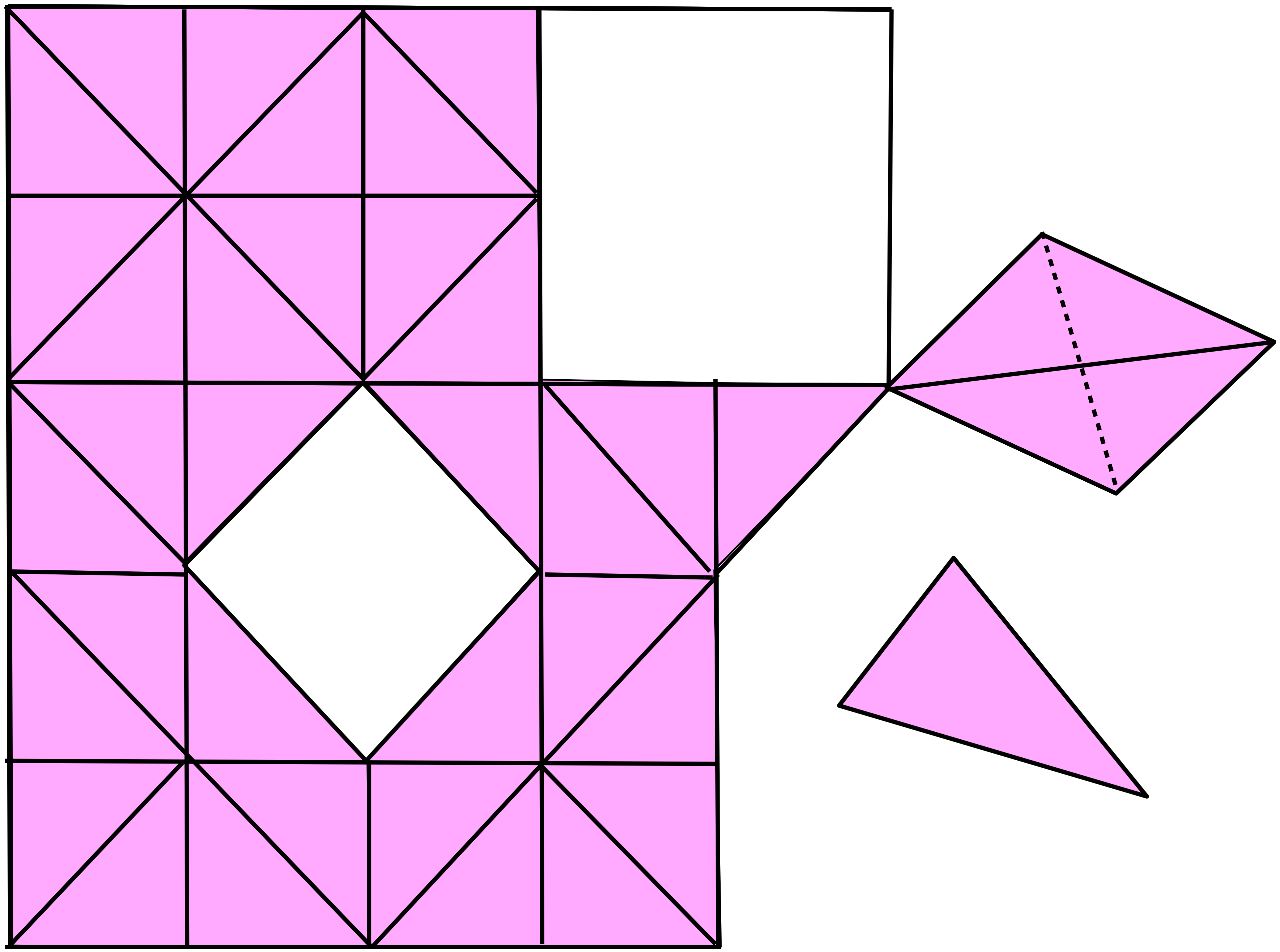
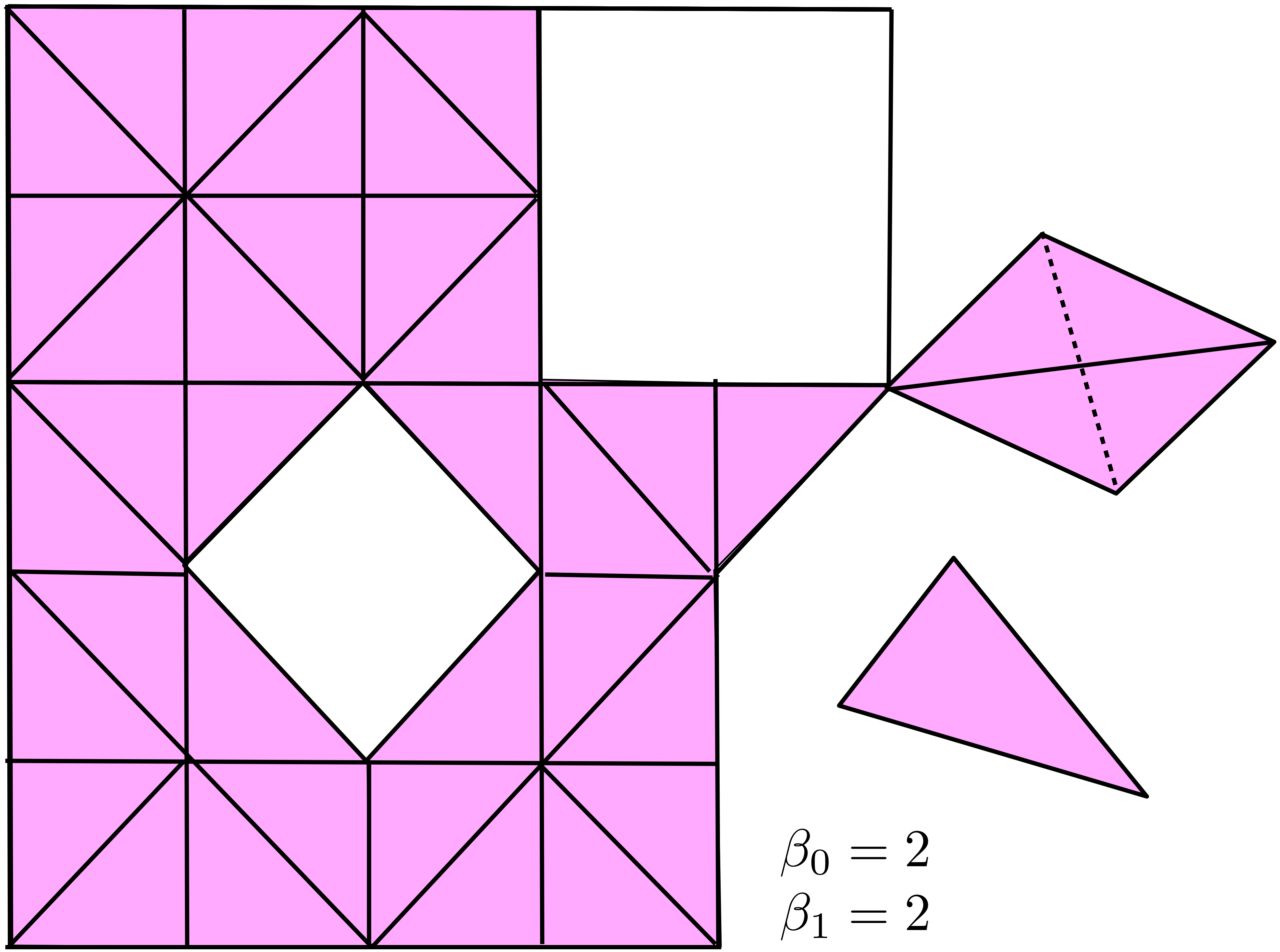


$\beta_0 = 1$
$\beta_1 = 0$
$\beta_2 = 1$ if empty and $\beta_2 = 0$ if filled
$\beta_3 = 0$

# Elementary examples

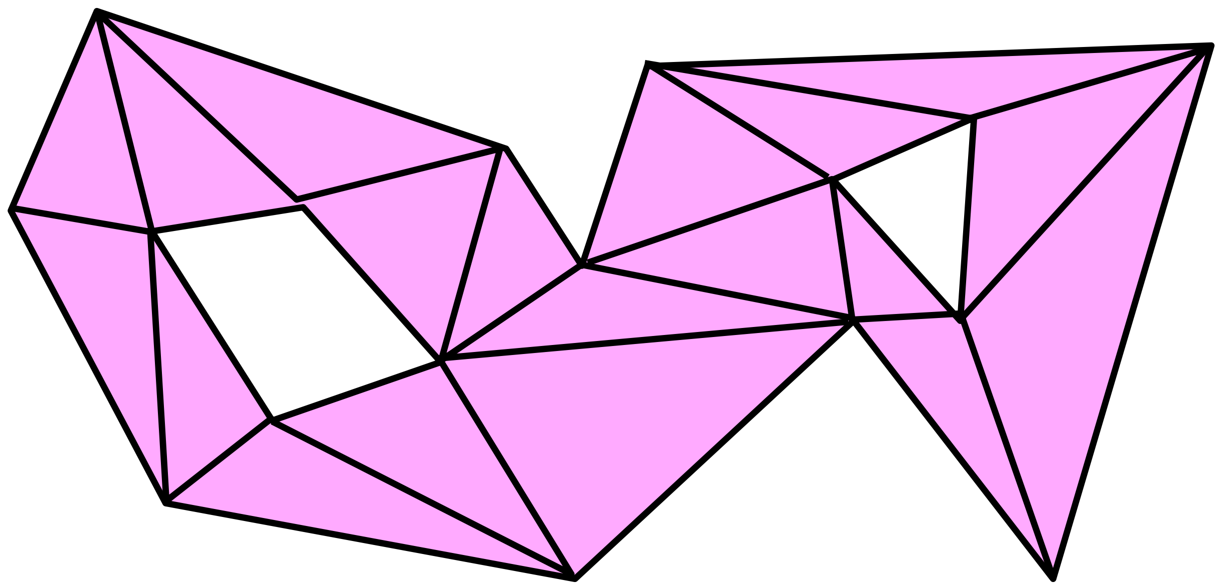


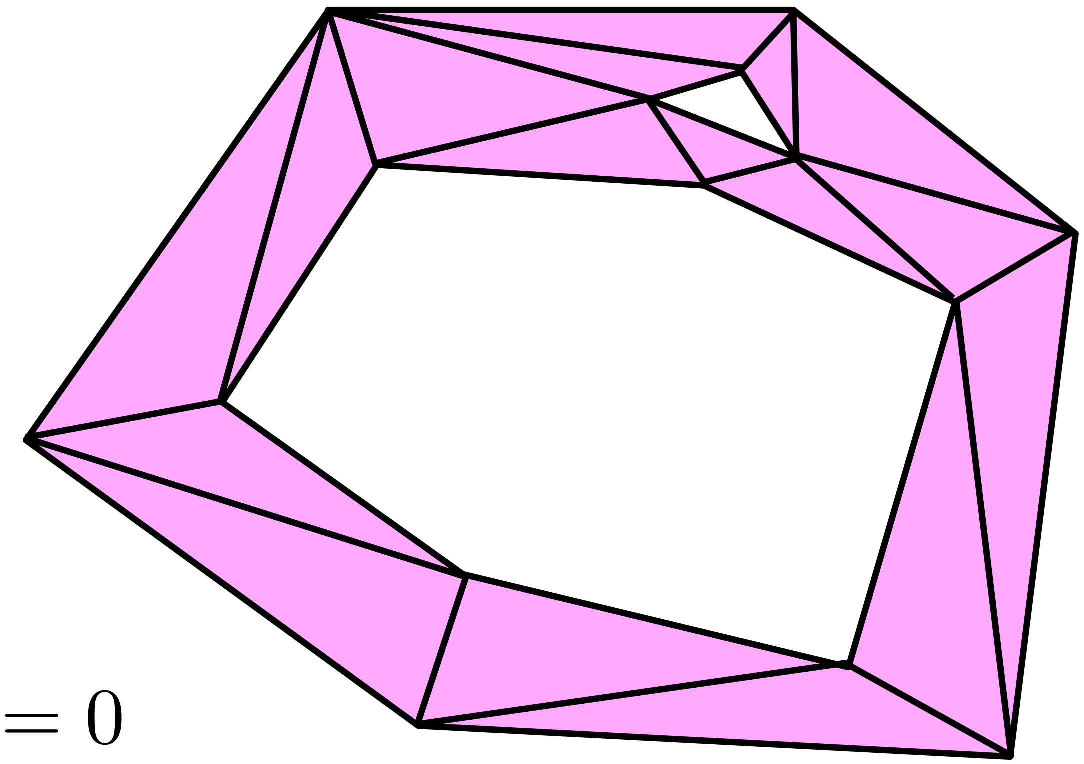
# Elementary examples



$\beta_0 = 2$
$\beta_1 = 2$
$\beta_2 = 1$ if empty and $\beta_2 = 0$ if filled
$\beta_3 = 0$

# Topological invariance and singular homology



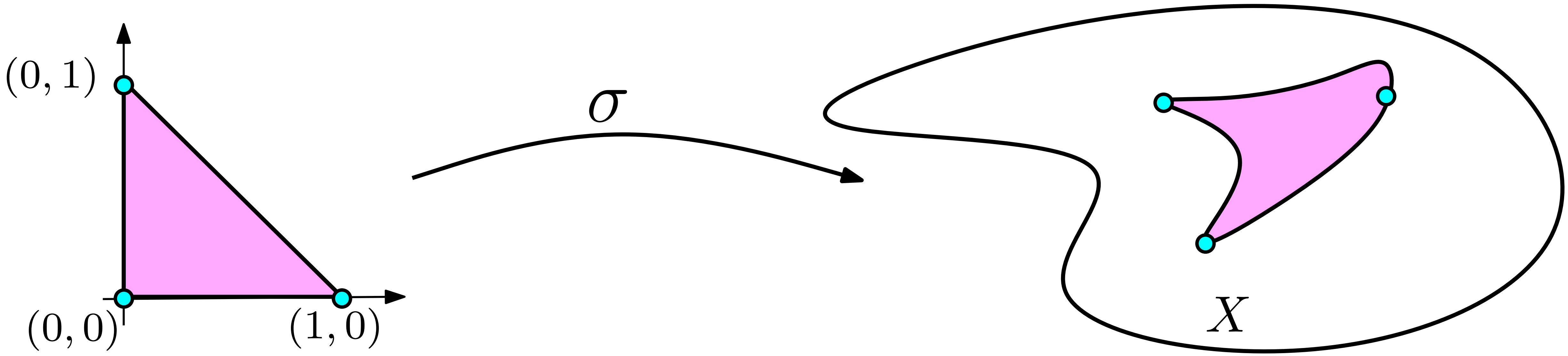


$\beta_0 = 1,\ \beta_1 = 2, \beta_2 = 0$

**Theorem:** If $K$ and $K'$ are two simplicial complexes with homeomorphic supports then their homology groups are isomorphic and their Betti numbers are equal.

- This is a classical result in algebraic topology but the proof is not obvious.
- Rely on the notion of singular homology $\rightarrow$ defined for any topological space.

# Topological invariance and singular homology



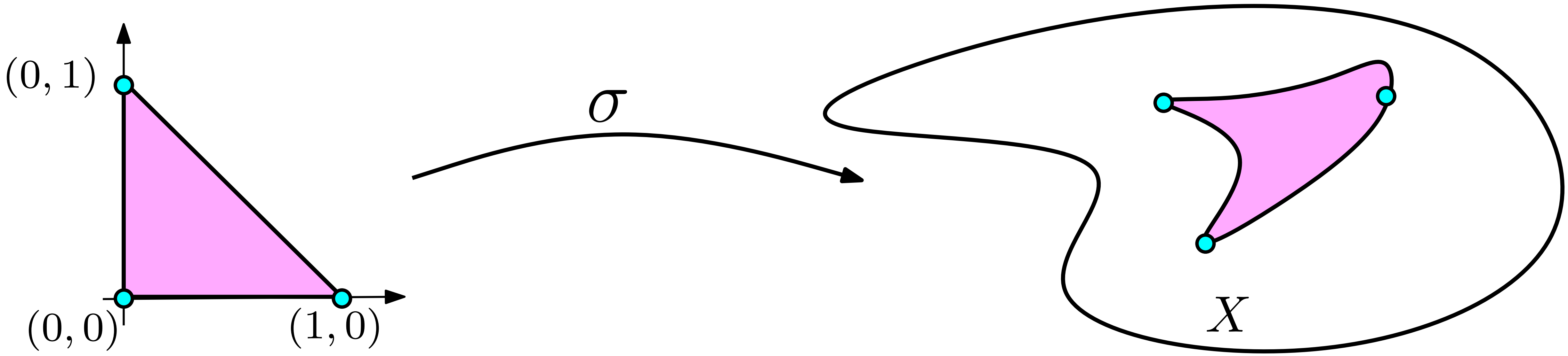

Let $\Delta_k$ be the standard simplex in $\mathbb{R}^k$. A singular $k$-simplex in a topological space $X$ is a continuous map $\sigma : \Delta_k \to X$.

The same construction as for simplicial homology can be done with singular complexes $\to$ Singular homology

Important properties:

- Singular homology is defined for any topological space $X$.
- If $X$ is homotopy equivalent to the support of a simplicial complex, then the singular and simplicial homology coincide!

# Topological invariance and singular homology




Let $\Delta_k$ be the standard simplex in $\mathbb{R}^k$. A singular $k$-simplex in a topological space $X$ is a continuous map $\sigma : \Delta_k \to X$.

Homology and continuous maps:

- if $f : X \to Y$ is a continuous map and $\sigma : \Delta_k \to X$ a simplex in $X$, then $f \circ \sigma : \Delta_k \to Y$ is a simplex in $Y \Rightarrow f$ induces a linear maps between homology groups:

$$f_\sharp : H_k(X) \to H_k(Y)$$

- if $f : X \to Y$ is an homeomorphism or an homotopy equivalence then $f_\sharp$ is an isomorphism.

# An algorithm for geometric inference

- $X \subset \mathbb{R}^d$ be a compact set such that $\mathrm{wfs}(X) > 0$.
- $L \subset \mathbb{R}^d$ be a finite set such that $d_H(X, L) < \varepsilon$ for some $\varepsilon > 0$.

# An algorithm for geometric inference

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**Goal:** Compute the Betti numbers of $X^r$ for $0 < r < \mathrm{wfs}(X)$ from $L$.

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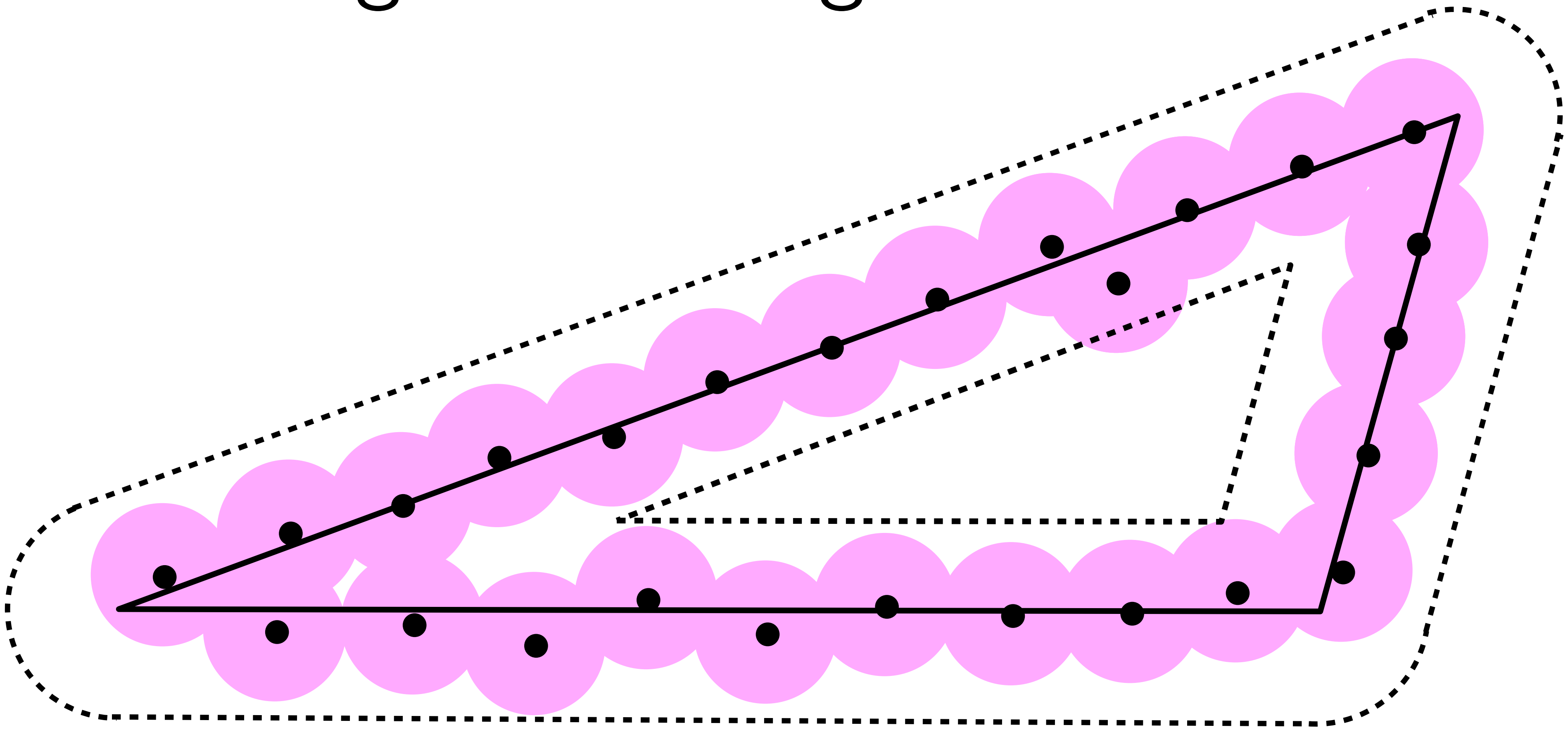
**Goal:** Compute the Betti numbers of $X^r$ for $0 < r < \mathrm{wfs}(X)$ from $L$.

