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Topological signature of Data : a (partial) introduction to Topological Data Analysis

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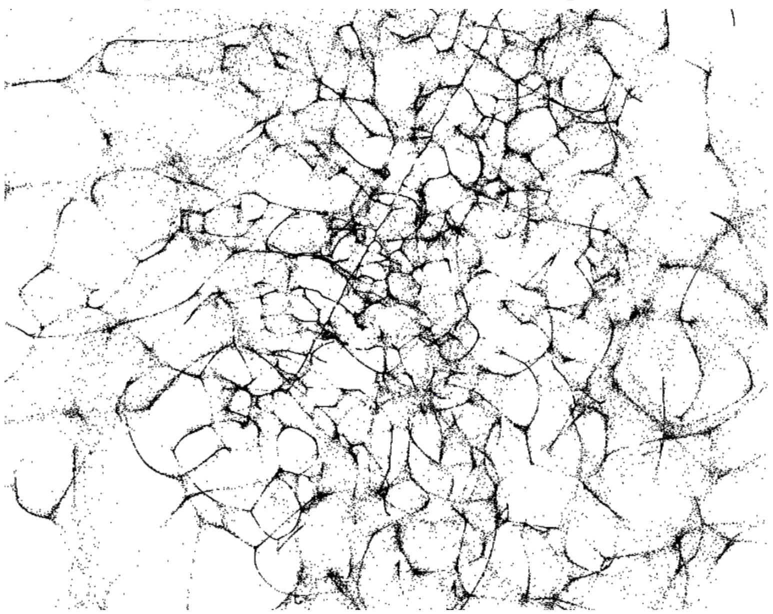
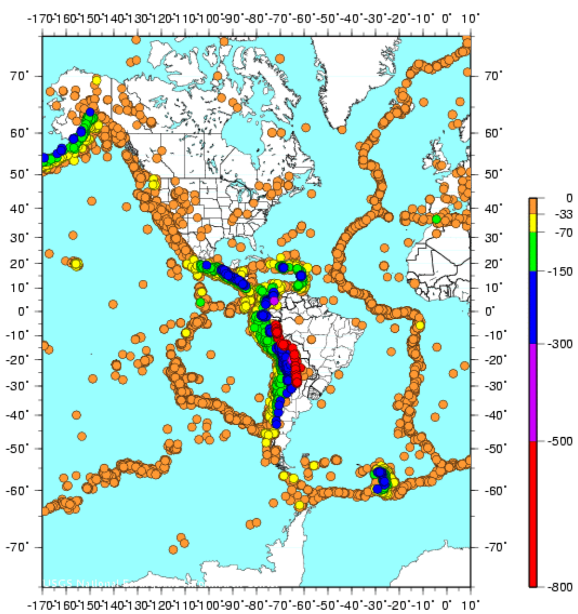
Introduction

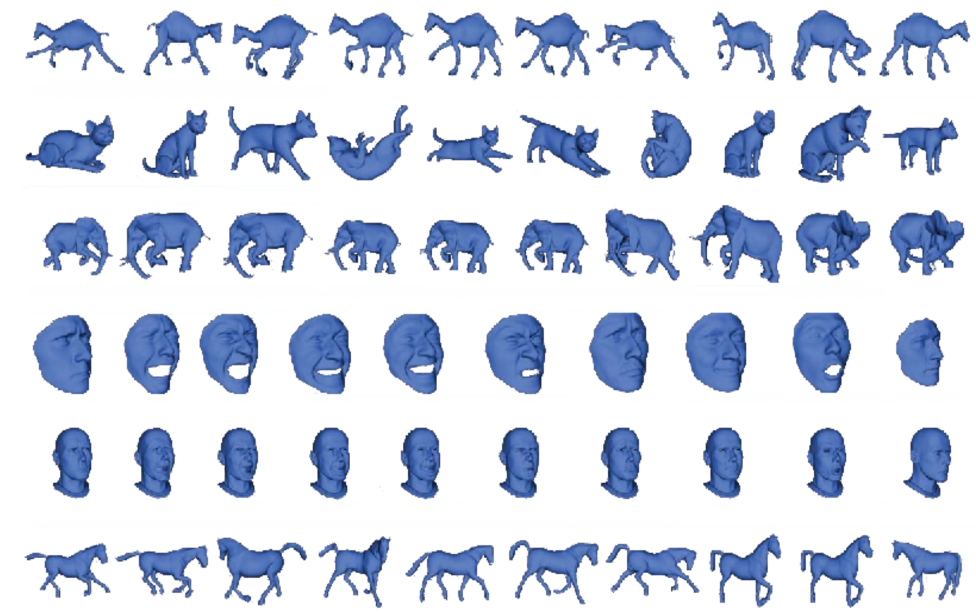
[distribution of galaxies]