**Theorem:** [CL'05 - CSEH'05]
Assume that wfs$(X) > 4\varepsilon$. For $\alpha > 0$ s.t. $\alpha + 4\varepsilon <$ wfs$(X)$, let $i : L^{\alpha+\varepsilon} \hookrightarrow L^{\alpha+3\varepsilon}$ be the canonical inclusion.For any $0 < r < \mathrm{wfs}(X)$,

$$H_k(X^r) \cong im\left(i_* : H_k(L^{\alpha+\varepsilon}) \to H_k(L^{\alpha+3\varepsilon})\right)$$

# An algorithm for geometric inference

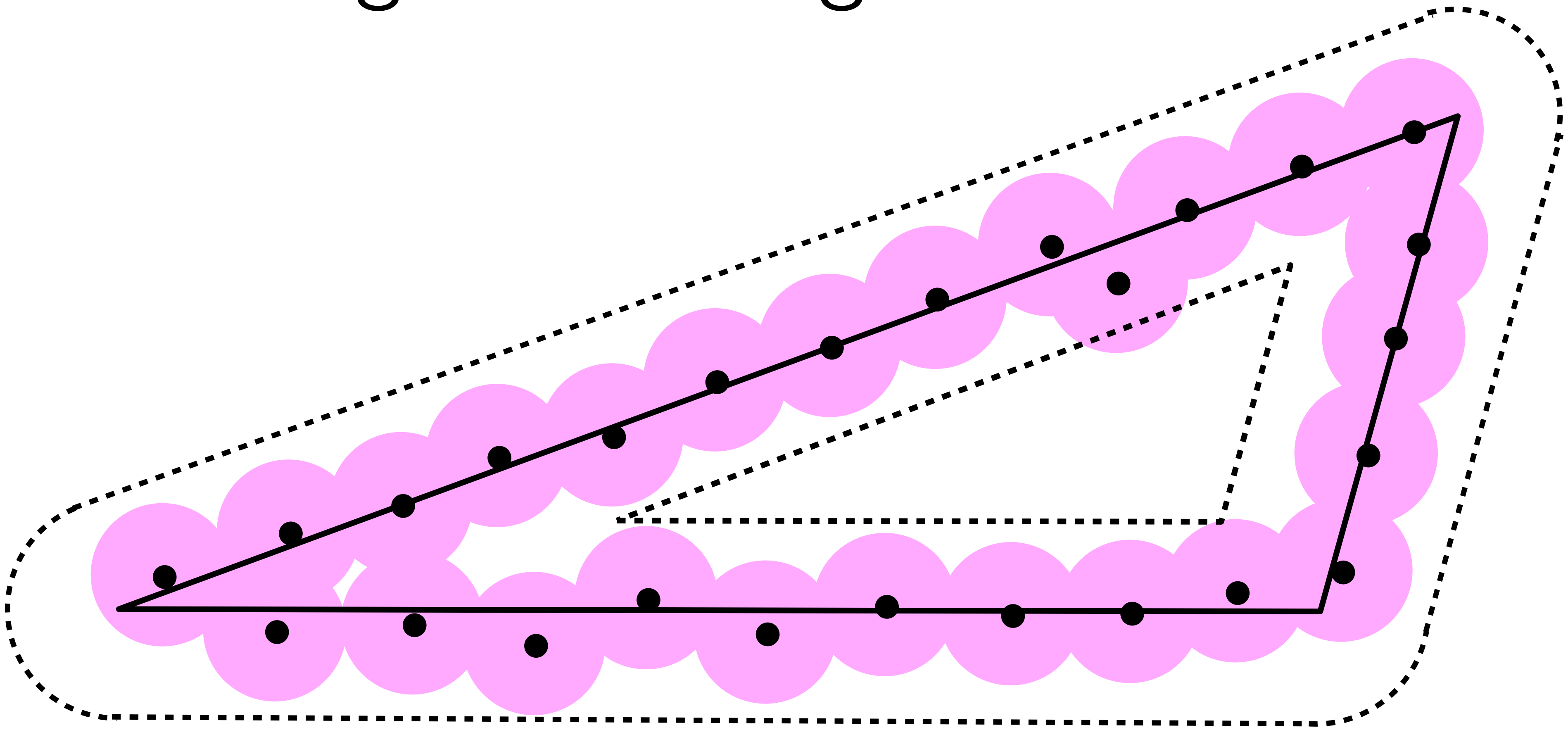
**Proof:**



For any $\alpha > 0$, $\quad X^{\alpha} \subseteq L^{\alpha+\varepsilon} \subseteq X^{\alpha+2\varepsilon} \subseteq L^{\alpha+3\varepsilon} \subseteq X^{\alpha+4\varepsilon} \subseteq \cdots$

# An algorithm for geometric inference

**Proof:**



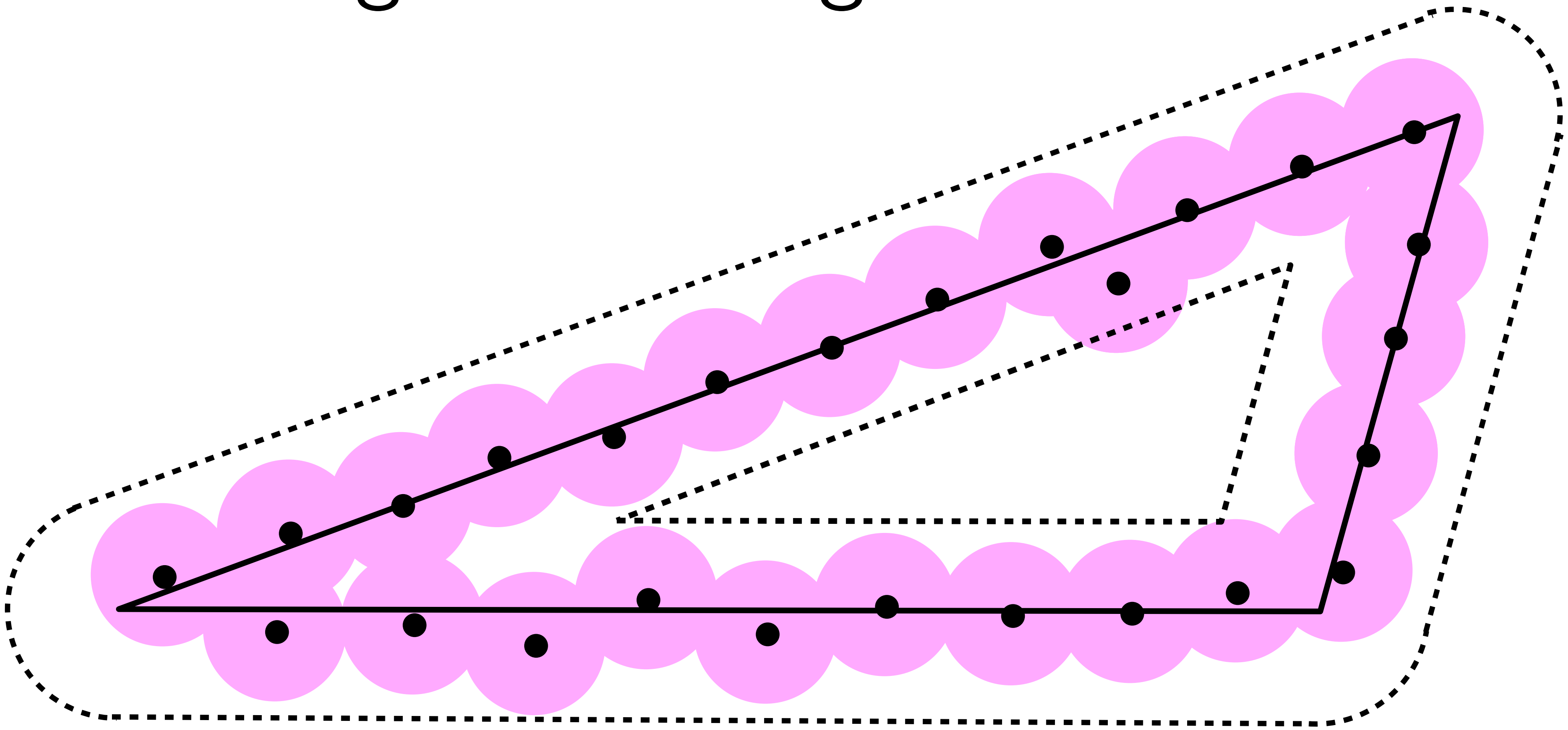
For any $\alpha > 0$, $\quad X^{\alpha} \subseteq L^{\alpha+\varepsilon} \subseteq X^{\alpha+2\varepsilon} \subseteq L^{\alpha+3\varepsilon} \subseteq X^{\alpha+4\varepsilon} \subseteq \cdots$

At homology level:

$$H_k(X^{\alpha}) \to H_k(L^{\alpha+\varepsilon}) \to H_k(X^{\alpha+2\varepsilon}) \to H_k(L^{\alpha+3\varepsilon}) \to H_k(X^{\alpha+4\varepsilon}) \to \cdots$$

# An algorithm for geometric inference

**Proof:**



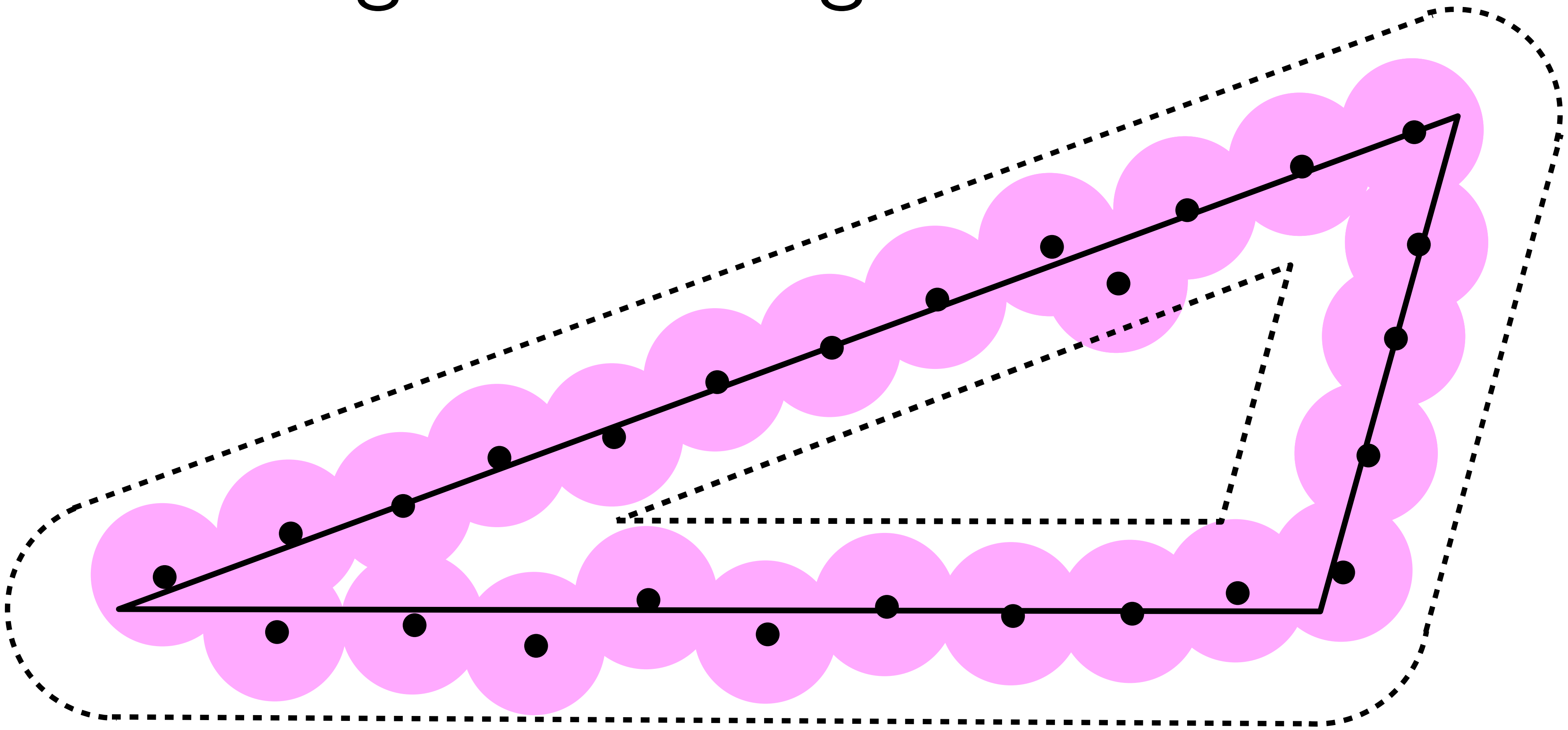
For any $\alpha > 0$, $\quad X^{\alpha} \subseteq L^{\alpha+\varepsilon} \subseteq X^{\alpha+2\varepsilon} \subseteq L^{\alpha+3\varepsilon} \subseteq X^{\alpha+4\varepsilon} \subseteq \cdots$

At homology level:

$$
\begin{tikzcd}
H_k(X^{\alpha}) \arrow[r] \arrow[rr, bend right, "\text{isomorphism}"'] & H_k(L^{\alpha+\varepsilon}) \arrow[r] \arrow[rr, bend left, "\text{rank} = \dim H_k(X^{\alpha})"] & H_k(X^{\alpha+2\varepsilon}) \arrow[r] \arrow[rr, bend right, "\text{isomorphism}"'] & H_k(L^{\alpha+3\varepsilon}) \arrow[r] & H_k(X^{\alpha+4\varepsilon}) \arrow[r] & \cdots
\end{tikzcd}
$$

# An algorithm for geometric inference

**Proof:**



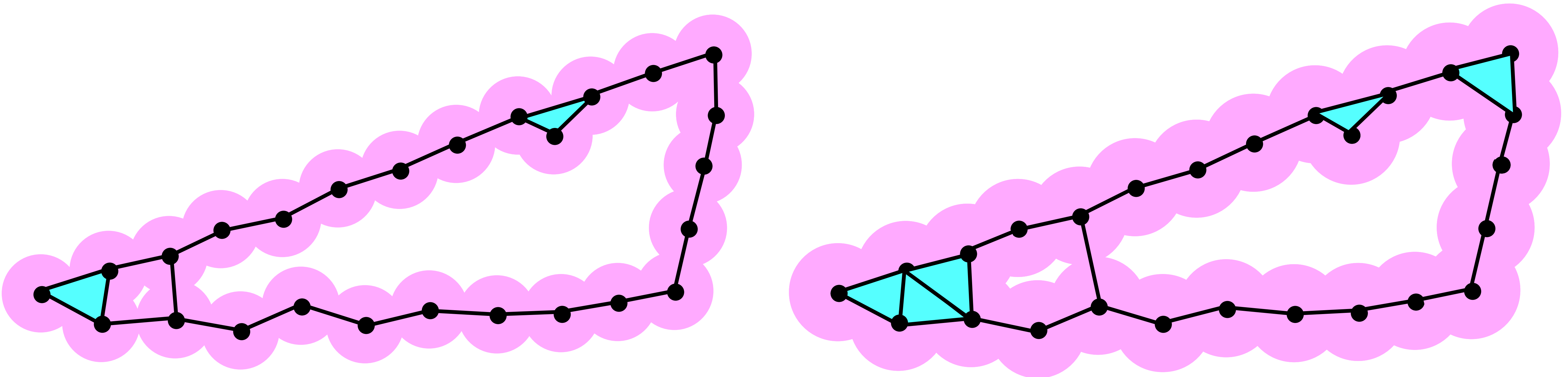
For any $\alpha > 0$, $\qquad X^\alpha \subseteq L^{\alpha+\varepsilon} \subseteq X^{\alpha+2\varepsilon} \subseteq L^{\alpha+3\varepsilon} \subseteq X^{\alpha+4\varepsilon} \subseteq \cdots$

At homology level:

Cannot be directly computed !

$$\begin{tikzcd} H_k(X^\alpha) \arrow[r] \arrow[rr, bend right, "\text{isomorphism}"'] & H_k(L^{\alpha+\varepsilon}) \arrow[r] \arrow[rr, bend left, "\text{rank} = \dim H_k(X^\alpha)"] & H_k(X^{\alpha+2\varepsilon}) \arrow[r] \arrow[rr, bend right, "\text{isomorphism}"'] & H_k(L^{\alpha+3\varepsilon}) \arrow[r] & H_k(X^{\alpha+4\varepsilon}) \arrow[r] & \cdots \end{tikzcd}$$

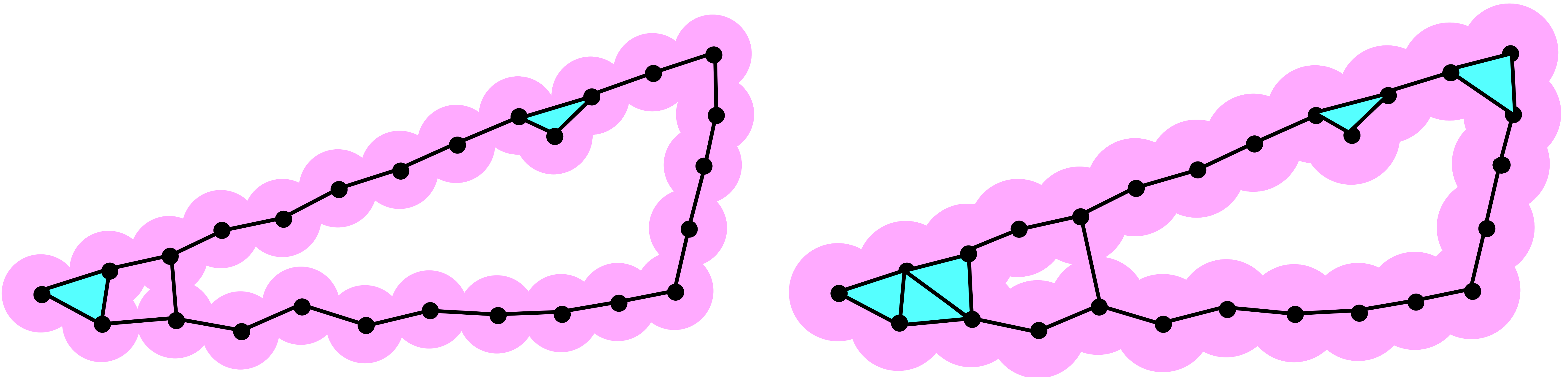
# Using the Čech complex



The Čech complex $\mathcal{C}^\alpha(L)$:

$$\text{for } p_0, \cdots p_k \in L, \quad \sigma = [p_0 p_1 \cdots p_k] \in \mathcal{C}^\alpha(L) \quad \text{iff} \quad \bigcap_{i=0}^{k} B(p_i, \alpha) \neq \emptyset$$

# Using the Čech complex



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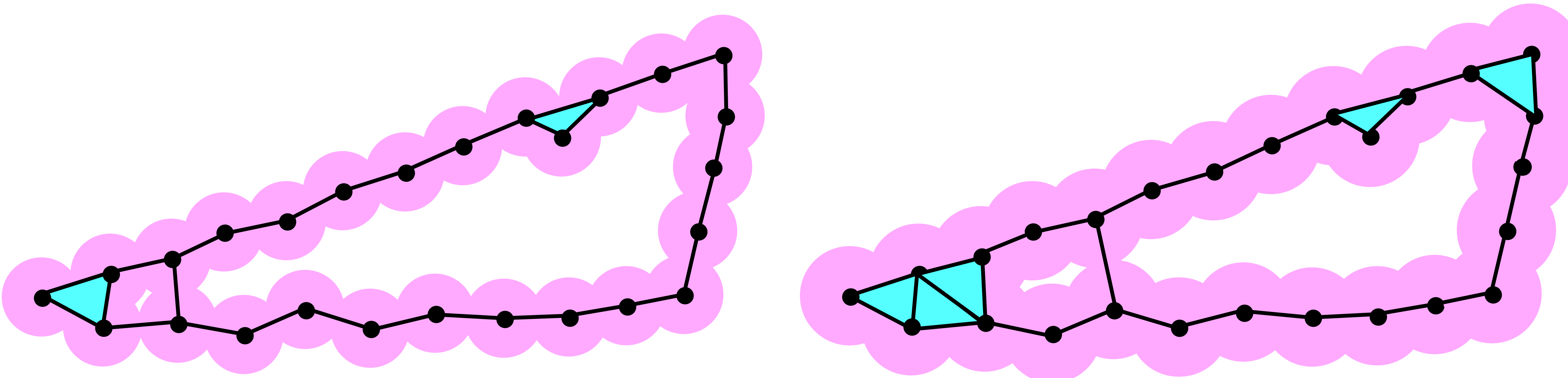
**Nerve theorem:** For any $\alpha > 0$, $L^\alpha$ and $\mathcal{C}^\alpha(L)$ are homotopy equivalent and the homotopy equivalences can be chosen to commute with inclusions.

# Using the Čech complex



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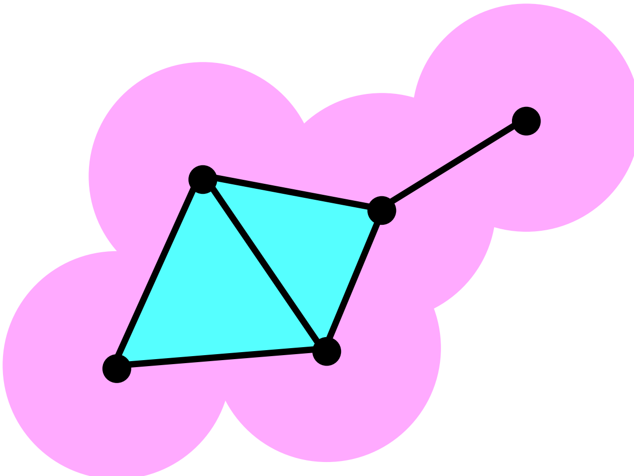
**Nerve theorem:** For any $\alpha > 0$, $L^\alpha$ and $\mathcal{C}^\alpha(L)$ are homotopy equivalent and the homotopy equivalences can be chosen to commute with inclusions.

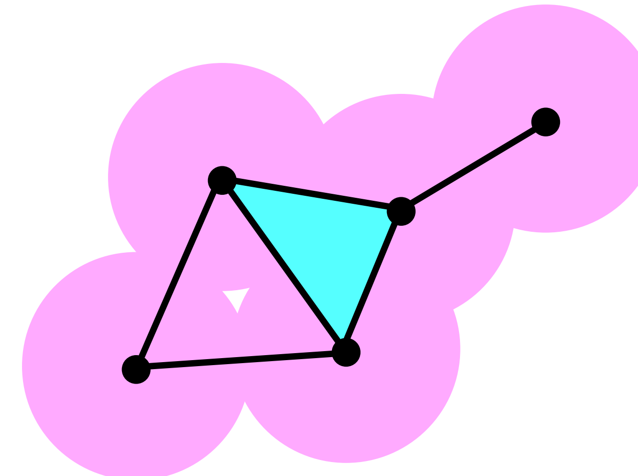
$$\begin{array}{ccccccc}
\cdots & \rightarrow & H_k(L^{\alpha+\varepsilon}) & \rightarrow & H_k(L^{\alpha+3\varepsilon}) & \rightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & H_k(\mathcal{C}^{\alpha+\varepsilon}(L)) & \rightarrow & H_k(\mathcal{C}^{\alpha+3\varepsilon}(L)) & \rightarrow & \cdots
\end{array}$$

Allow to work with simplicial complexes but... still too difficult to compute

# Using the Rips complex



Rips vs Čech



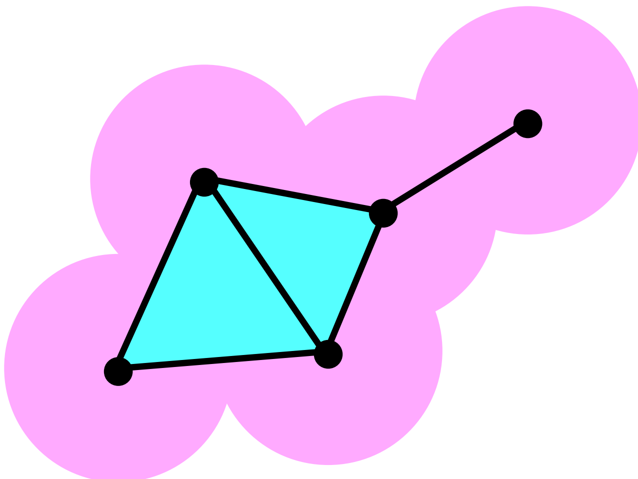
The Rips complex $\mathcal{R}^\alpha(L)$: for $p_0, \cdots p_k \in L$,

$$\sigma = [p_0 p_1 \cdots p_k] \in \mathcal{R}^\alpha(L) \text{ iff } \forall i, j \in \{0, \cdots k\},\ d(p_i, p_j) \leq \alpha$$

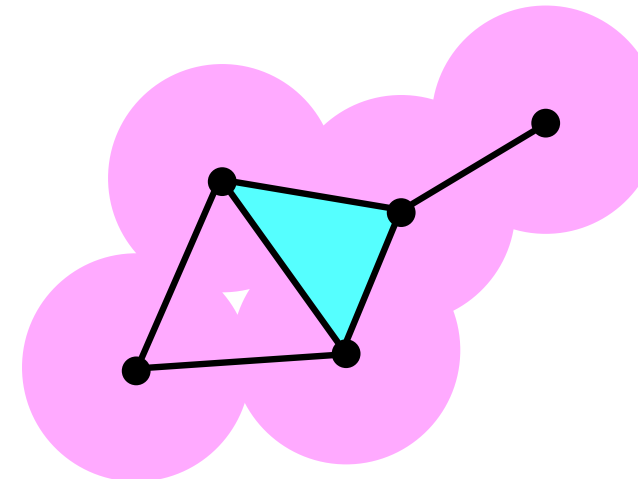
- Easy to compute and fully determined by its $1$-skeleton
- Rips-Čech interleaving: for any $\alpha > 0$,

$$\mathcal{C}^{\frac{\alpha}{2}}(L) \subseteq \mathcal{R}^\alpha(L) \subseteq \mathcal{C}^\alpha(L) \subseteq \mathcal{R}^{2\alpha}(L) \subseteq \cdots$$

# Using the Rips complex



Rips vs Čech



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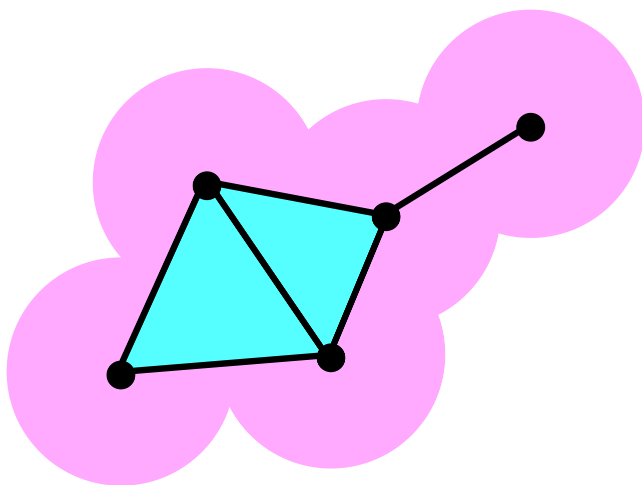
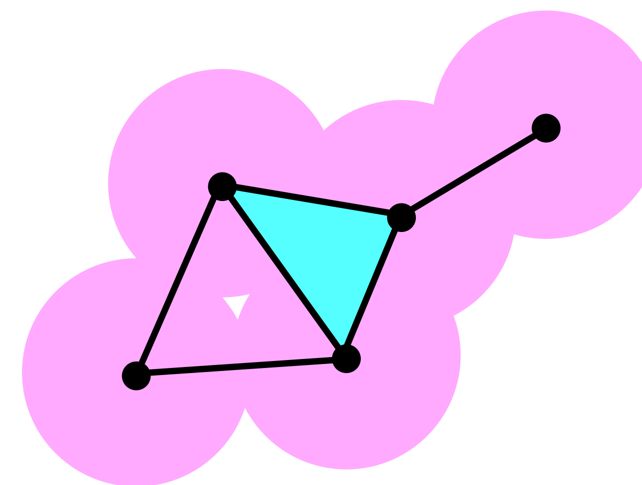
**Theorem:** [C-Oudot'08]
Let $X \subset \mathbb{R}^d$ be a compact set and $L \subset \mathbb{R}^d$ a finite set such that $d_H(X, L) < \varepsilon$ for some $\varepsilon < \frac{1}{9}$ wfs$(X)$. Then for all $\alpha \in [2\varepsilon, \frac{1}{4}(\mathrm{wfs}(X) - \varepsilon)]$ and all $\lambda \in (0, \mathrm{wfs}(X))$, one has: $\forall k \in \mathbb{N}$

$$\beta_k(X^\lambda) = \dim(H_k(X^\lambda)) = \mathsf{rk}(\mathcal{R}^\alpha(L) \to \mathcal{R}^{4\alpha}(L))$$

# Using the Rips complex



Rips vs Čech



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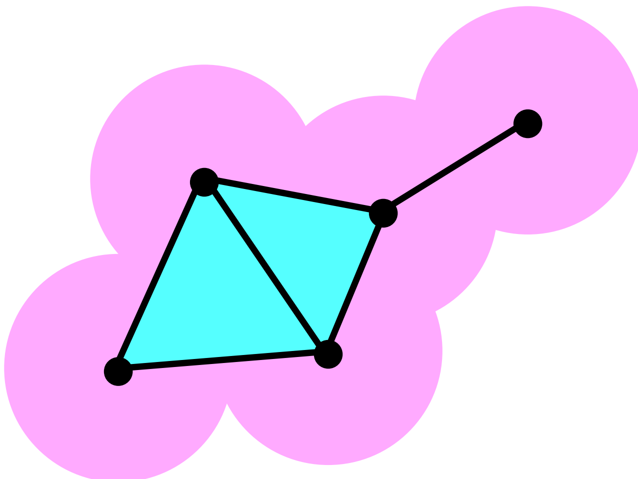
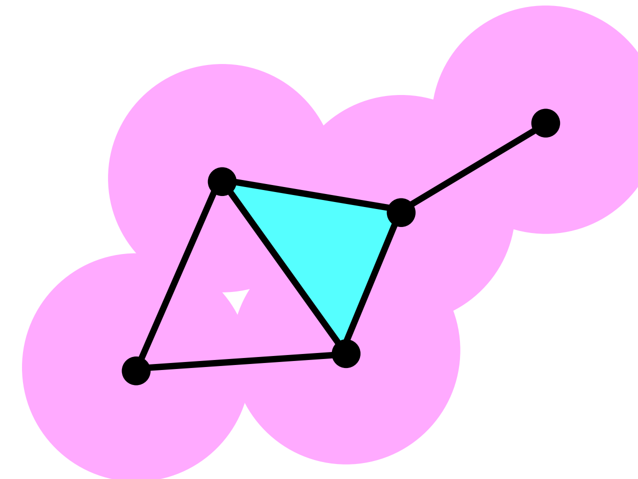
Easy to compute using persistence algo.

# Using the Rips complex



Rips vs Čech



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**Theorem:** [C-Oudot'08]
Let $X \subset \mathbb{R}^d$ be a compact set and $L \subset \mathbb{R}^d$ a finite set such that $d_H(X, L) < \varepsilon$ for some $\varepsilon < \frac{1}{9}$ wfs$(X)$. Then for all $\alpha \in [2\varepsilon, \frac{1}{4}(\mathrm{wfs}(X) - \varepsilon)]$ and all $\lambda \in (0, \mathrm{wfs}(X))$, one has: $\forall k \in \mathbb{N}$

$$\beta_k(X^\lambda) = \dim(H_k(X^\lambda)) = \mathrm{rk}(\mathcal{R}^\alpha(L) \to \mathcal{R}^{4\alpha}(L))$$

**Pb:** Choice of $\alpha$ when wfs$(X)$ is unknown?

# Multiscale inference

**Input:** A point cloud $W$ and its pairewise distances $\{d(w,w')\}_{w,w'\in W}$.
$\to$ Maintain a nested pair $\mathcal{R}^{4\varepsilon}(L) \hookrightarrow \mathcal{R}^{16\varepsilon}(L)$ where $L = L(\varepsilon)$.

Init.: $L = \emptyset$; $\varepsilon = +\infty$

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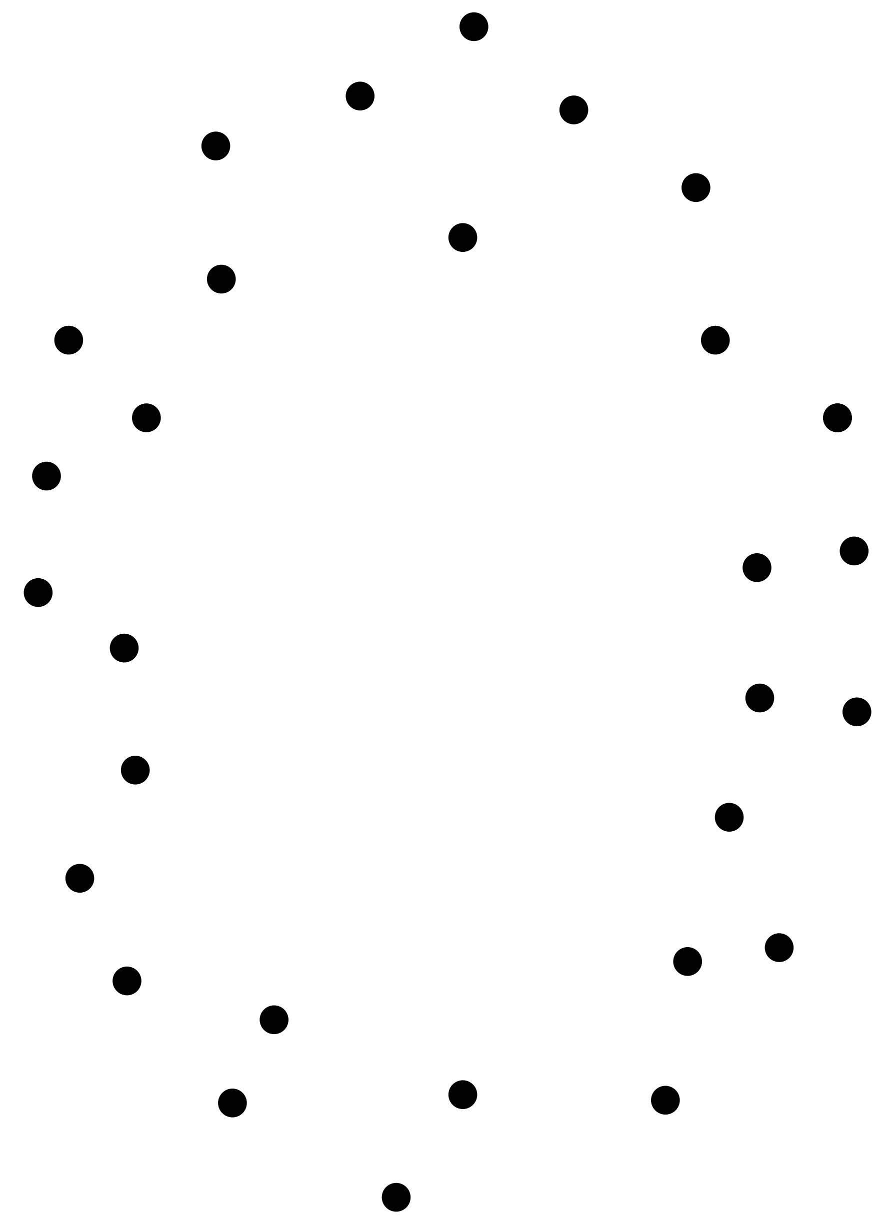
insert $p = argmax_{w\in W} d(w, L)$ in $L$
update $\varepsilon = \max_{w\in W} d(w, L)$
update $\mathcal{R}^{4\varepsilon}(L)$ and $\mathcal{R}^{16\varepsilon}(L)$
Persistence( $\mathcal{R}^{4\varepsilon}(L) \hookrightarrow \mathcal{R}^{16\varepsilon}(L)$)
END_WHILE

Rank of the map induced at homology level

Output: Sequence of persistent Betti numbers of $\mathcal{R}^{4\varepsilon}(L) \hookrightarrow \mathcal{R}^{16\varepsilon}(L)$



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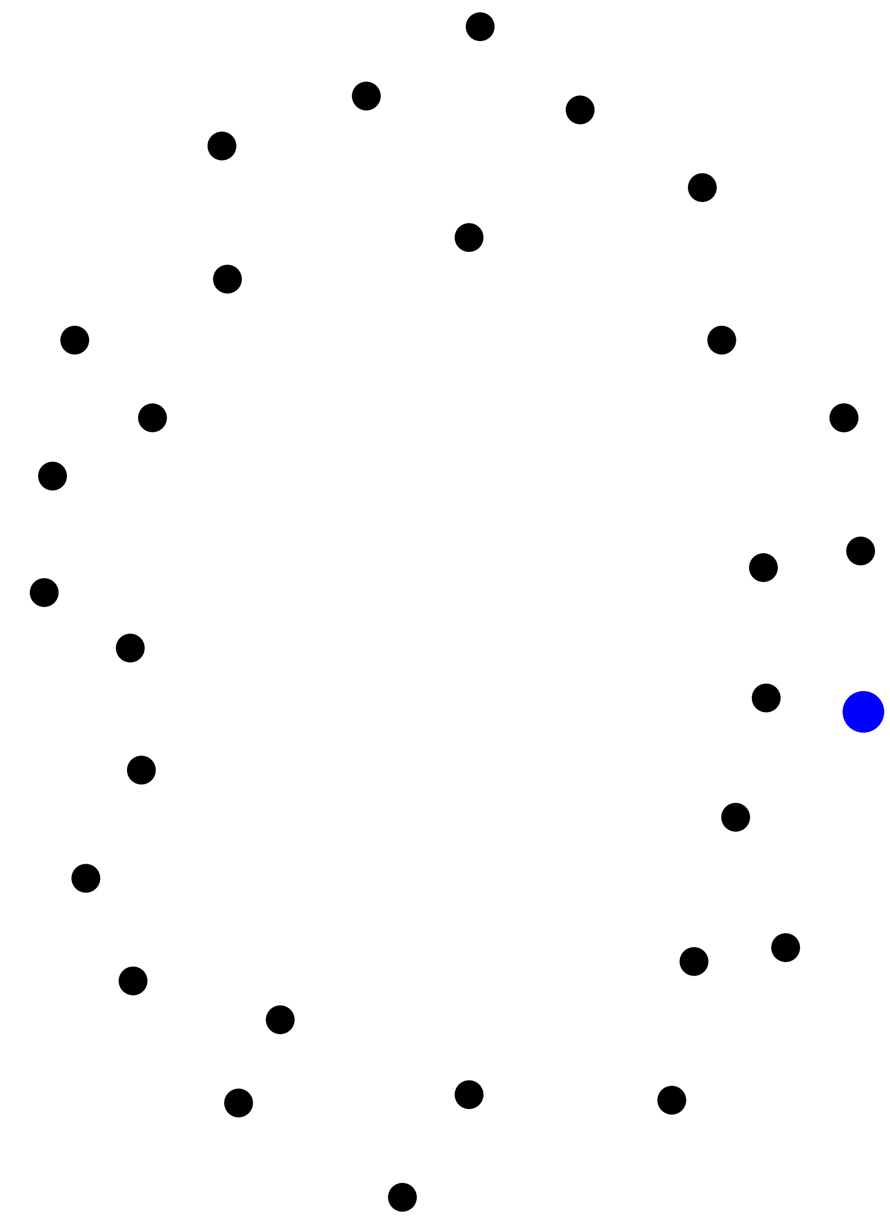

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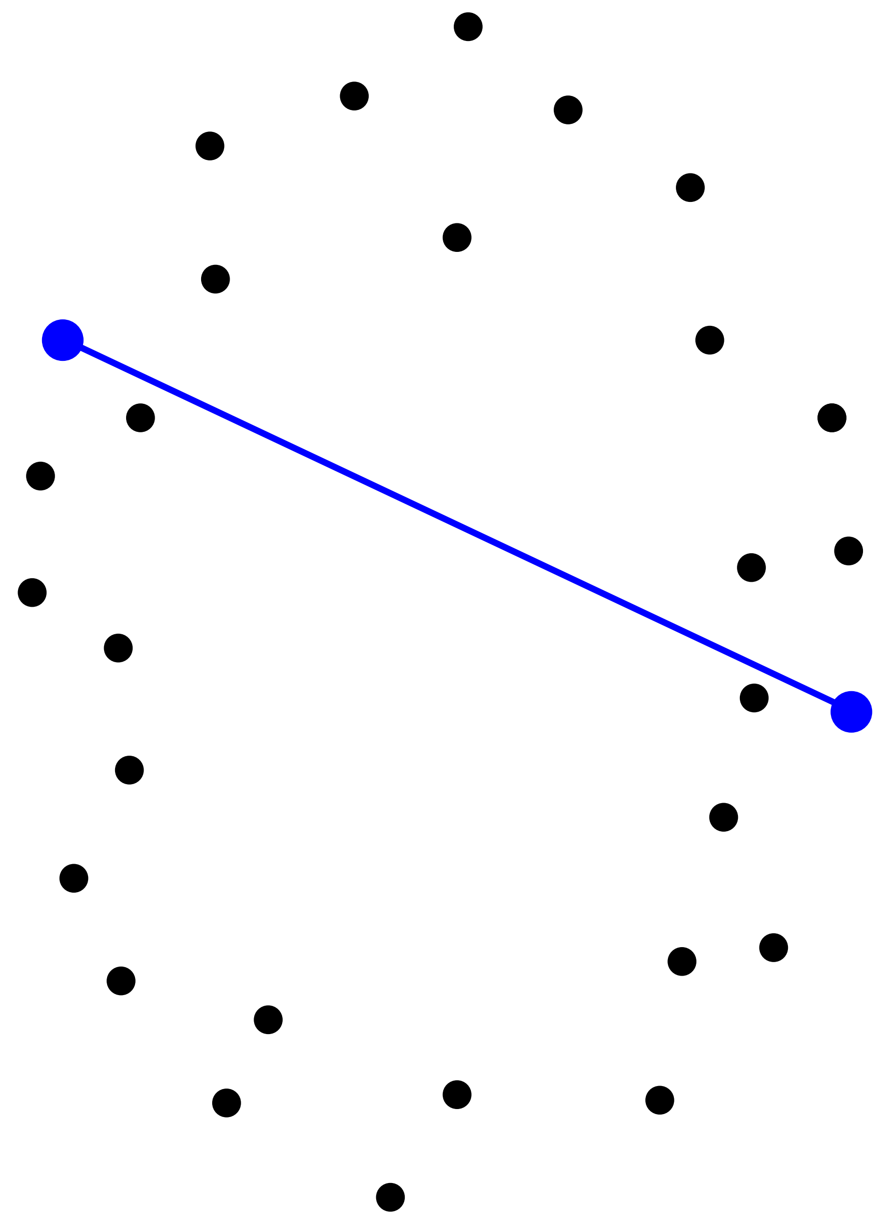

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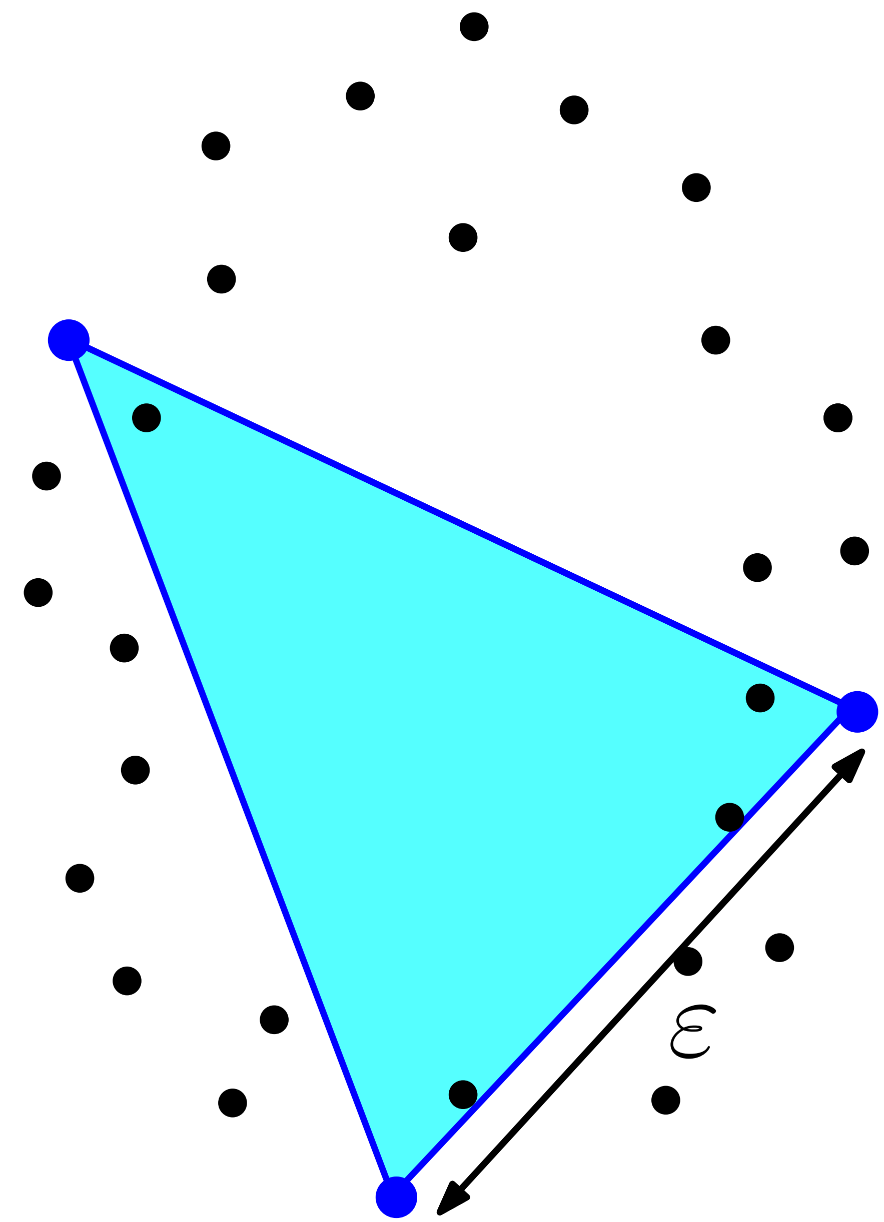

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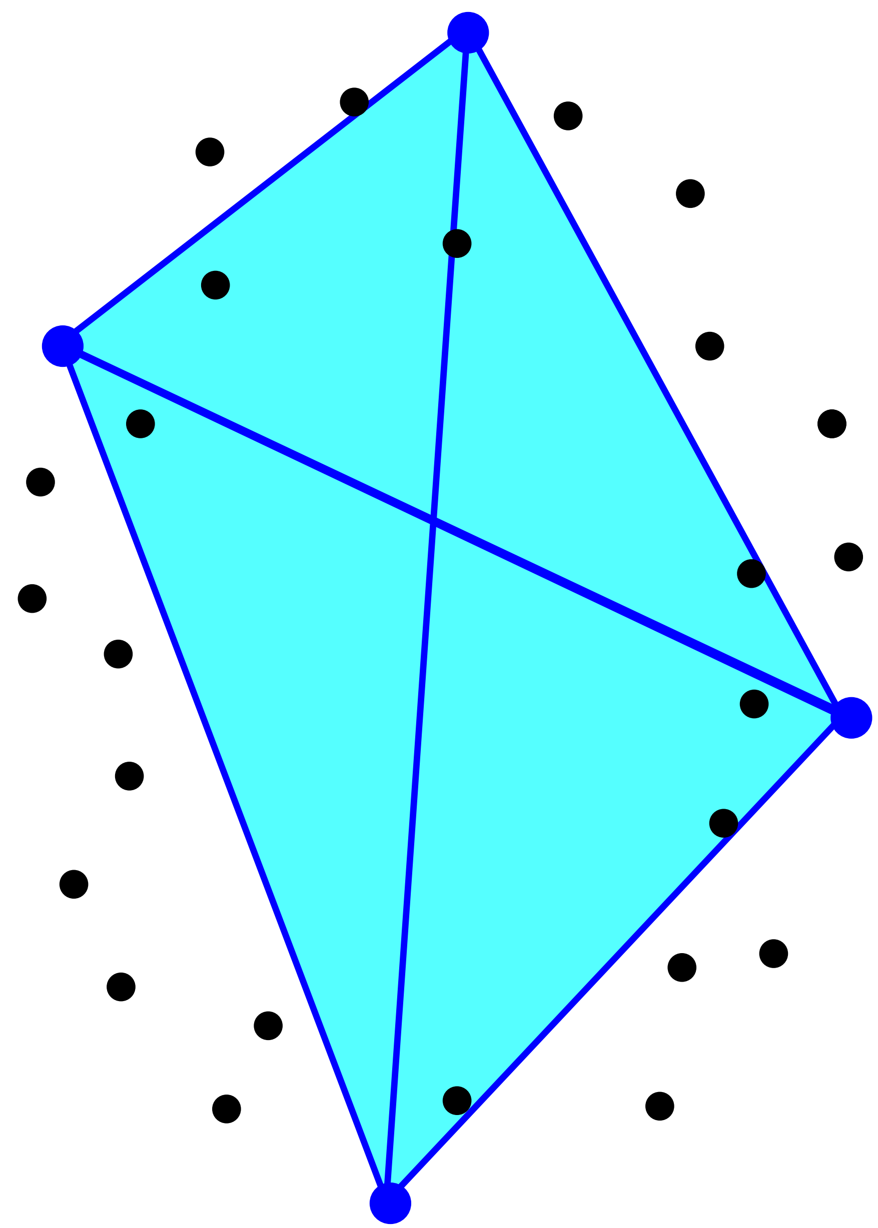