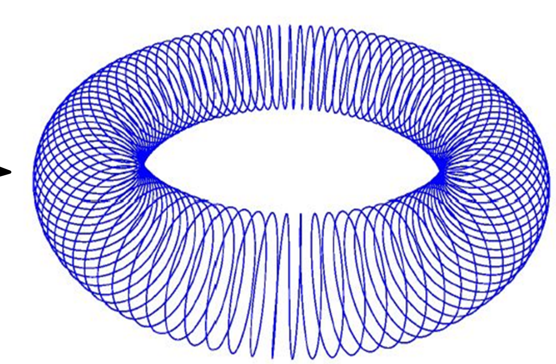
[Earthquake epicenters]



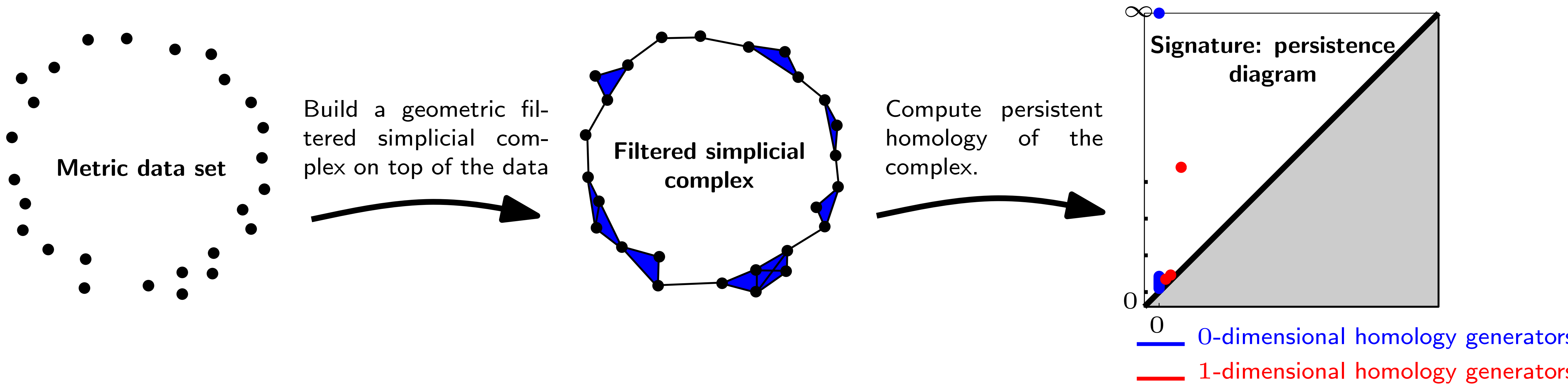
[3D shape database]



- Data often come as (sampling of) metric spaces or sets/spaces endowed with a similarity measure with, possibly complex, topological/geometric structure.
- Goal of TDA: infer relevant topological and geometric features of these spaces.
- Challenges and goals:
  $\rightarrow$ no direct access to topological/geometric information: need of intermediate constructions (simplicial complexes);
  $\rightarrow$ distinguish topological "signal" from noise;
  $\rightarrow$ topological information may be multiscale;
  $\rightarrow$ statistical analysis of topological information.