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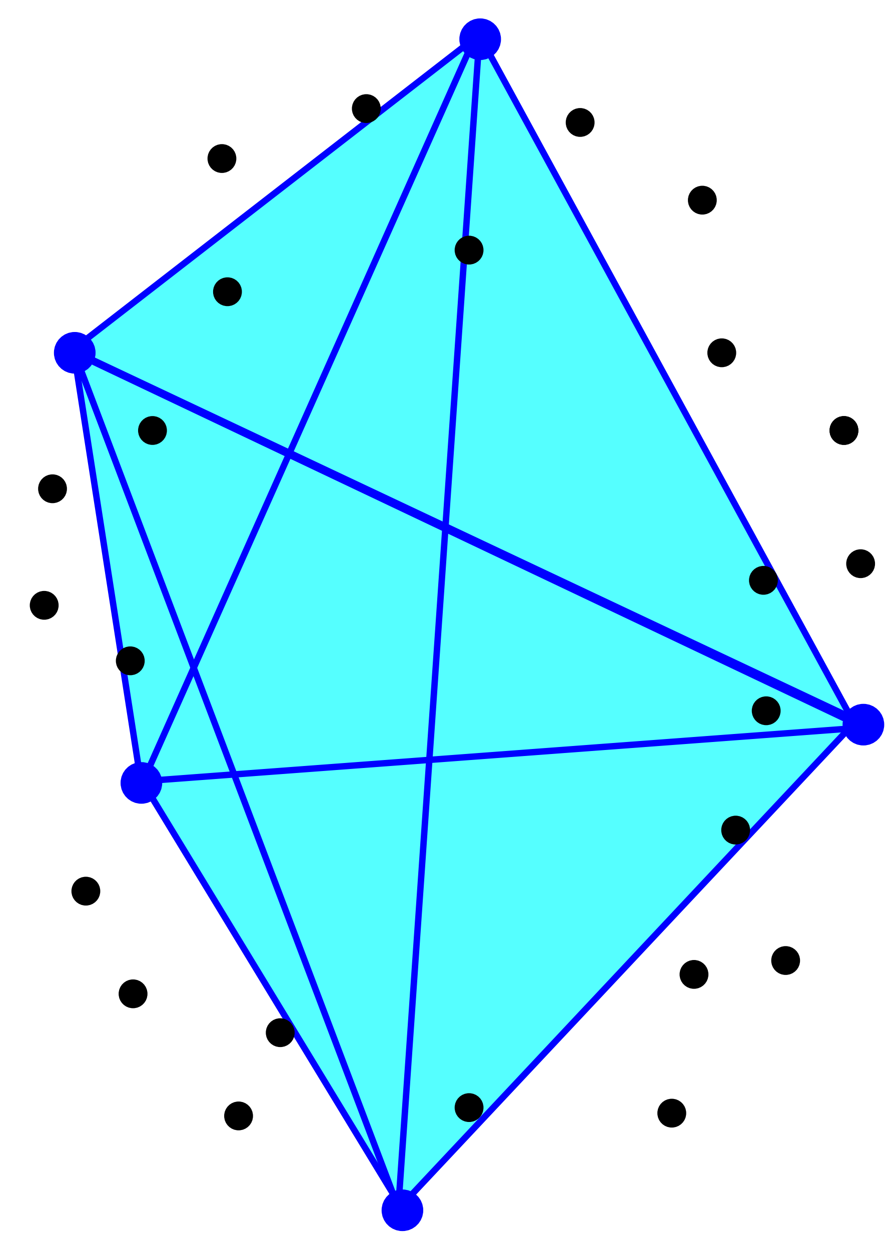

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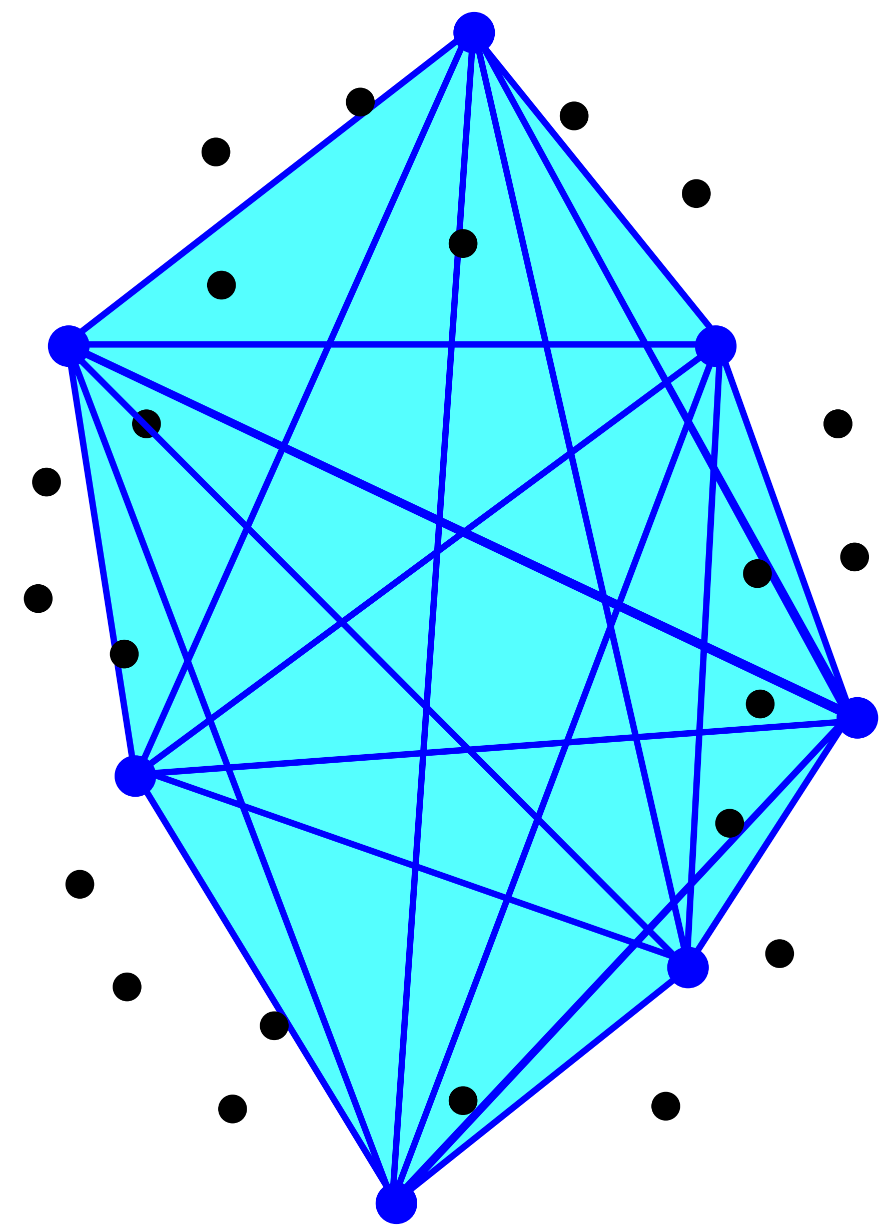

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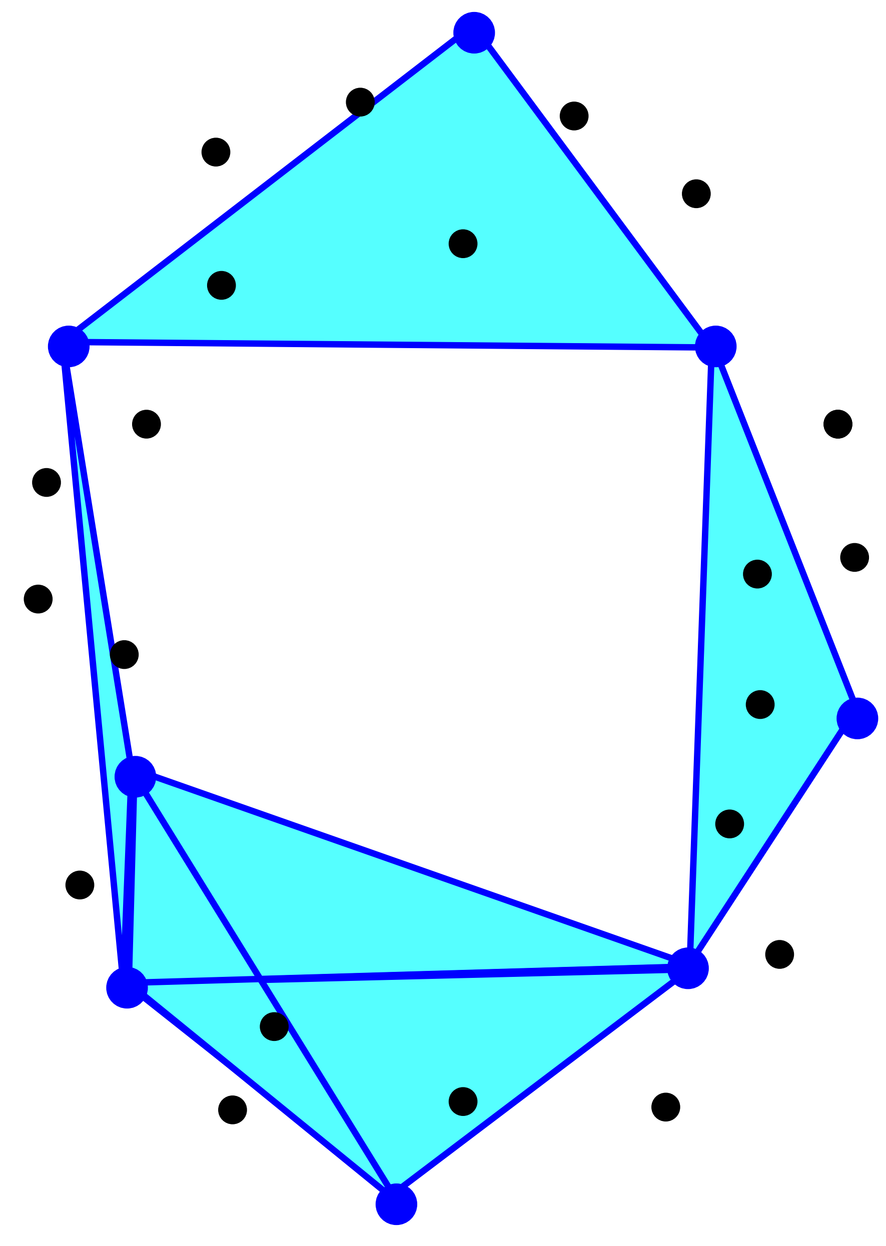

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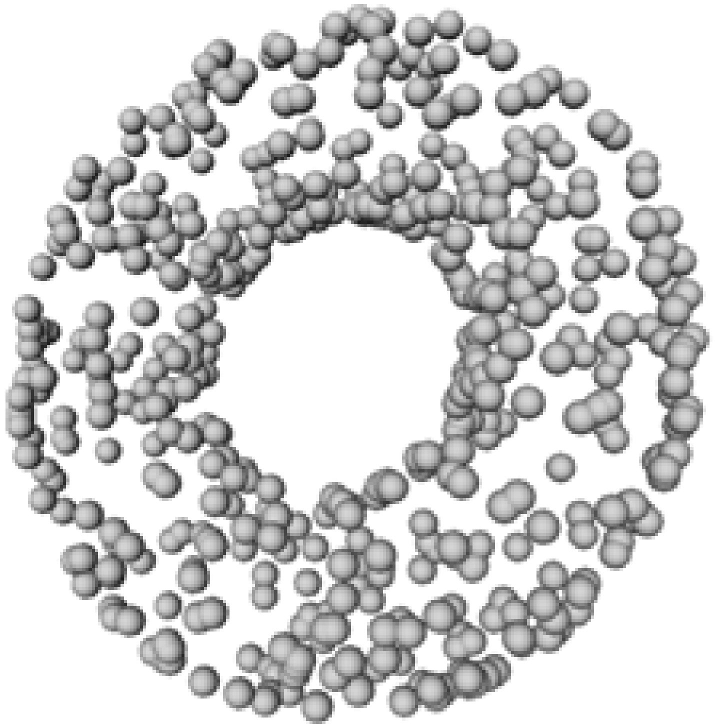
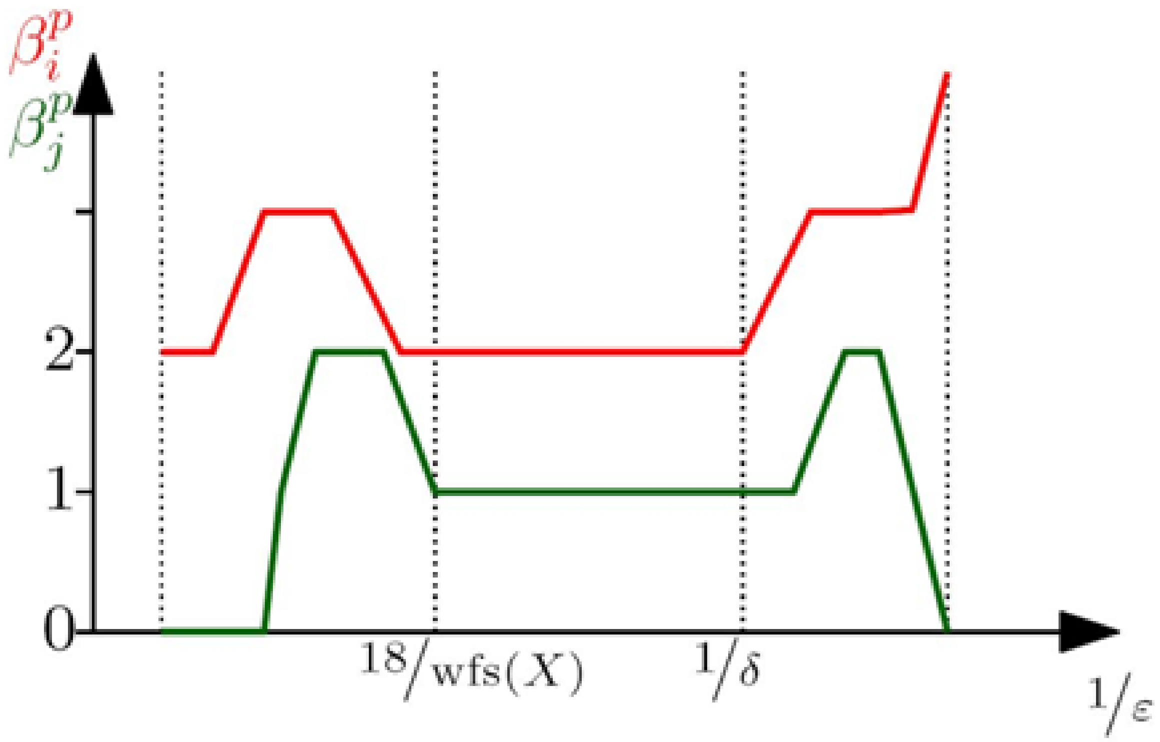


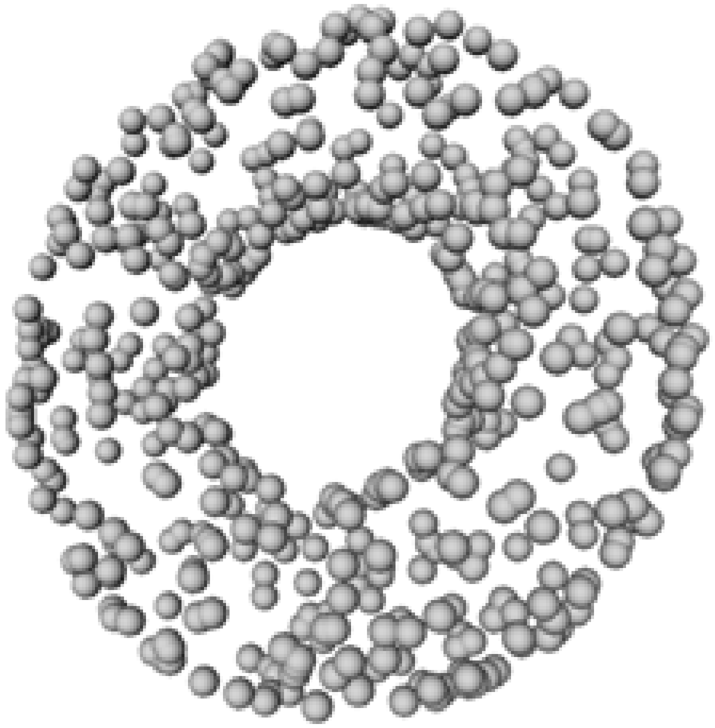


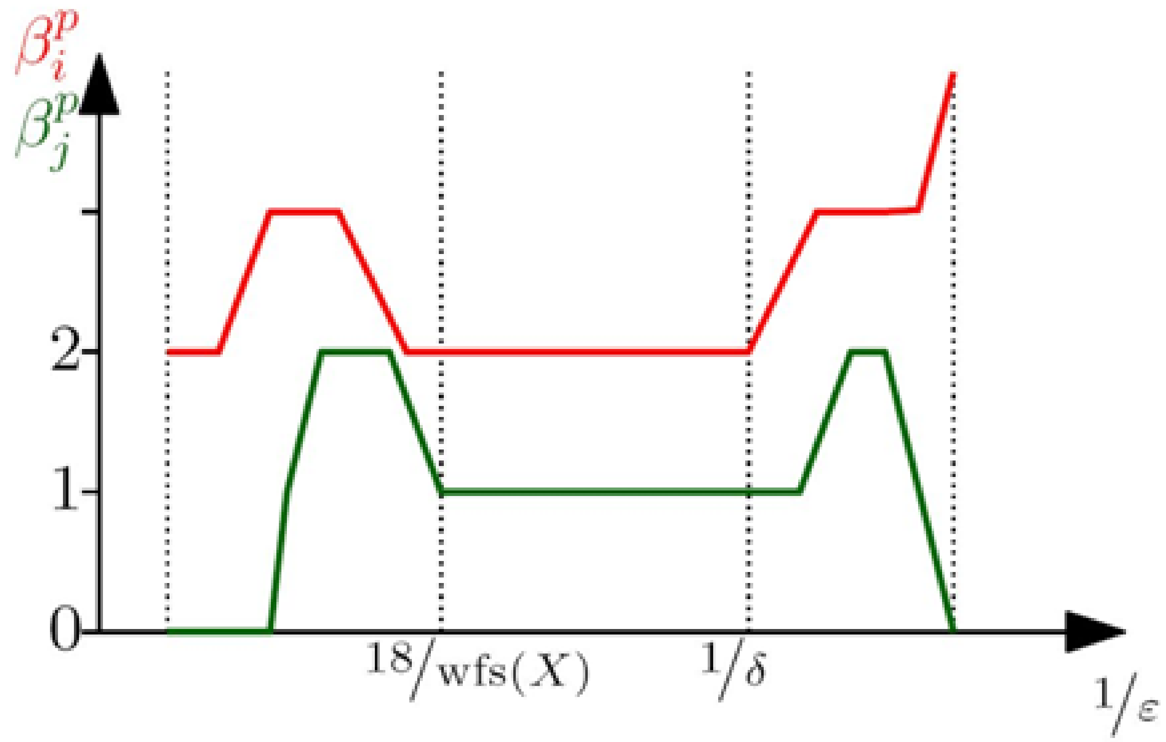

**Theorem:** [C-Oudot'08]
If $d_H(W, X) < \delta$ for $\delta < \frac{1}{18}\mathrm{wfs}(X)$, then at every iteration of the algorithm such that $\delta < \varepsilon < \frac{1}{18}\mathrm{wfs}(X)$,

$$\beta_k(X^\lambda) = \dim H_k(X^\lambda) = rk(H_k(\mathcal{R}^{4\varepsilon}(L)) \to H_k(\mathcal{R}^{4\varepsilon}(L)))$$

for any $\lambda \in (0, \mathrm{wfs}(X))$ and any $k \in \mathbb{N}$.