Topological signatures for data



A "classical" approach:

- Build a geometric filtered simplicial complex on top of $(\mathbb{X}, \rho_{\mathbb{X}})$ (ρ_X being a metric/similarity on $\mathbb{X}$).

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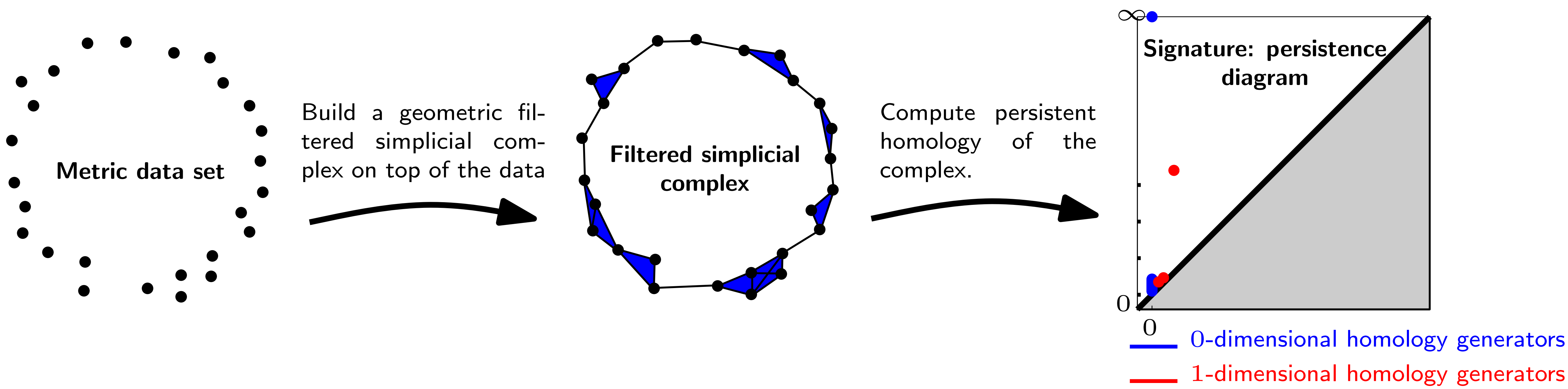