# Multiscale inference






**Complexity of the algorithm:**

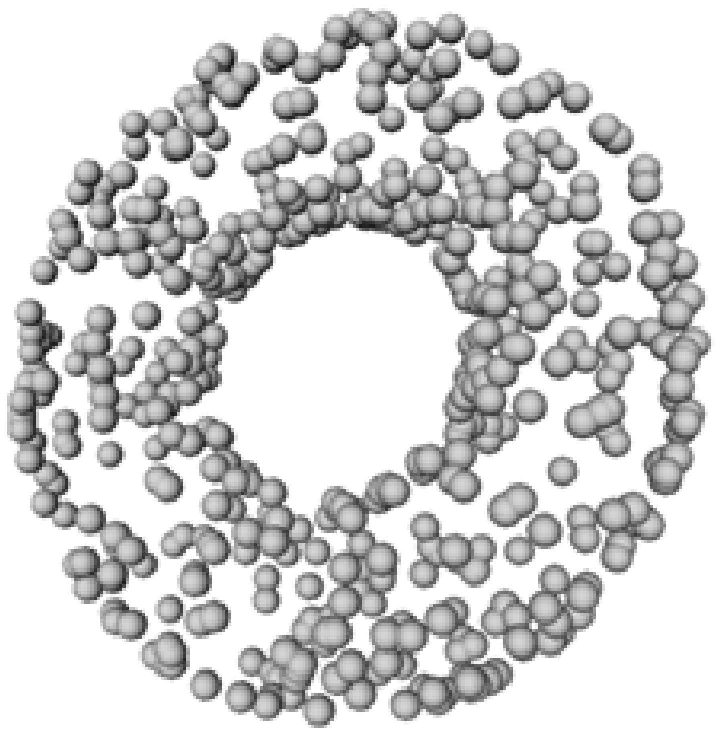
- If $X \subset \mathbb{R}^d$ is non smooth the running time of the algorithm is

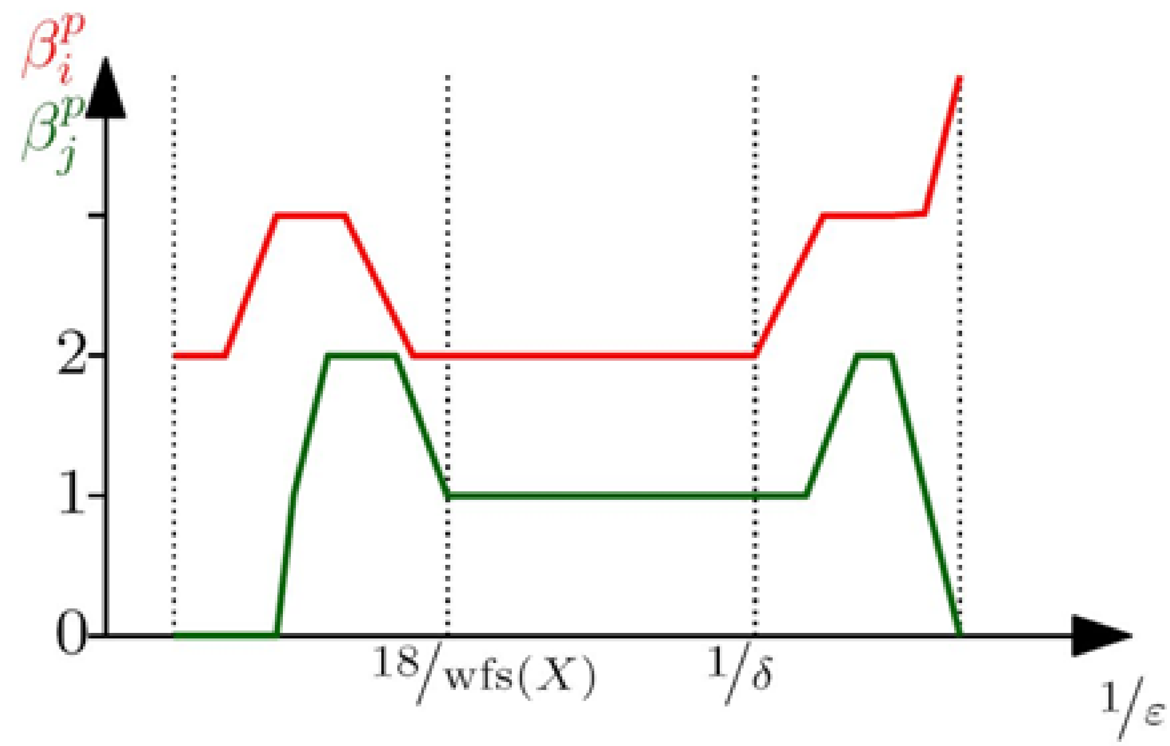
$$O(8^{33^d}|W|^5)$$

- If $X$ is a smooth submanifold of $\mathbb{R}^d$ dimension $m$ the running time is

$$O(8^{35^m}|W|)$$

# Multiscale inference



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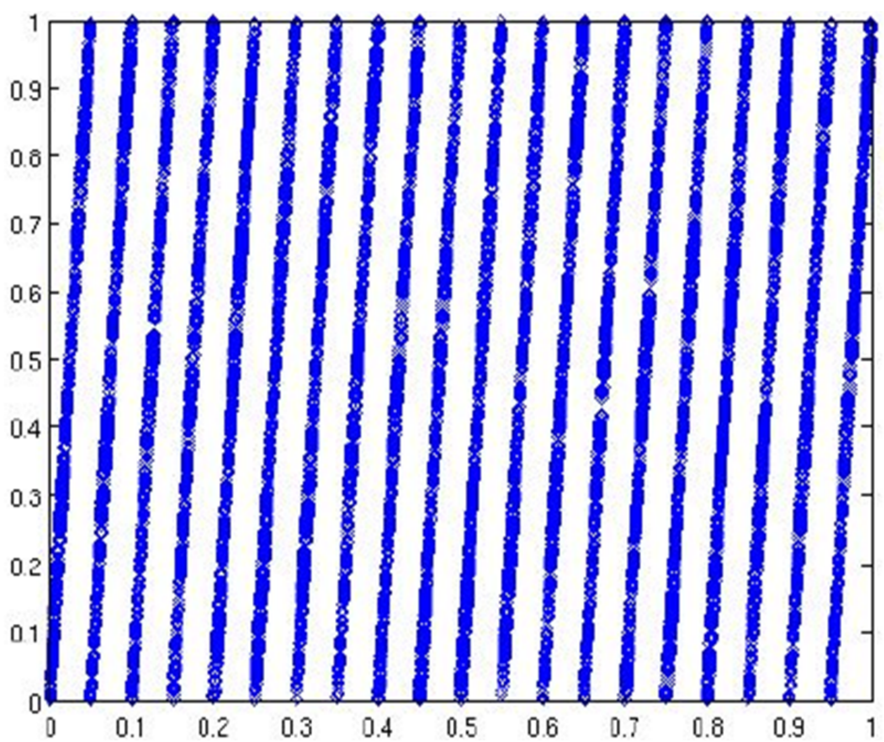
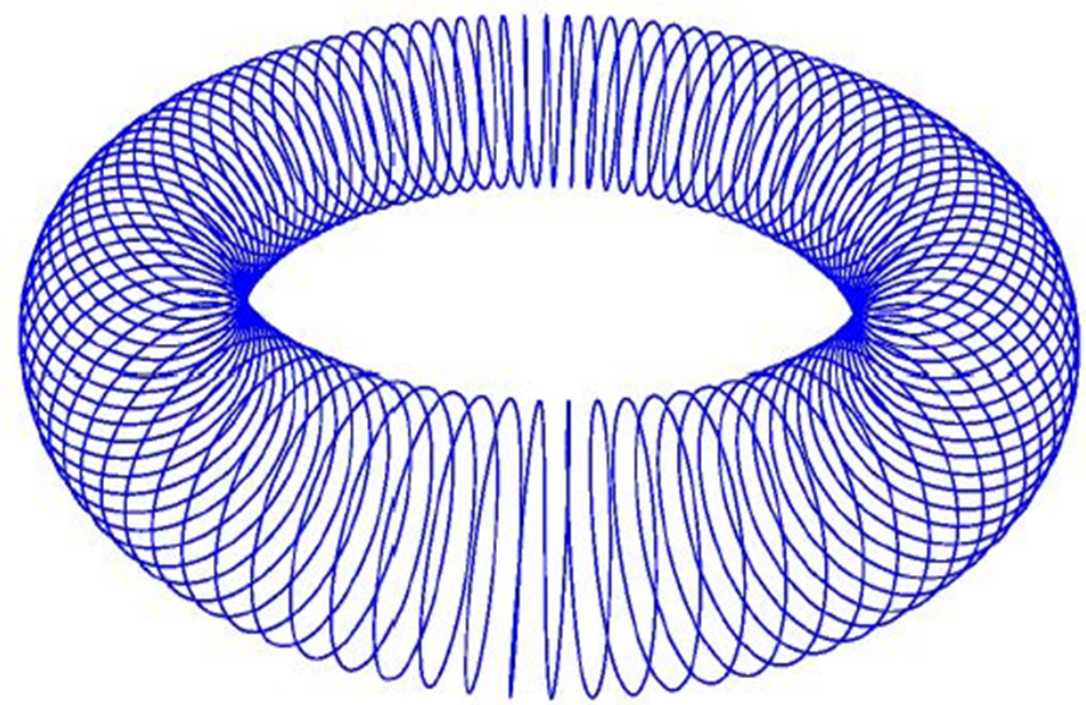
Depend on the intrinsic dimension of $X$

# A synthetic example

$[0,1] \times [0,1]$



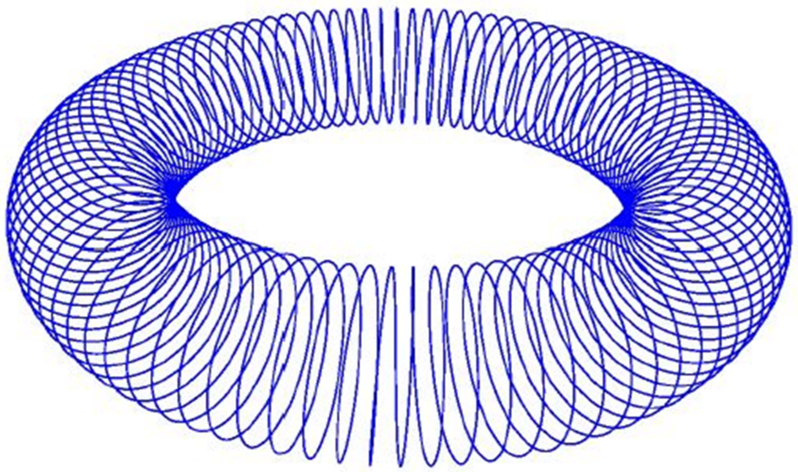

$\mathbb{R}^{1000}$

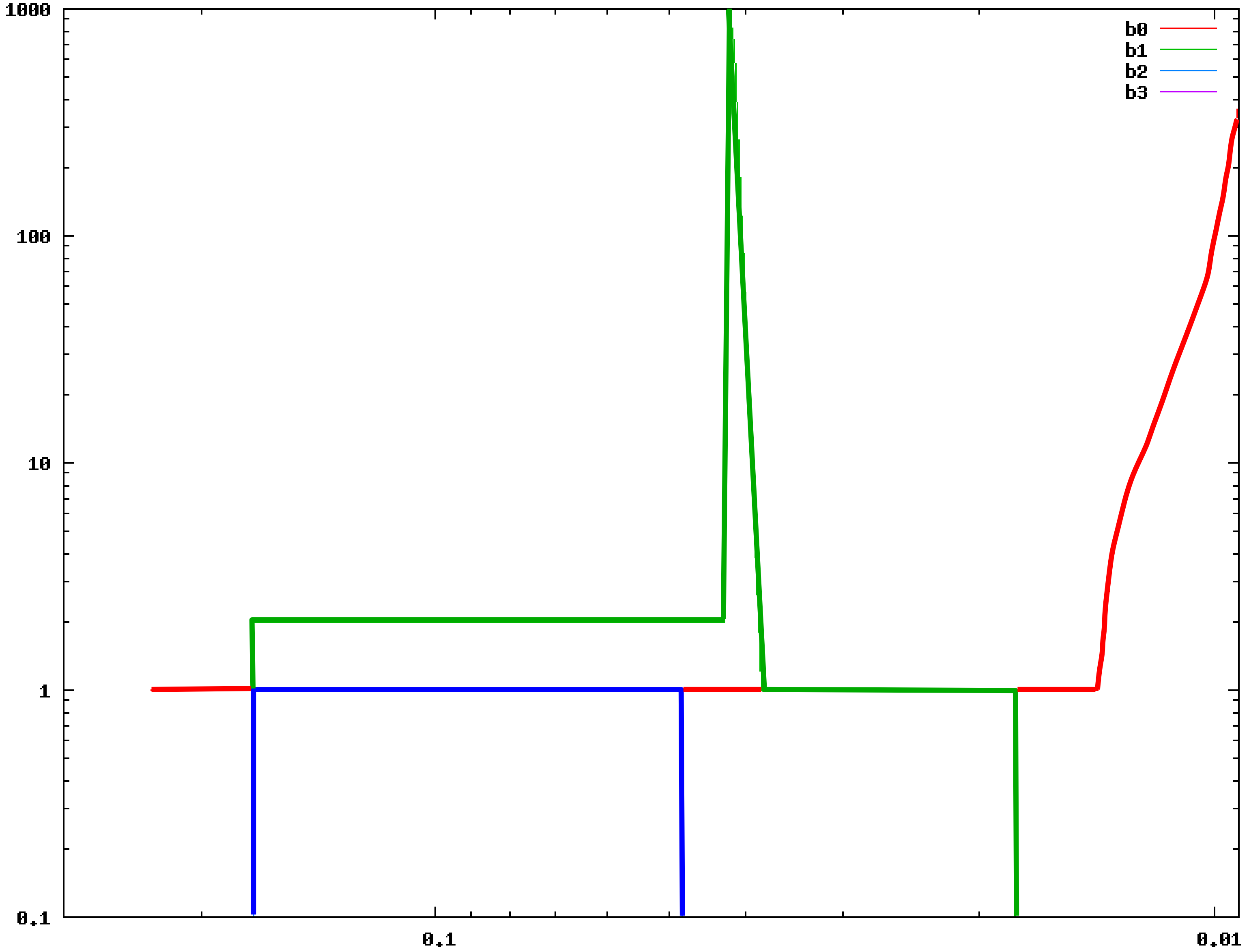


Non-linear embedding of $\mathbb{S}^1 \times \mathbb{S}^1$ in $\mathbb{R}^{1000}$

$50,000$ points sampled uniformly at random from a curve drawn on the $2$-torus $\mathbb{S}^1 \times \mathbb{S}^1$.

# A synthetic example






Output: sequence of Betti numbers on a log-log scale

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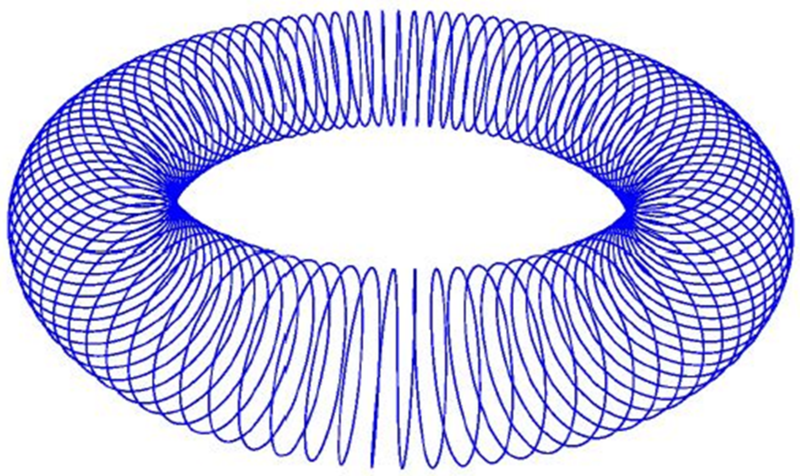

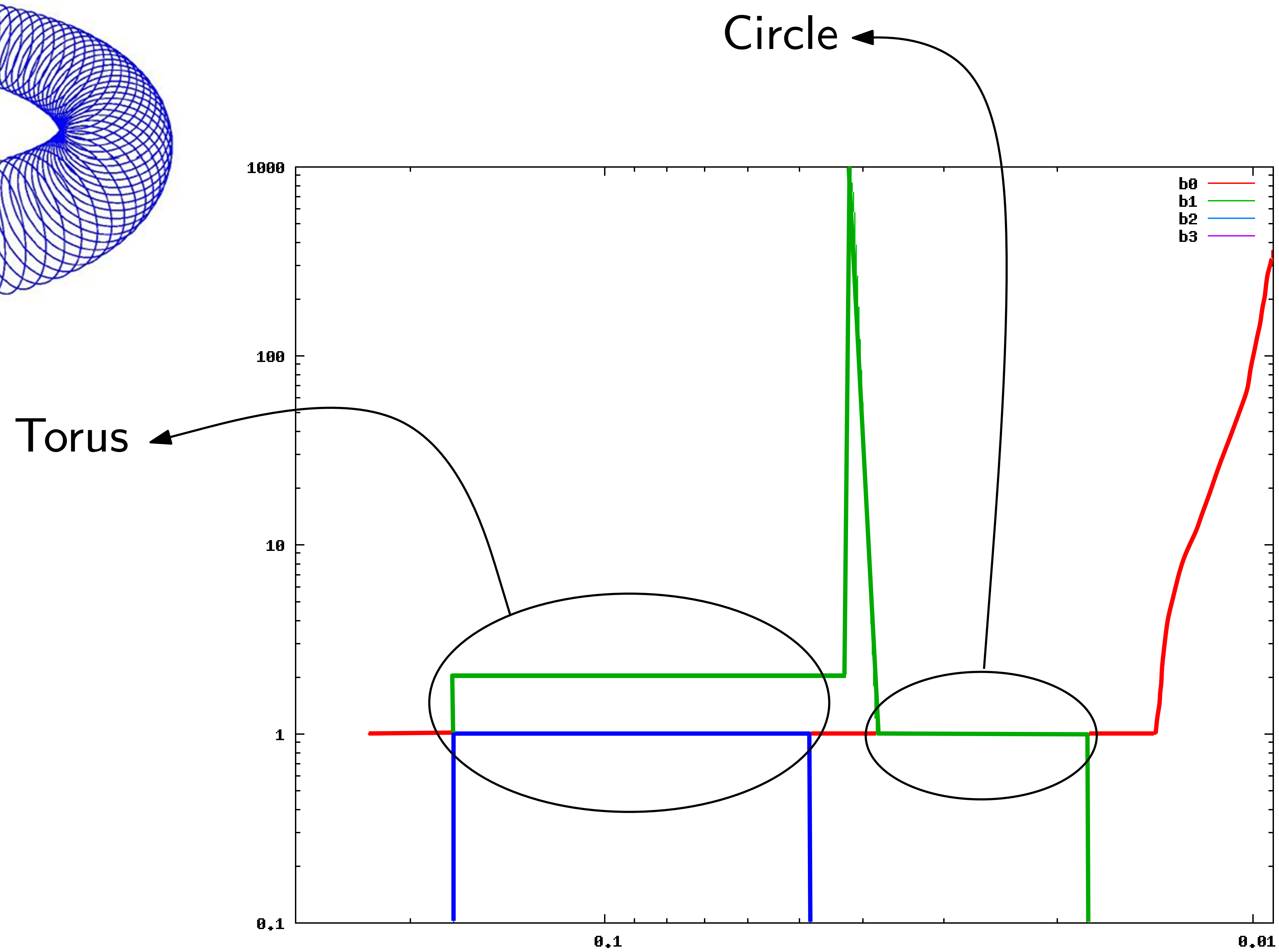


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# An algorithm to compute Betti numbers

**Input:** A filtration of a simplicial complex $\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$, s. t. $K^{i+1} = K^i \cup \sigma^{i+1}$ where $\sigma^{i+1}$ is a simplex of $K$.

**Output:** The Betti numbers $\beta_0, \beta_1, \cdots, \beta_d$ of $K$.

```
β_0 = β_1 = ··· = β_d = 0;
for i = 1 to m
   k = dim σ^i − 1;
   if σ^i is contained in a (k+1)-cycle in K^i
      then β_{k+1} = β_{k+1} + 1;
      else β_k = β_k − 1;
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end for;
output (β_0, β_1, ··· , β_d);
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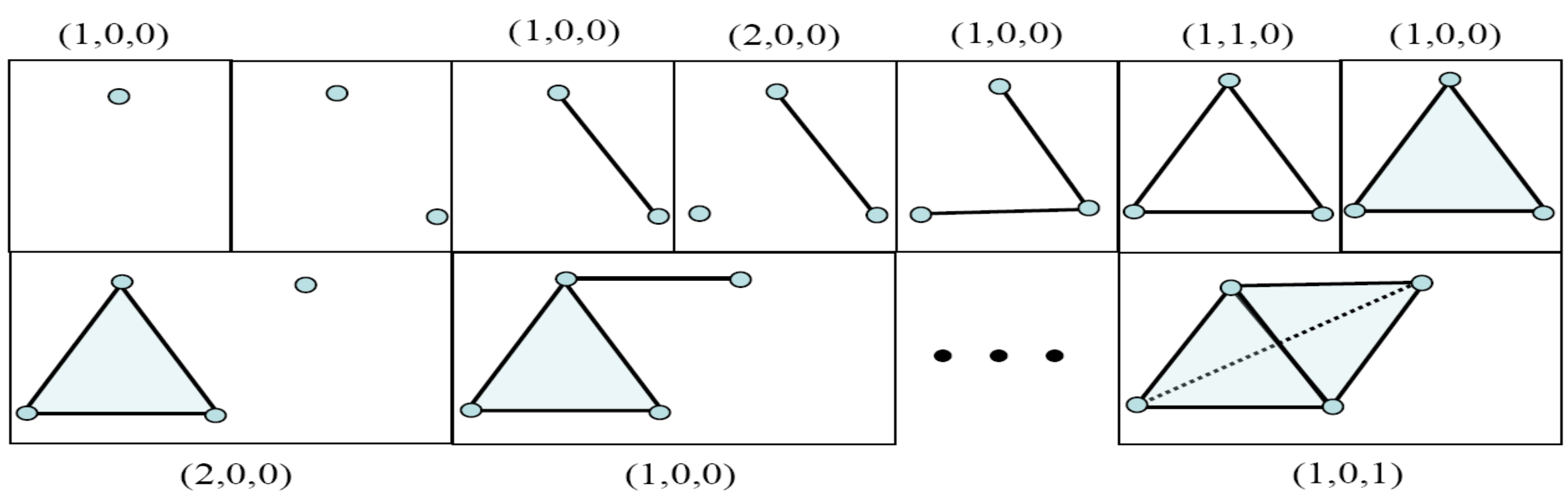

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**Remark:** At the $i^{th}$ step of the algorithm, the vector $(\beta_0, \cdots, \beta_d)$ stores the Betti numbers of $K^i$.

# Proof

- If $\sigma^i$ is contained in a $(k+1)$-cycle in $K^i$, this cycle is not a boundary in $K^i$.
- If $\sigma^i$ is contained in a $(k+1)$-cycle $c$ in $K^i$, then $c$ cannot be homologous to a cycle in $K^{i-1}$

$$\Rightarrow \beta_{k+1}(K^i) \geq \beta_{k+1}(K^{i-1}) + 1$$

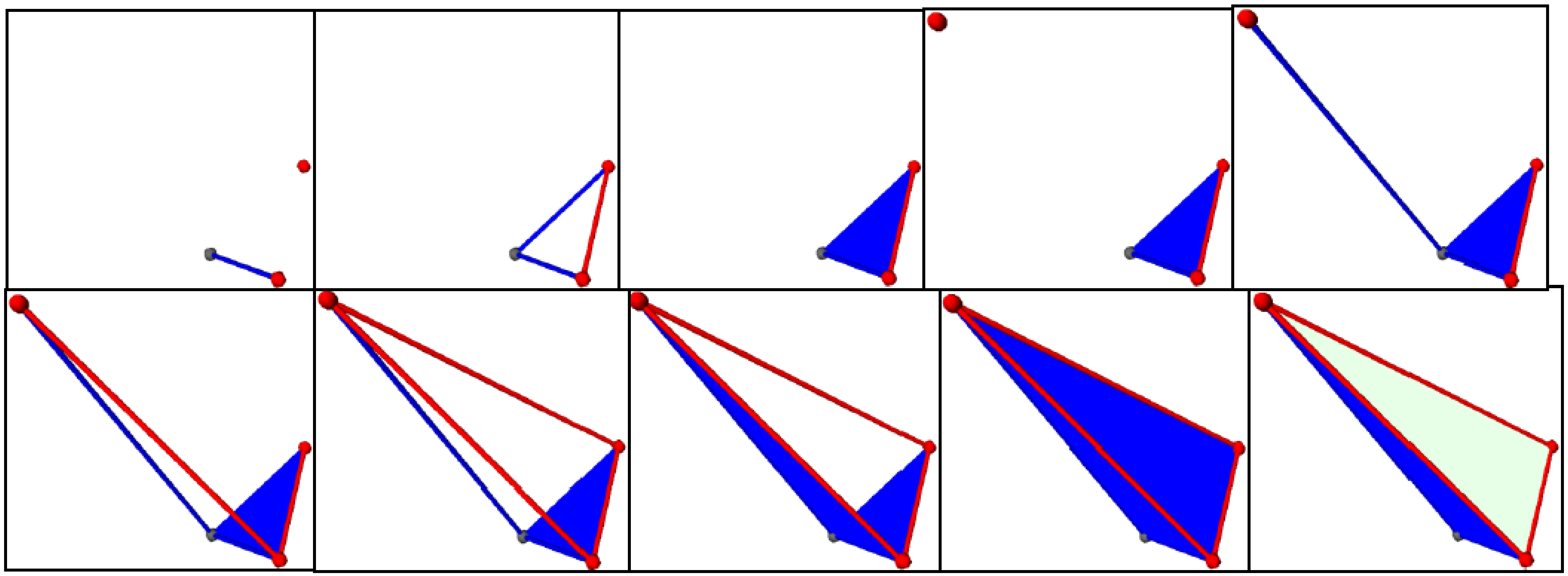
- If $\sigma^i$ is not contained in a $(k+1)$-cycle $c$ in $K^i$, then $\partial\sigma^i$ is not a boundary in $K^{i-1}$

$$\Rightarrow \beta_k(K^i) \leq \beta_k(K^{i-1}) - 1$$

- the previous inequalities are equalities.

# Positive and negative simplices

Let $\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$ be a filtration of a simplicial complex $K$ s. t. $K^{i+1} = K^i \cup \sigma^{i+1}$ where $\sigma^{i+1}$ is a simplex of $K$.



**Definition:** A $(k+1)$-simplex $\sigma^i$ is positive if it is contained in a $(k+1)$-cycle in $K^i$. It is negative otherwise.

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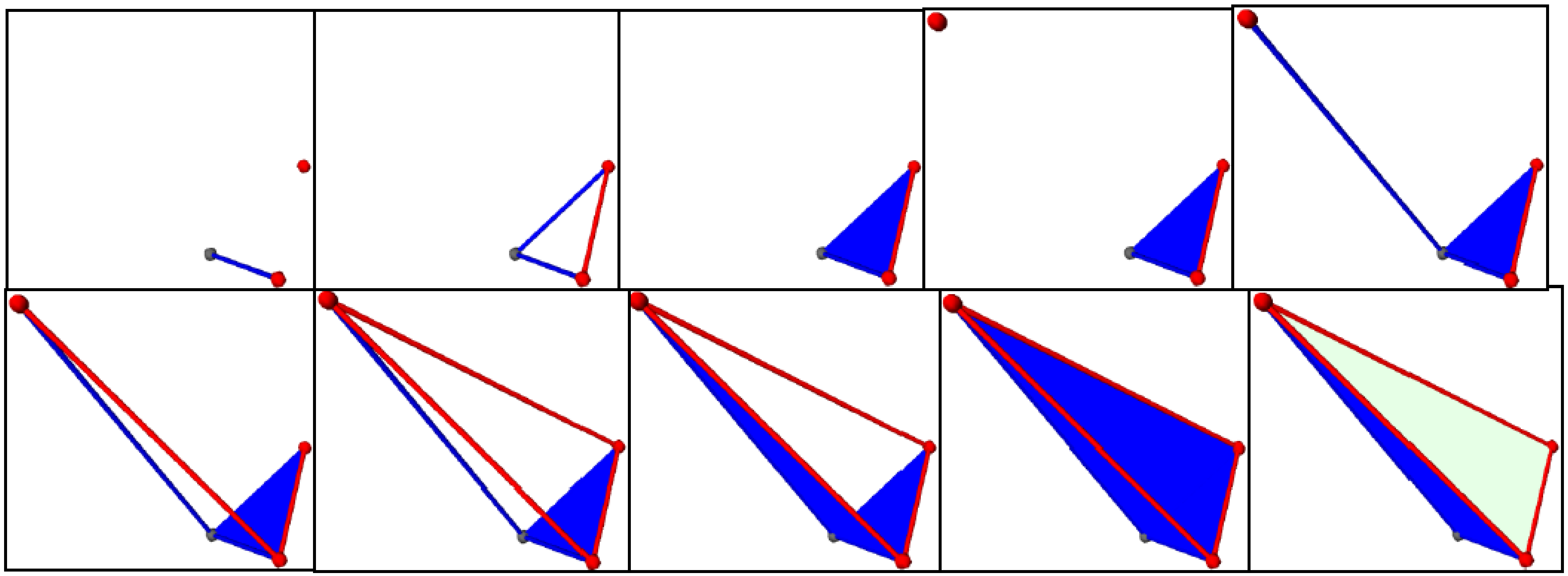

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Create a new $(k+1)$-cycle in $K^i$

Destroy a $k$-cycle in $K^i$

$$\beta_k(K) = \sharp(\text{positive simplices}) \; - \; \sharp(\text{negative simplices})$$

# Getting more information

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- How to keep track of the evolution of the topology all along the filtration?
- What are the created/destroyed cycles?
- What is the lifetime of a cycle?
- How to compute rank$(H_k(K^i) \to H_k(K^j))$?

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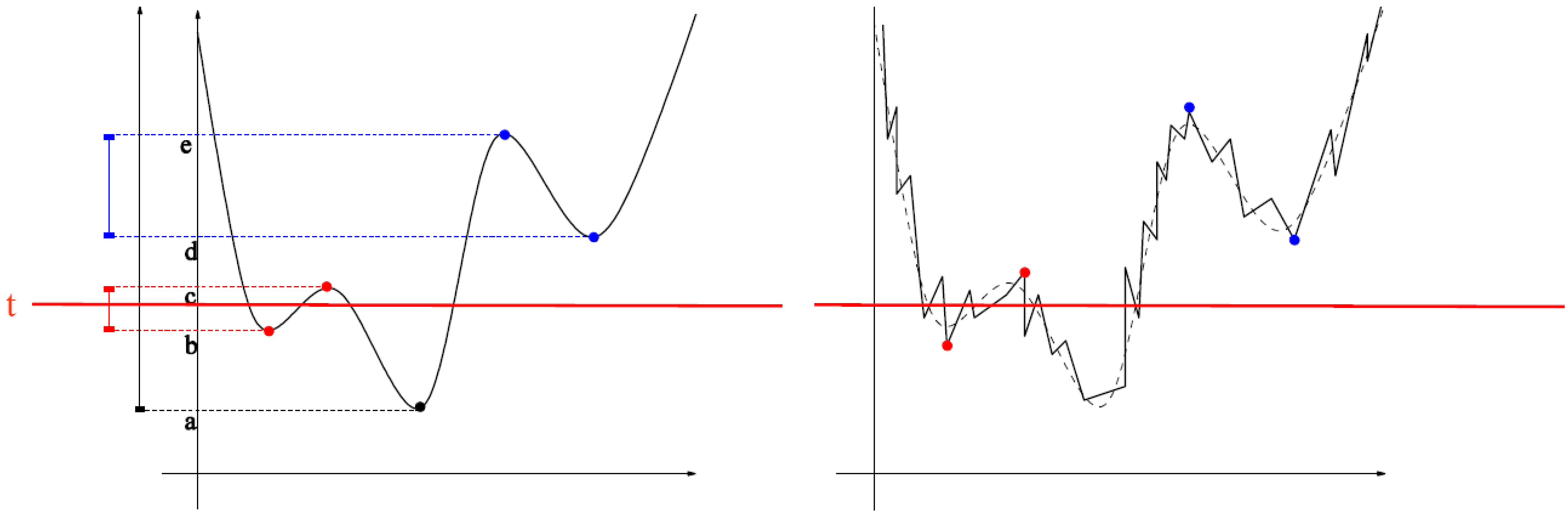
This is where topological persistence comes into play!

# Topological persistence

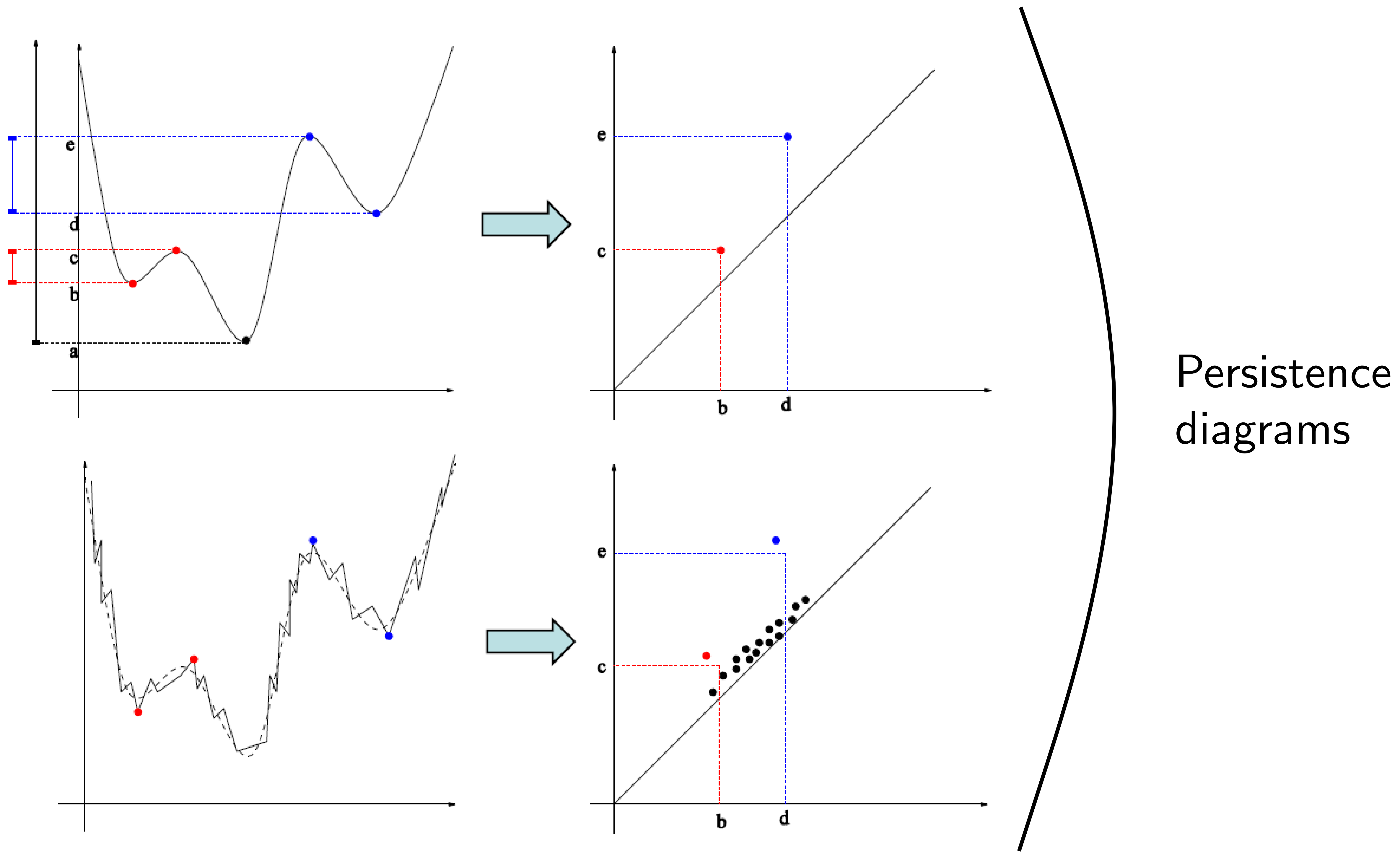
- a tool to study topological properties of data (represented by real valued functions on topological spaces).
- A method that allow to separate information from topological noise.
- References:
  - H. Edelsbrunner, D. Letscher and A. Zomorodian. *Topological persistence and simplification*. Discrete Comput. Geom., 28:511-533, 2002.
  - D. Cohen-Steiner and H. Edelsbrunner and J. Harer, *Stability of Persistence Diagrams*, Proc. 21st ACM Sympos. Comput. Geom. 2005.
  - F. Chazal and D. Cohen-Steiner and L. J. Guibas and M. Glisse and S. Y. Oudot, *Proximity of Persistence Modules and their Diagrams*, Proc. 25th ACM Sympos. Comput. Geom. 2009.

# A simple example




- What is the relevant number of connected components of $f^{-1}((-\infty, t])$?
- More generally, study the topology of the sublevel sets $f^{-1}((-\infty, t])$ as $t$ varies.

# A simple example: filter out topological noise



# Functions defined over higher dimensional spaces

- $f : X \to \mathbb{R}$ continuous where $X$ is a topological space
- Not only connected components but also cycles, voids, etc... $\to$ persistence of homological features / evolution of $H_k(f^{-1}((-\infty, t]))$

Relation between fonctions and filtrations:

- $\forall t \leq t' \in \mathbb{R},\ f^{-1}((-\infty, t]) \subseteq f^{-1}((-\infty, t']) \to$ filtration of $X$ by the sublevel sets of $f$.
- If $f$ is defined at the vertices of a simplicial complex $K$ , the sublevel sets filtration is a filtration of the simplicial complex $K$.

- For $\sigma = [v_0, \cdots, v_k] \in K,\ f(\sigma) = \max_{i=0,\cdots,k} f(v_i)$
- The simplices of $K$ are ordered according increasing $f$ values (and dimension in case of equal values on different simplices).

# Notations

In the following:

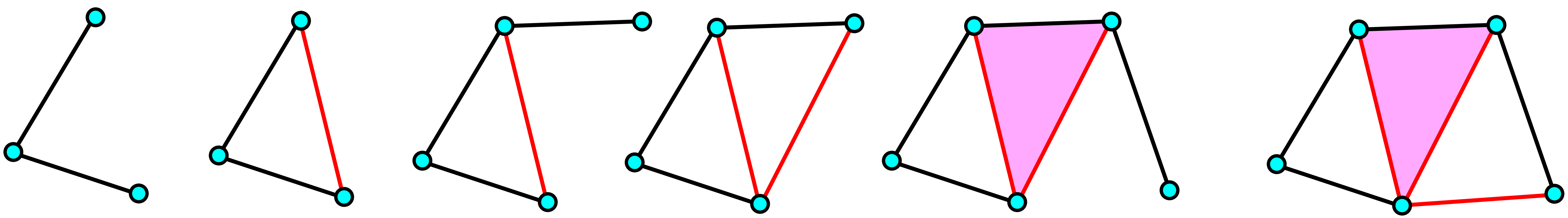
- Let $\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$ be a filtration of a simplicial complex $K$ s. t. $K^{i+1} = K^i \cup \sigma^{i+1}$ where $\sigma^{i+1}$ is a simplex of $K$.

- $Z_k^i =$ the $k$-cylcles of $K^i$, $B_k^i =$ the $k$-boundaries of $K^i$ and $H_k^i =$ the $k^{th}$-homology group of $K^i$.

- $Z_k^0 \subseteq Z_k^1 \subseteq \cdots \subseteq Z_k^i \subseteq \cdots \subseteq Z_k^m = Z_k(K)$

- $B_k^0 \subseteq B_k^1 \subseteq \cdots \subseteq B_k^i \subseteq \cdots \subseteq B_k^m = B_k(K)$

# Cycle associated to a positive simplex

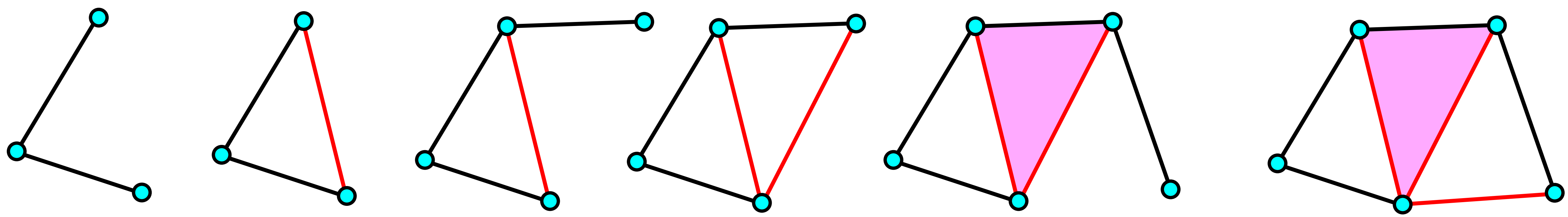


**Lemma:** If $\sigma^i$ is a positive $k$-cycle, then there exists a $k$-cycle $c_\sigma$ s.t.:
- $c_\sigma$ is not a boundary in $K^i$,
- $c_\sigma$ contains $\sigma^i$ but no other positive $k$-simplex.

The cycle $c^\sigma$ is unique.