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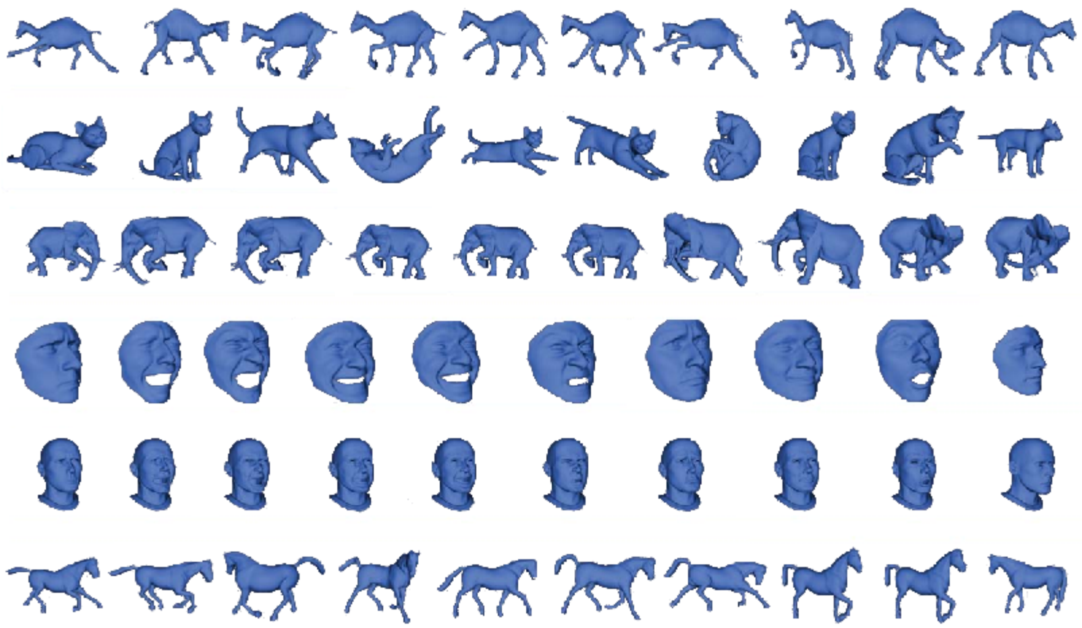
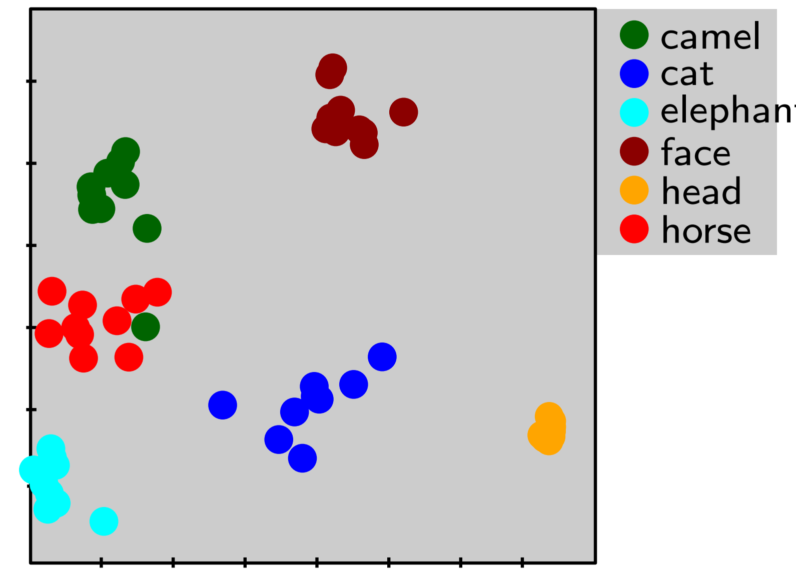
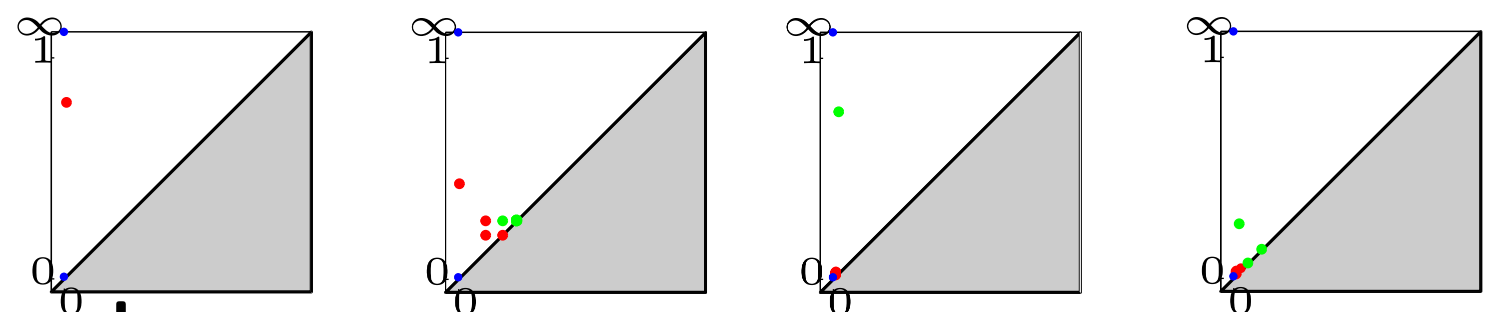


[C., Cohen-Steiner, Guibas, Mémoli, Oudot '09]



Use the metric on the space of persistence diagrams.