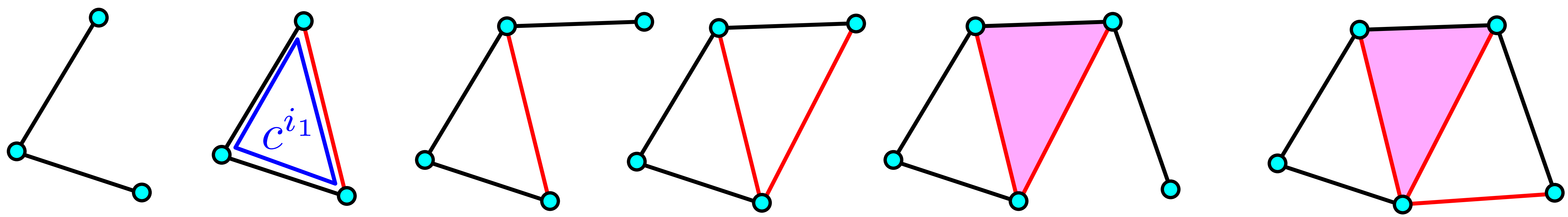
**Proof:**
By induction on the order of appearence of the simplices in the filtration.

# Homology basis



- At the beginning: the basis of $H_k^0$ is empty.
- If a basis of $H_k^{i-1}$ has been built and $\sigma^i$ is a positive $k$-simplex then one adds the homology class of the cycle $c^i$ associated to $\sigma^i$ to the basis of $H_k^{i-1} \Rightarrow$ basis of $H_k^i$.
- If a basis of $H_k^{j-1}$ has been built and $\sigma^j$ is a negative $(k+1)$-simplex:
  - let $c^{i_1}, \cdots, c^{i_p}$ be the cycles associated to the positive simplices $\sigma^{i_1}, \cdots, \sigma^{i_p}$ that form a basis of $H_k^{j-1}$
  - $d = \partial\sigma^j = \sum_{k=1}^{p} \varepsilon_k c^{i_k} + b$
  - $l(j) = \max\{i_k : \varepsilon_k = 1\}$
  - Remove the homology class of $c^{l(j)}$ from the basis of $H_k^{j-1} \Rightarrow$ basis of $H_k^j$.

# Homology basis

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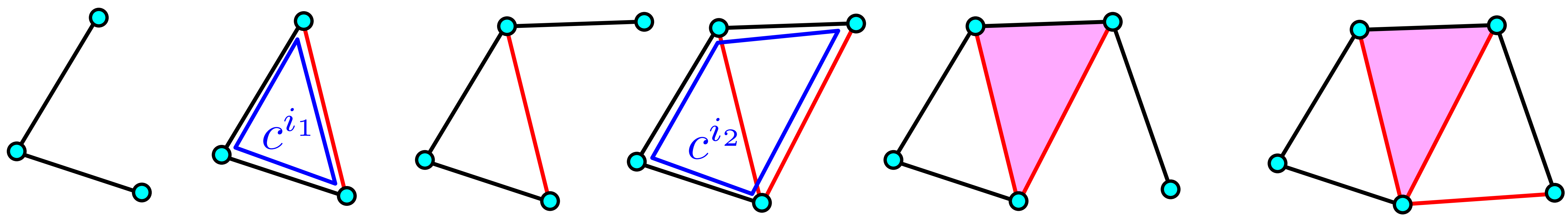


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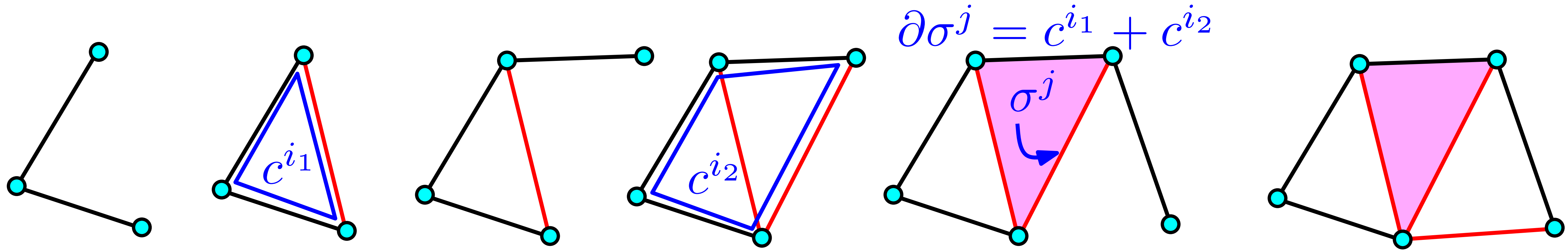


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# Pairing simplices

- If a basis of $H_k^{j-1}$ has been built and $\sigma^j$ is a negative $(k+1)$-simplex:
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  - $l(j) = \max\{i_k : \varepsilon_k = 1\}$
  - Remove the homology class of $c^{l(j)}$ from the basis of $H_k^{j-1} \Rightarrow$ basis of $H_k^j$.

The simplices $\sigma^{l(j)}$ and $\sigma^j$ are paired to form a persistent pair $(\sigma^{l(j)}, \sigma^j)$.
$\rightarrow$ The homology class created by $\sigma^{l(j)}$ in $K^{l(j)}$ is killed by $\sigma^j$ in $K^j$. The persistence (or life-time) of this cycle is : $j - l(j) - 1$.

**Remark:** filtrations of $K$ can be indexed by increasing sequences $\alpha_i$ of real numbers (useful when working with a function defined on the vertices of a simplicial complex).

# The persistence algorithm: first version

**Input:** $\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$ a $d$-dimensional filtration of a simplicial complex $K$ s. t. $K^{i+1} = K^i \cup \sigma^{i+1}$ where $\sigma^{i+1}$ is a simplex of $K$.

$L_0 = L_1 = \cdots = L_{d-1} = \emptyset$
For $j = 0$ to $m$
  $k = \dim \sigma^j - 1$;
  if $\sigma^j$ is a negative simplex
    $l(j) =$ highest index of the positive simplices associated to $\partial\sigma^j$;
    $L_k = L_k \cup \{(\sigma^{l(j)}, \sigma^j)\}$;
  end if
end for
output $L_0, L_1, \cdots, L_{d-1}$ ;

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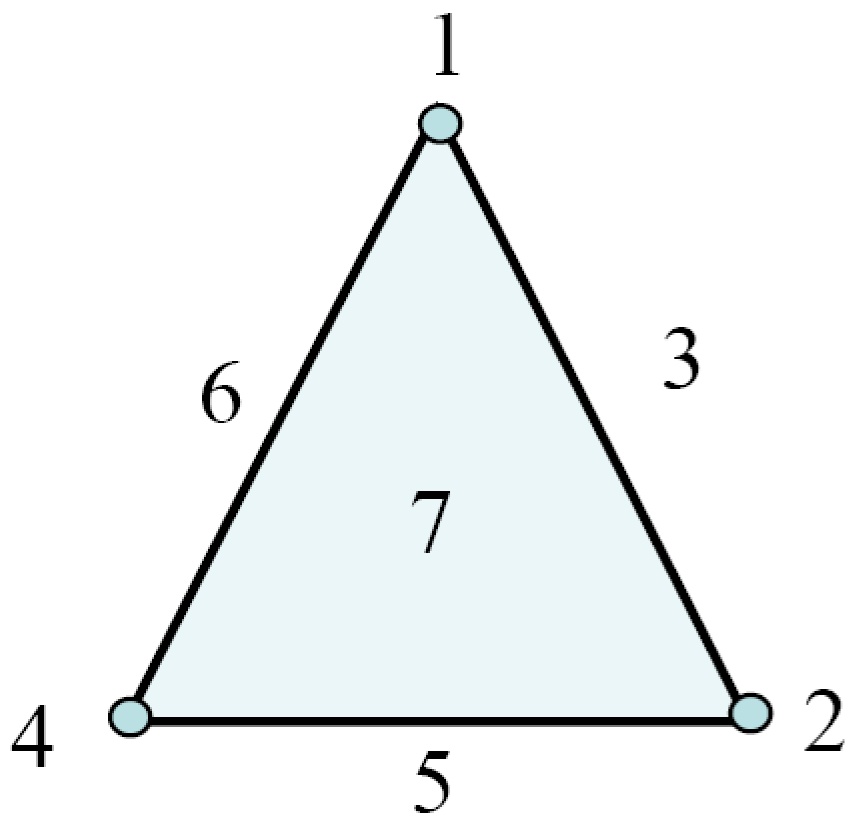
How to test this condition?

# The matrix of the boundary operator




$$\begin{array}{cc} \begin{array}{ccccccc} & & & & j & & \end{array} & \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \\ \\ \\ l(j) \\ \\ \\ \\ \end{array} \end{array}$$

- $M = (m_{ij})_{i,j=1,\cdots,m}$ with coefficient in $\mathbb{Z}/2$ defined by

$$m_{ij} = 1 \text{ if } \sigma^i \text{ is a face of } \sigma^j \text{ and } m_{ij} = 0 \text{ otherwise}$$

- For any column $C_j$, $l(j)$ is defined by

$$(i = l(j)) \Leftrightarrow (m_{ij} = 1 \text{ and } m_{i'j} = 0 \forall i' > i)$$

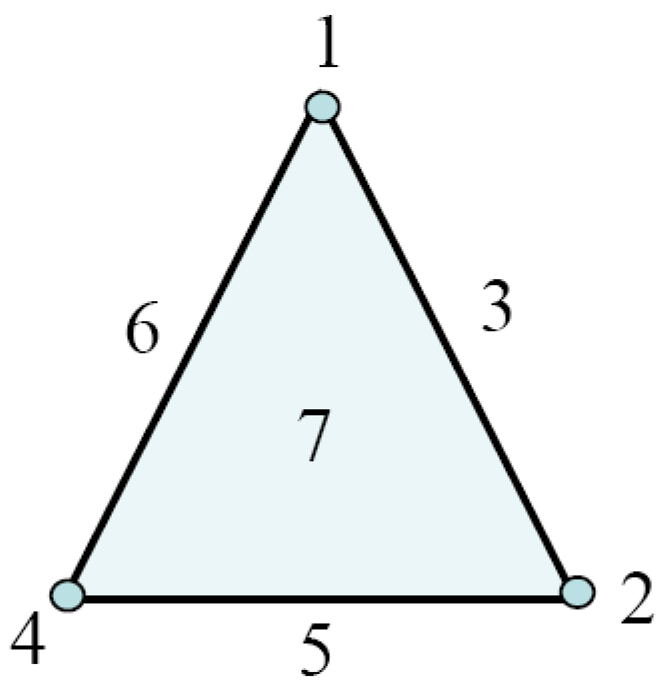
# The persistence algorithm: second version

**Input:** $\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K$ a $d$-dimensional filtration of a simplicial complex $K$ s. t. $K^{i+1} = K^i \cup \sigma^{i+1}$ where $\sigma^{i+1}$ is a simplex of $K$.

For $j = 0$ to $m$
    While (there exists $j' < j$ such that $l(j') == l(j)$)
        $C_j = C_j + C_{j'}$ mod$(2)$;
    End while
End for
Output the pairs $(l(j), j)$;

**Remark:** The worst case complexity of the algorithm is $O(m^3)$ but much lower in most practical cases.

# A very simple example




$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{c} \begin{array}{ccccccc} & & & & & C_5 + C_6 & \\ & & & & & \downarrow & \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 1 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

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Pairs: $(2,3)$ $(4,5)$ $(7,7)$

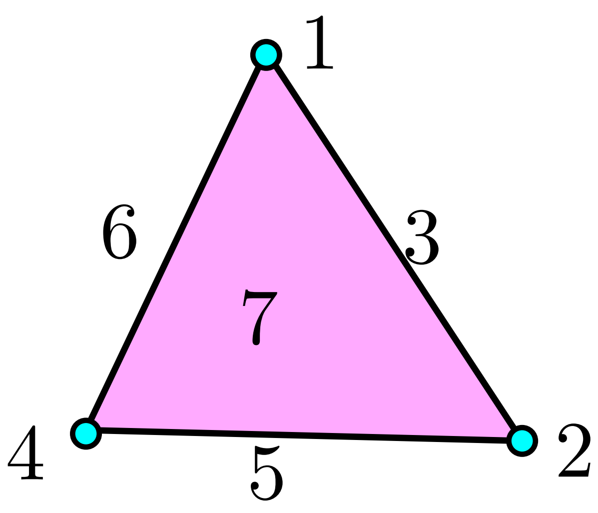
# Correctness of the second algorithm

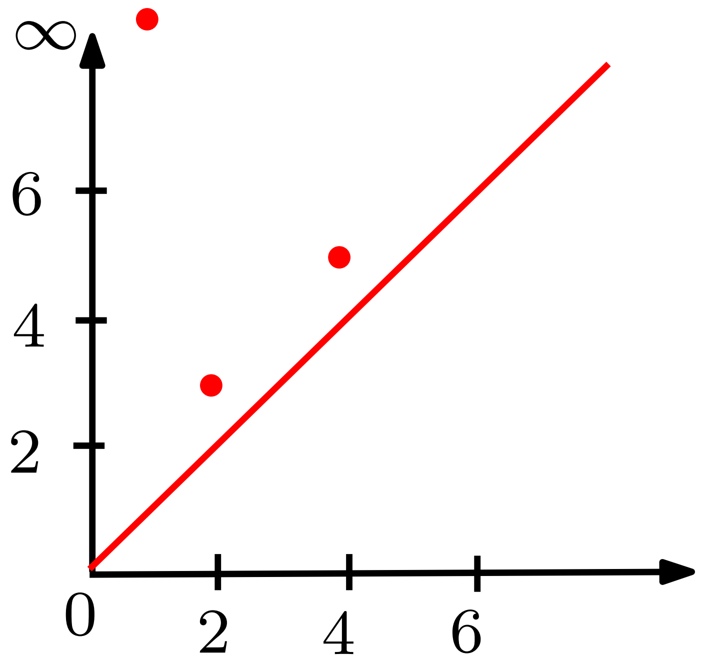
**Proposition:** the second algorithm outputs the persistence pairs.

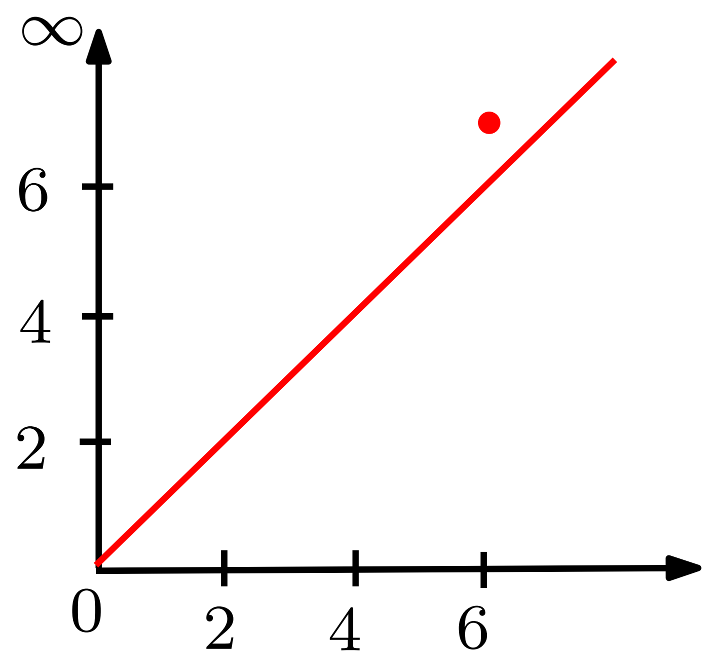
**Proof:** follows from the four remarks below.

1. At each step of the algorithm, the column $C_j$ represents a chain of the form
$$\partial\left(\sigma^j + \sum_{i<j} \varepsilon_i \sigma^i\right) \text{ with } \varepsilon_i \in \{0,1\}$$
2. At this end of the algorithm, if $j$ is s.t. $l(j)$ is defined then $\sigma^{l(j)}$ is a positive simplex.
3. If at the end of the algorithm if the column $C_j$ is zero then $\sigma^j$ is positive.
4. If at the end of the algorithm the column $C_j$ is not zero then $(\sigma^{l(j)}, \sigma^j)$ is a persistence pair.

# Persistence diagrams

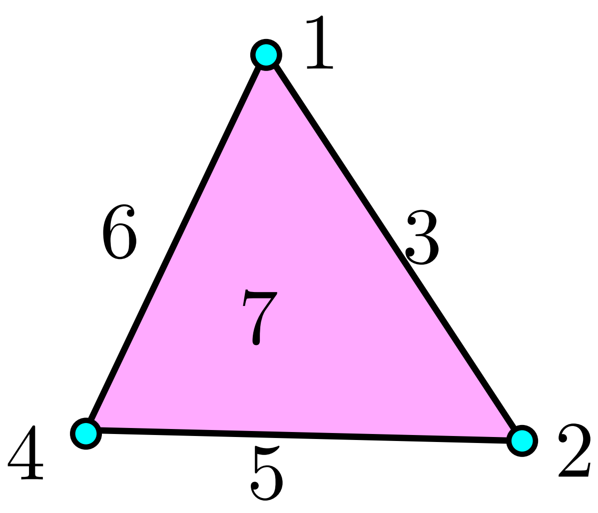