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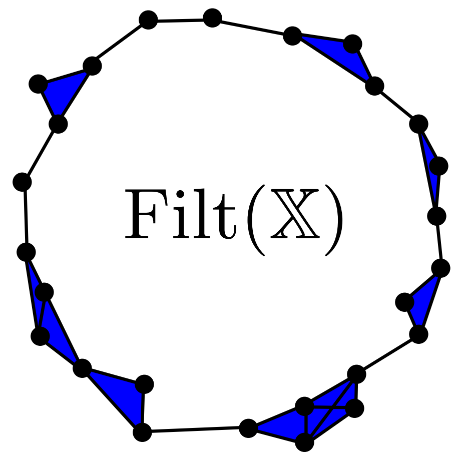
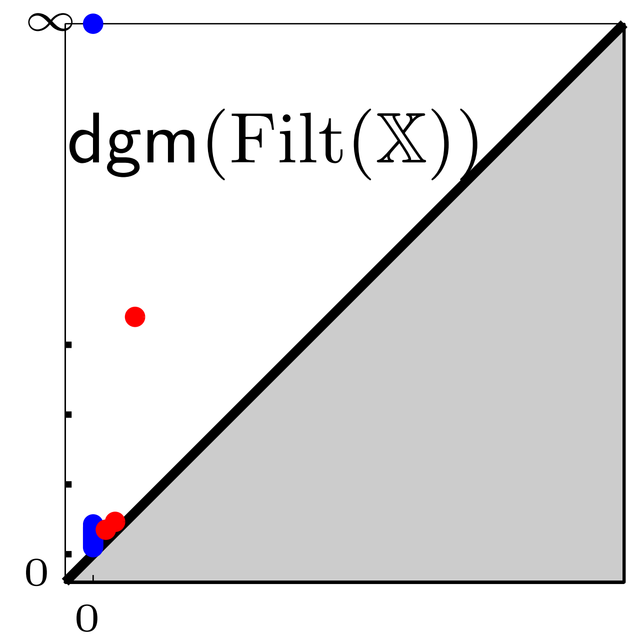
- Compare the signatures of "close" data sets $\rightarrow$ robustness and stability results.

Topological signatures for data





Examples:
- $\mathrm{Filt}(\mathbb{X}) = \mathrm{Rips}_\alpha(\mathbb{X})$
- $\mathrm{Filt}(\mathbb{X}) = \check{\mathrm{C}}\mathrm{ech}_\alpha(\mathbb{X})$
- $\mathrm{Filt}(\mathbb{X}) =$ sublevelset filtration of $\rho(., \mathbb{M})$.

Questions:

- Is dgm$(\mathrm{Filt}(\mathbb{X}))$ well-defined? ($\mathbb{X}$ may not be finite)
- Stability properties of dgm$(\mathrm{Filt}(\mathbb{X}))$?

Tame persistent modules

Definition: A persistence module $\mathbb{V}$ is an indexed family of vector spaces $(V_a \mid a \in \mathbf{R})$ and a doubly-indexed family of linear maps $(v_a^b : V_a \to V_b \mid a \leq b)$ which satisfy the composition law $v_b^c \circ v_a^b = v_a^c$ whenever $a \leq b \leq c$, and where v_a^a is the identity map on V_a.

Examples:

- Let $\mathbb{S}$ be a filtered simplicial complex. If $V_a = \mathrm{H}(\mathbb{S}_a)$ and $v_a^b : \mathrm{H}(\mathbb{S}_a) \to \mathrm{H}(\mathbb{S}_b)$ is the linear map induced by the inclusion $\mathbb{S}_a \hookrightarrow \mathbb{S}_b$ then $(\mathrm{H}(\mathbb{S}_a) \mid a \in \mathbf{R})$ is a persistence module.
- Given a metric space $(\mathbb{X}, \rho)$, $\mathrm{H}(\mathrm{Rips}(\mathbb{X}))$ is a persistence module.
- Given a metric space $(\mathbb{X}, \rho)$, $\mathrm{H}(\check{\mathrm{C}}\mathrm{ech}(\mathbb{X}))$ is a persistence module.
- If $f : X \to \mathbf{R}$ is a function, then the filtration defined by the sublevel sets of f, $\mathbb{F}_a = f^{-1}((-\infty, a])$, induces a persistence module at homology level.

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Definition: A persistence module $\mathbb{V}$ is q-tame if for any $a < b$, v_a^b has a finite rank.

Theorem [CCGGO'09-CdSGO'12]:
q-tame persistence modules have well-defined persistence diagrams.