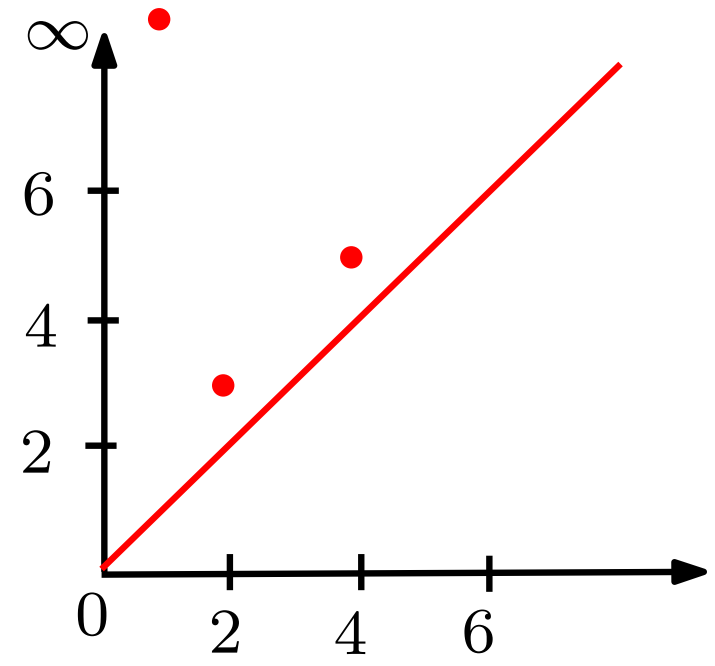



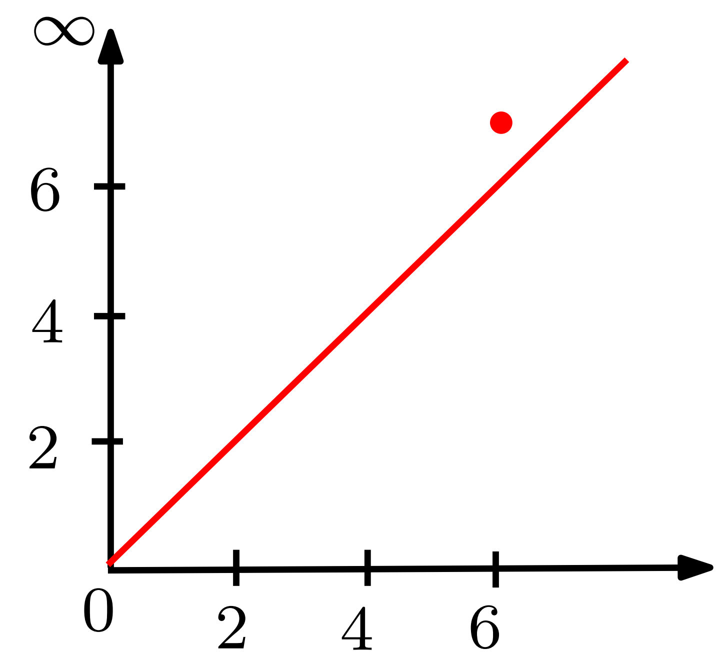




- each pair $(\sigma^{l(j)}, \sigma^j)$ is represented by $(l(j), j)$ or $(f(\sigma^{l(j)}), f(\sigma^j)) \in \mathbb{R}^2$ when considering filtrations induced by functions.
- The diagonal $\{y = x\}$ is added to the persistence diagram.
- Unpaired positive simplex $\sigma^i \to (i, +\infty)$.

# Persistence diagrams

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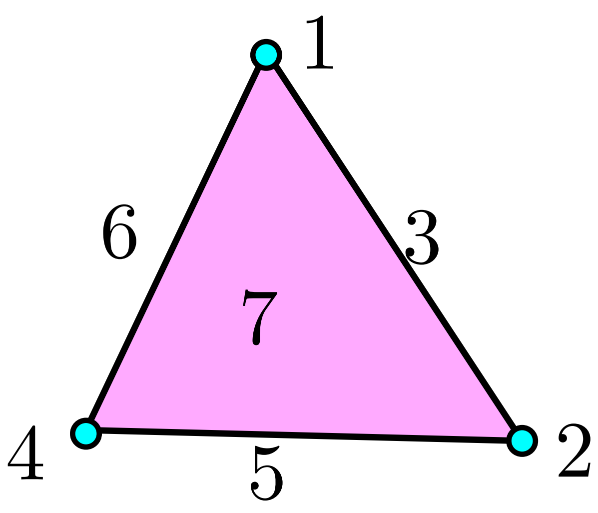
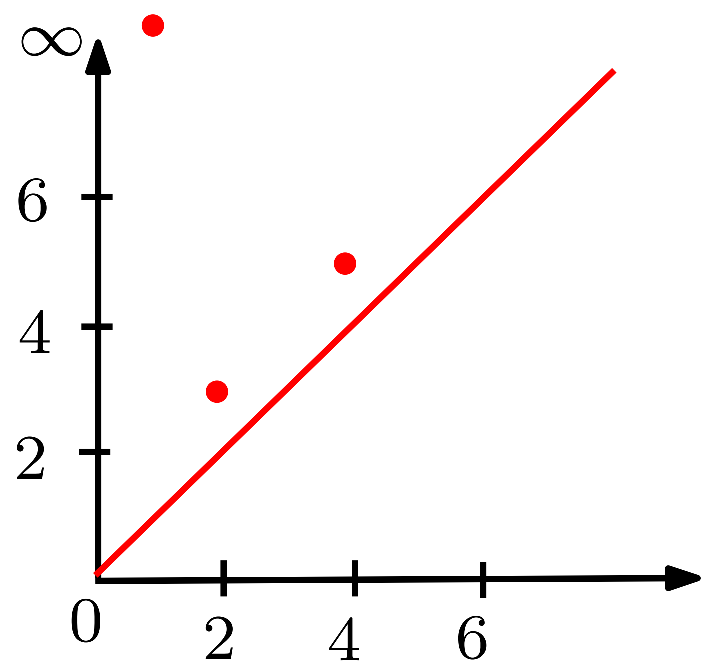
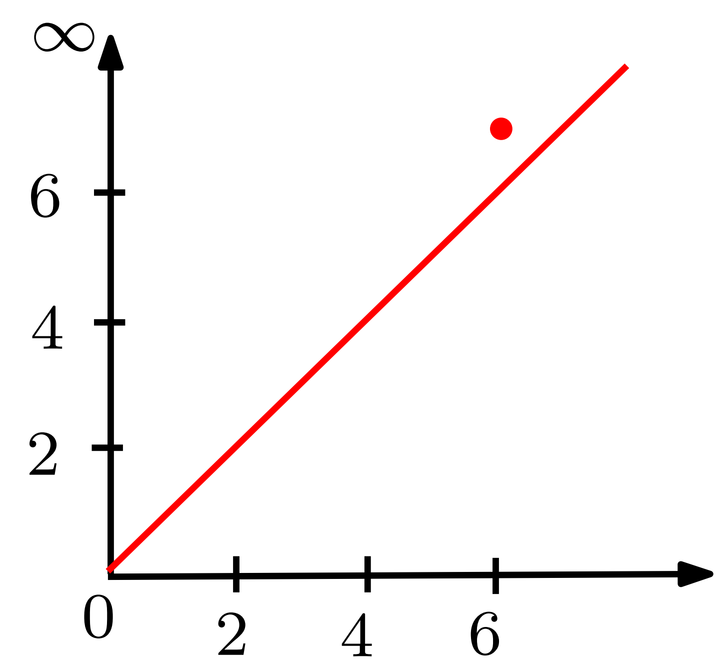
Warning: in this case, points may have multiplicity.

# Persistence diagrams

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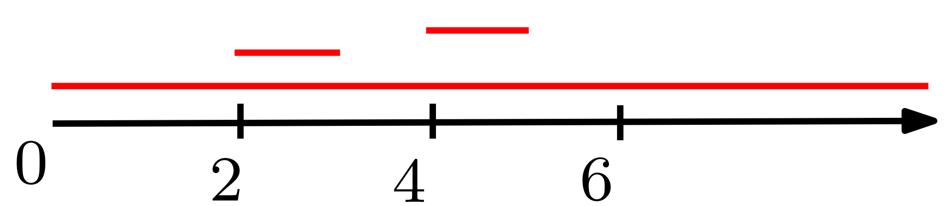
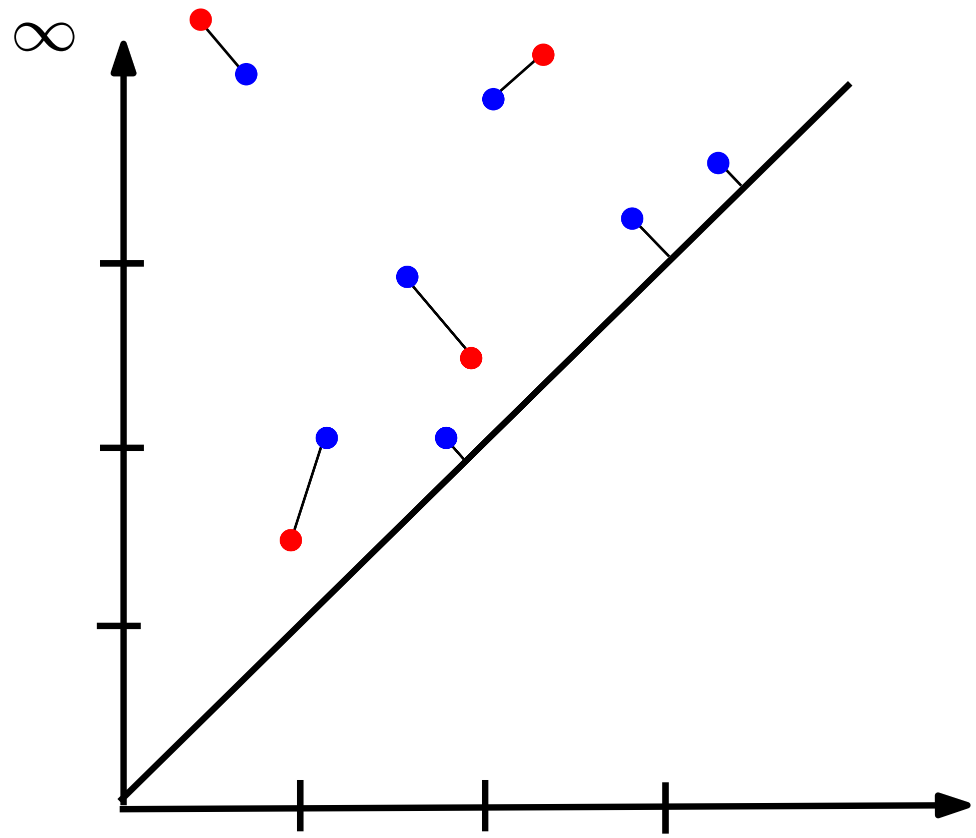
- Unpaired positive simplex $\sigma^i \to (i, +\infty)$.

**Barcodes:** an alternative (equivalent) representation where each pair $(i, j)$ is represented by the interval $[i, j]$



# Distance between persistence diagrams



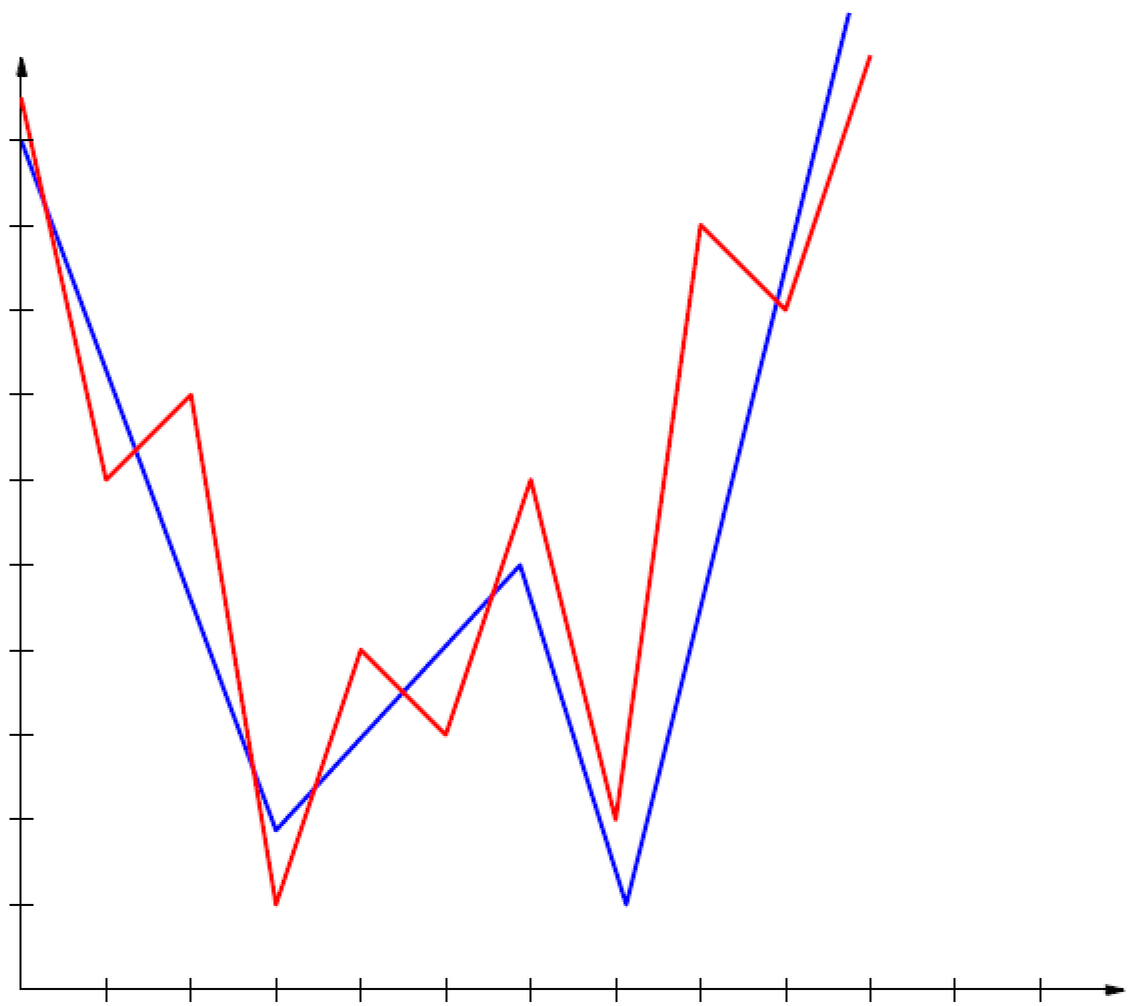

Let $K$ be a simplicial complex and $f, g$ two functions defined on the vertices of $K$. Let $D_f$ and $D_g$ be the persistence diagrams of $f$ and $g$.

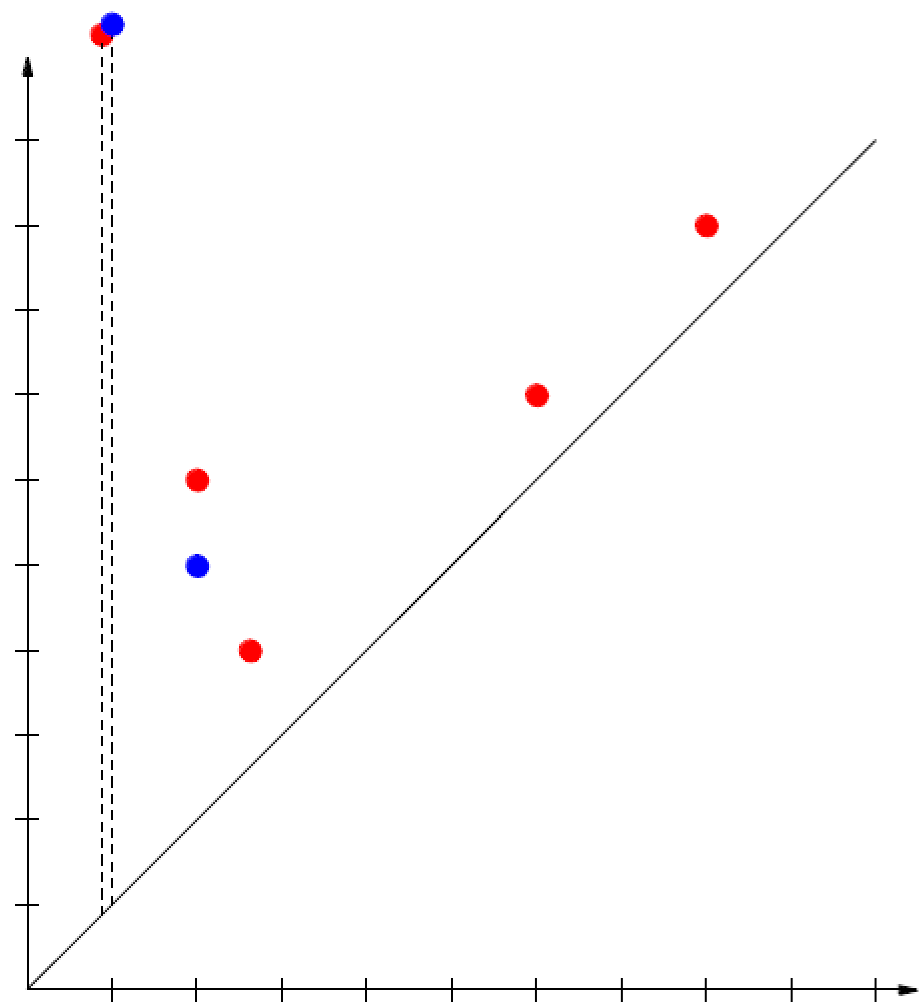
The bottleneck distance between $D_f$ and $D_g$ is

$$d_B(D_f, D_g) = \inf_{\gamma \in \Gamma} \sup_{p \in D_f} \|p - \gamma(p)\|_\infty$$

where $\Gamma$ is the set of all the bijections between $D_f$ and $D_g$ and $\|p - q\|_\infty = \max(|x_p - x_q|, |y_p - y_q|)$.

# Stability of persistence diagrams





**Theorem:** Let $K$ be a simplicial complex and let $f, g : K \to \mathbb{R}$.

$$d_B(D_f, D_g) \leq \|f - g\|_\infty$$

where $\|f - g\|_\infty = \sup_{v \in vertices(K)} |f(v) - g(v)|$.

# Stability of persistence diagrams

- Let $K$ and $K'$ be two simplicial complexes homeomorphic to a topological space $X$.
- Let $\phi : K \to X$ and $\phi' : K' \to X$ be homeomorphisms
- Let $f : X \to \mathbb{R}$ be a continuous function and $D_f(K)$ (resp. $D_f(K')$) the persistence diagram of $f \circ \phi$ (resp. $f \circ \phi'$).

**Theorem:** Let $\varepsilon > 0$ be such that for any simplex $\sigma \in K$ (resp. $\in K'$), $\sup_{x,y\in\sigma} |f \circ \phi(x) - f \circ \phi(y)| < \varepsilon$ (resp. $\sup_{x,y\in\sigma} |f \circ \phi'(x) - f \circ \phi'(y)| < \varepsilon$). Then one has

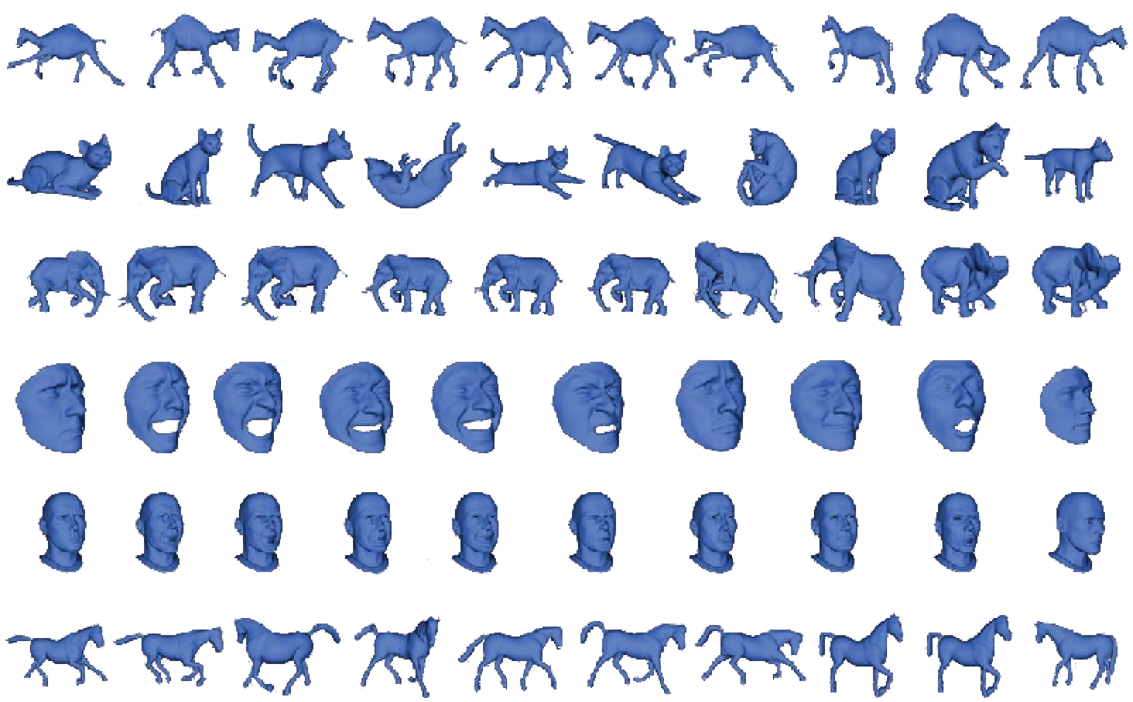
$$d_B(D_f(K), D_f(K')) \leq 2\varepsilon$$

**Remark:** this is a particular (and weaker) version of a much more general result. See:

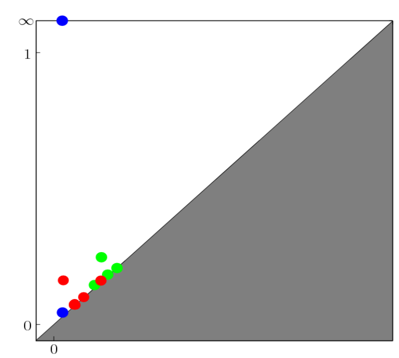

- D. Cohen-Steiner and H. Edelsbrunner and J. Harer, *Stability of Persistence Diagrams*, Proc. 21st ACM Sympos. Comput. Geom. 2005.
- F. Chazal and D. Cohen-Steiner and L. J. Guibas and M. Glisse and S. Y. Oudot, *Proximity of Persistence Modules and their Diagrams*, Proc. 25th ACM Sympos. Comput. Geom. 2009.

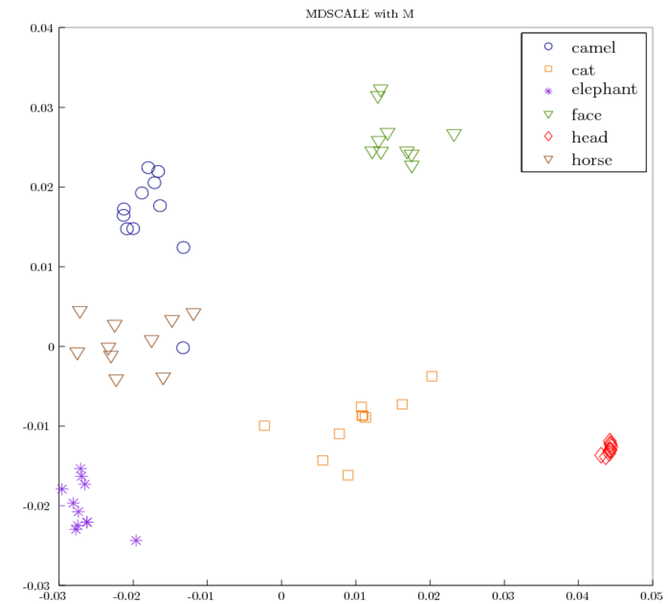
# Consequences of the stability

- Persistence diagrams are defined and stable for a large class of continuous functions defined over (pre-)compact metric spaces.



3D shapes database



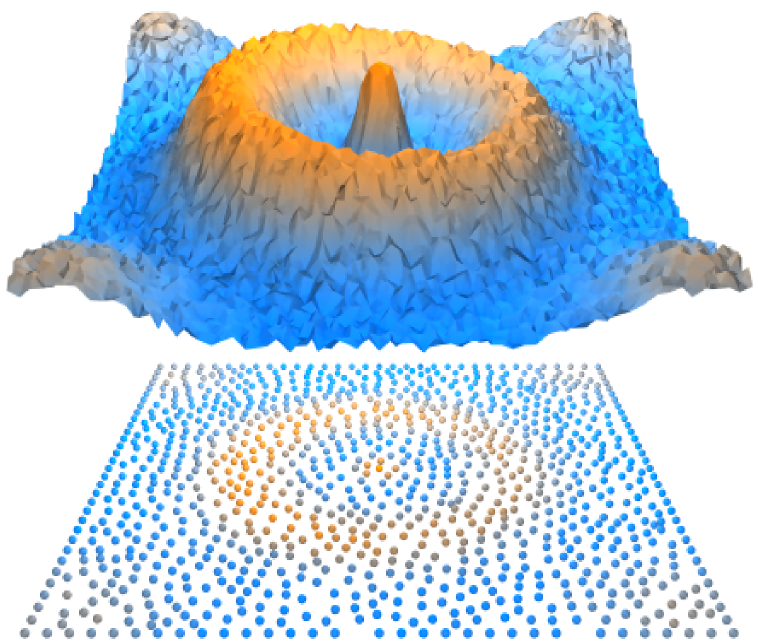
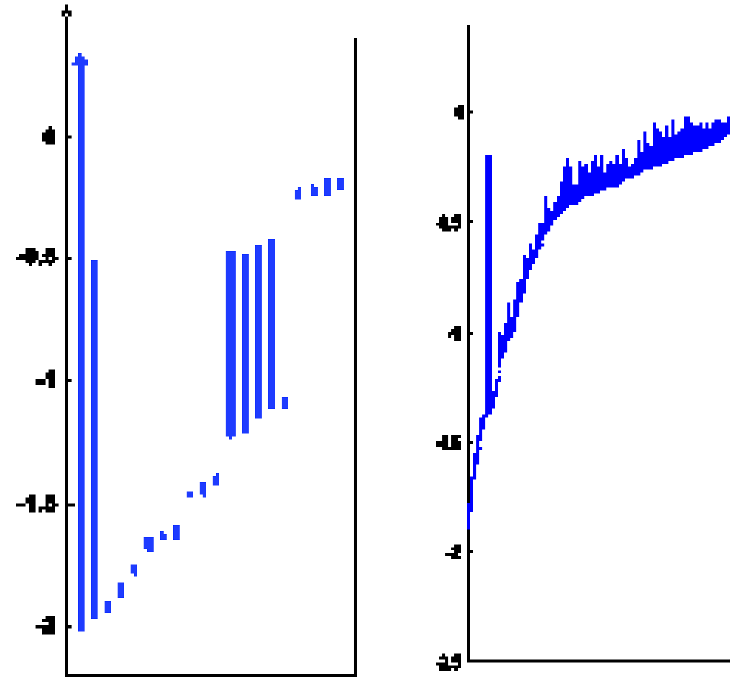
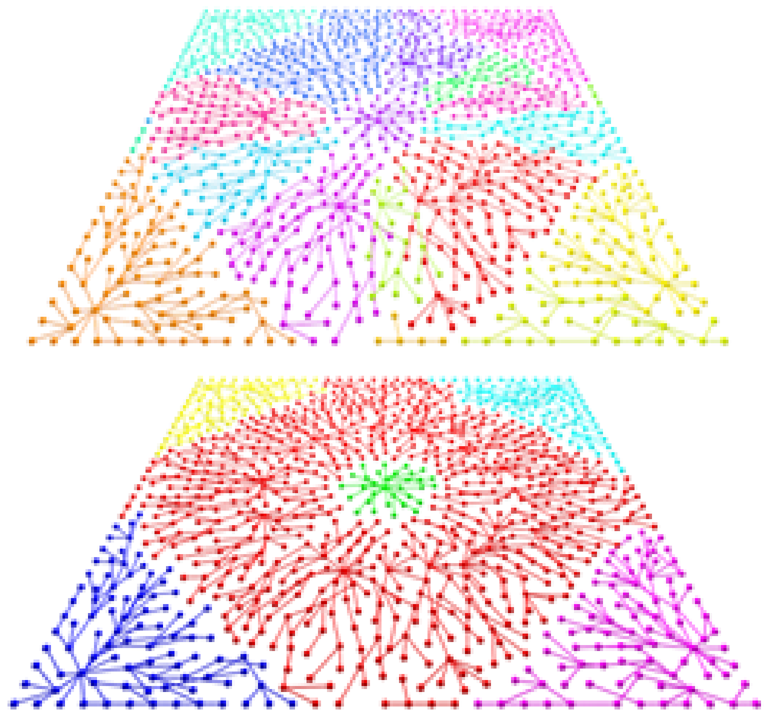



→ definition stable (Gromov-Hausdorff distance) topological signatures for compact metric spaces.
→ Efficient algorithm to compute signatures.
→ applications to shape classification.

**Ref:** F. Chazal, D. Cohen-Steiner, L. J. Guibas, F. Mémoli, S. Oudot, Gromov-Hausdorff Stable Signatures for Shapes using Persistence, Computer GraphicsForum (proc. SGP 2009), pp. 1393-1403, 2009.

# Consequences of the stability

- Persistence diagrams can be reliably estimated from data (functions known through a point cloud data set approximating a topological space).

Previous approach can be generalized, leading to robust algorithms to compute the topological persistence of functions defined over point clouds sampled around unknown shapes

**Ref:**


- F. Chazal, L. Guibas, S. Oudot, P. Skraba, *Analysis of Scalar Fields over Point Cloud Data*, proc. ACM Symposium on Discrete Algorithms 2009.

- F. Chazal, S. Oudot, *Toward Persistence-Based Reconstruction in Euclidean Spaces*, proc. ACM Symposium on Computational Geometry 2008.

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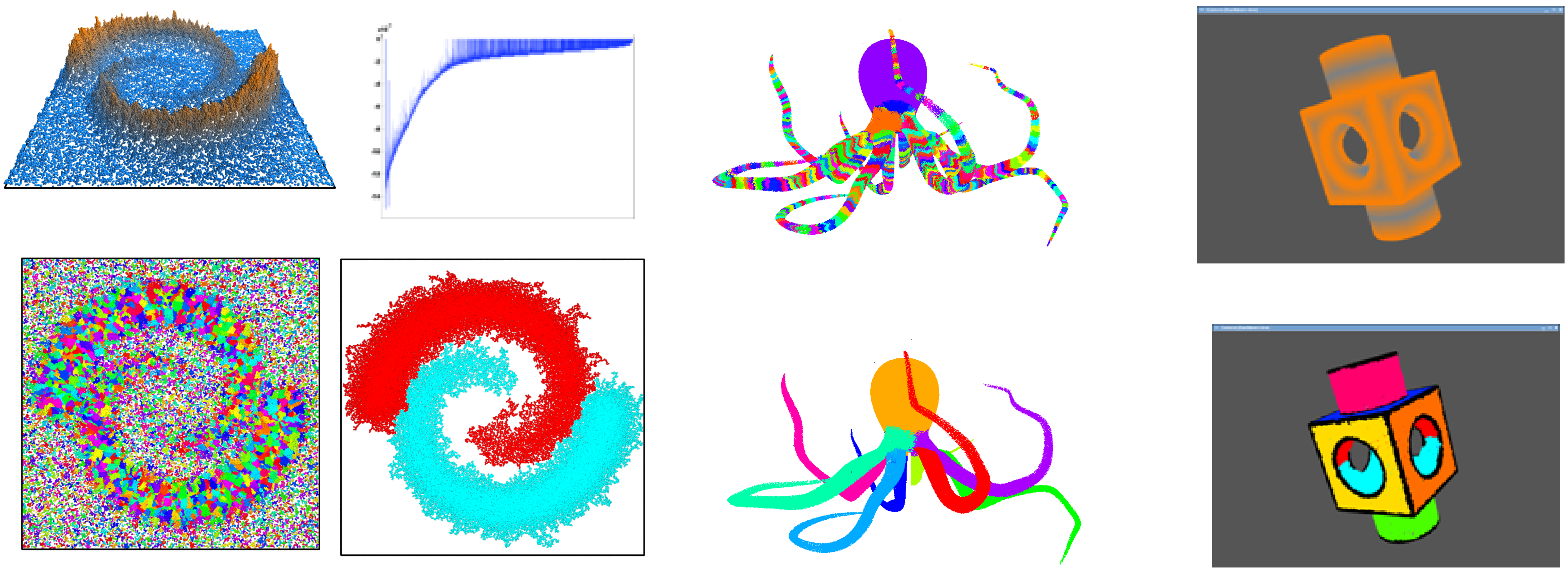

Applications to clustering, segmentations, sensor networks,...

**Ref:**


- F. Chazal, L. Guibas, S. Oudot, P. Skraba, *Analysis of Scalar Fields over Point Cloud Data*, proc. ACM Symposium on Discrete Algorithms 2009.

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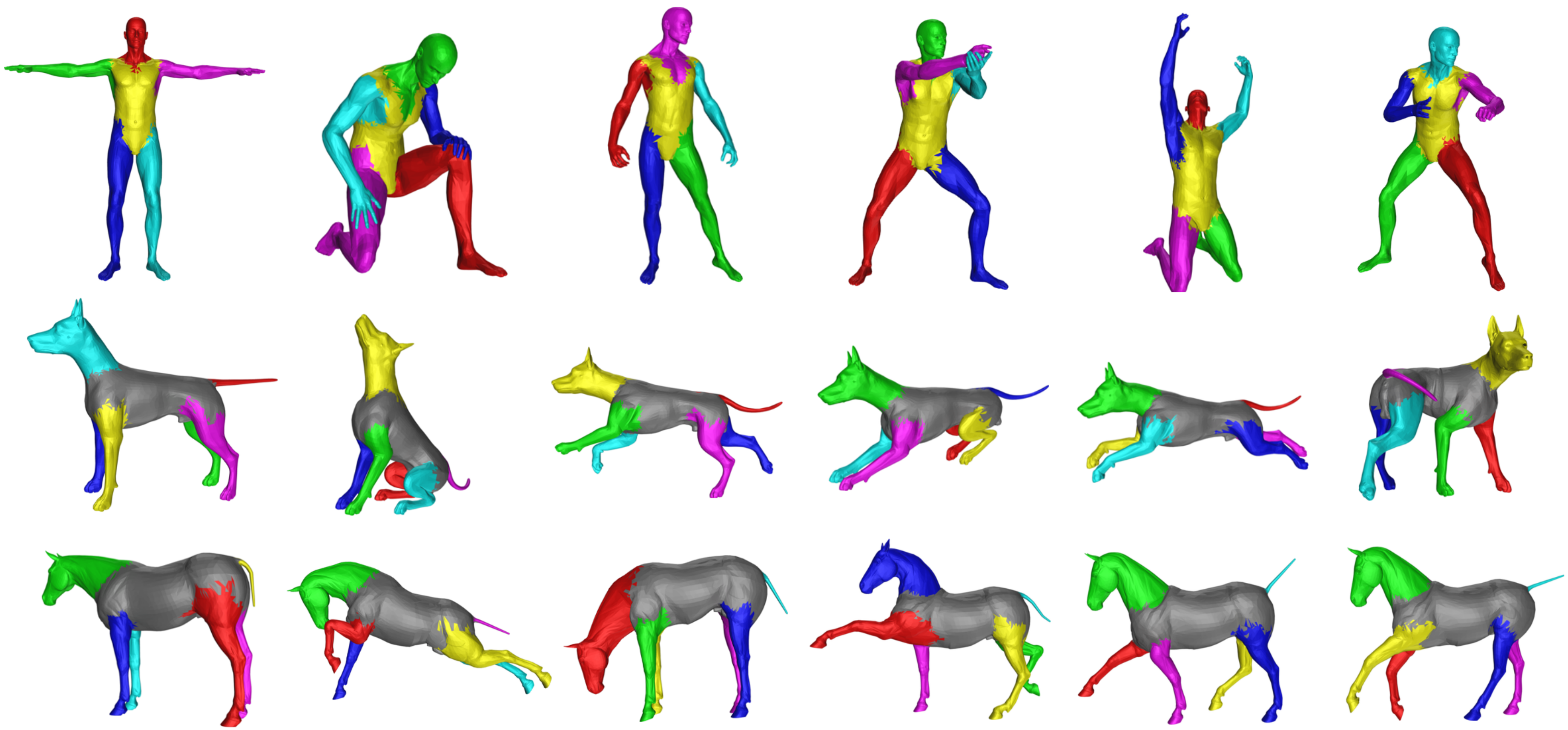

Applications to non rigid shapes segmentation

**Ref:**


- P. Skraba, M. Ovsjanikov, F. Chazal, L. Guibas, Persistence-Based Segmentation of Deformable Shapes, Proc. Workshop on Nonrigid Shape Analysis and Deformable Image Alignment (NORDIA), Proc. CVPR 2010