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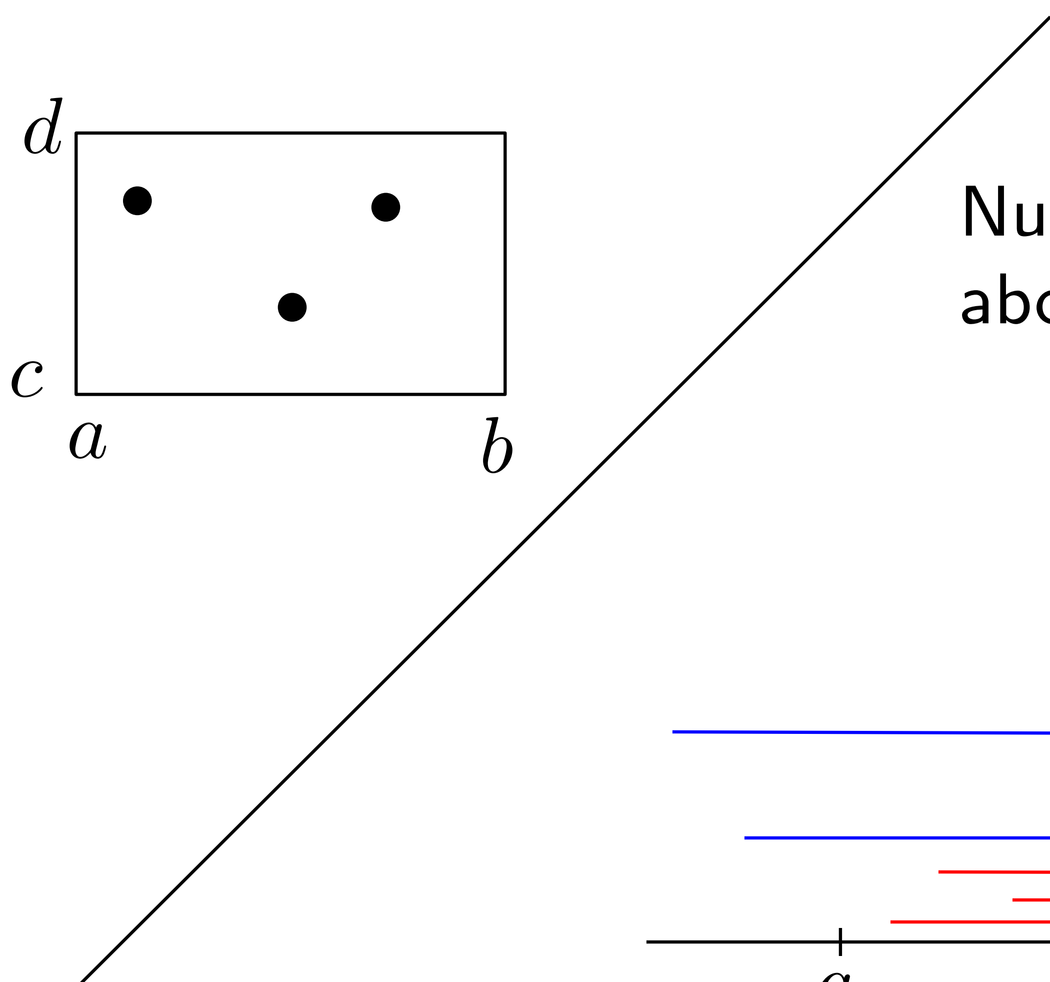
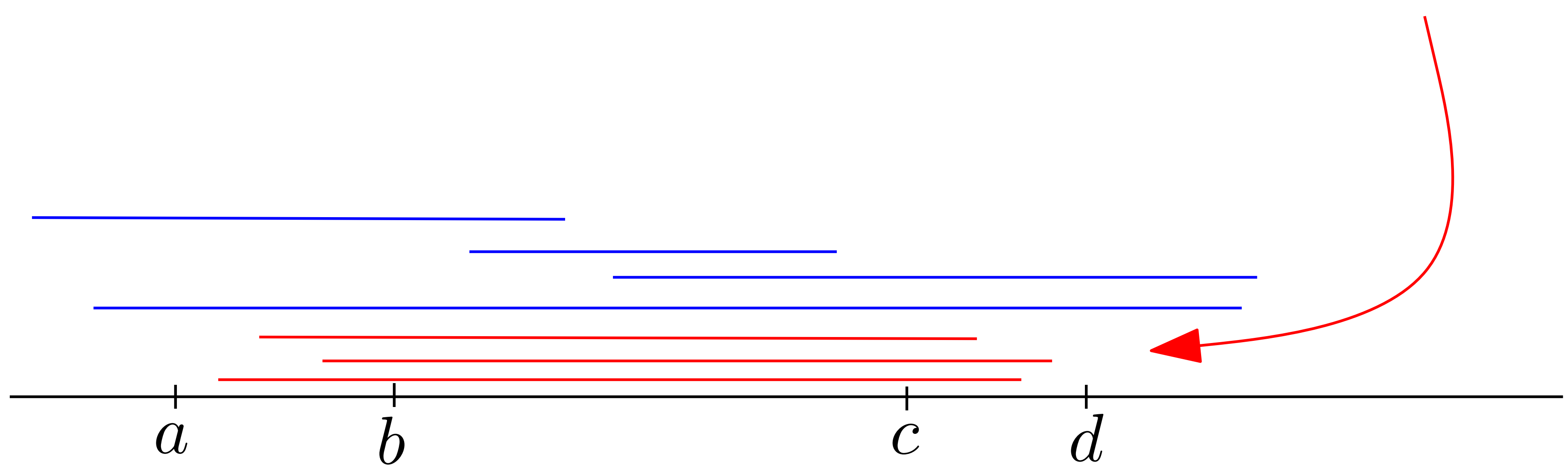
An idea about the definition of persistence diagrams:



Measures on rectangles:

Number of points in any rectangle $[a, b] \times [c, d]$ above the diagonal:

$$rk(v_b^c) - rk(v_b^d) + rk(v_a^d) - rk(v_a^c)$$



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Theorem [CCGGO'09-CdSGO'12]:
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Theorem[CdSO'12]: Let $\mathbb{X}$ be a precompact metric space. Then $\mathrm{H}(\mathrm{Rips}(\mathbb{X}))$ and $\mathrm{H}(\check{\mathrm{C}}\mathrm{ech}(\mathbb{X}))$ are q-tame.

As a consequence dgm$(\mathrm{H}(\mathrm{Rips}(\mathbb{X})))$ and dgm$(\mathrm{H}(\check{\mathrm{C}}\mathrm{ech}(\mathbb{X})))$ are well-defined!

Recall that a metric space $(\mathbb{X}, \rho)$ is precompact if for any $\epsilon > 0$ there exists a finite subset $F_\epsilon \subset \mathbb{X}$ such that $d_H(\mathbb{X}, F_\epsilon) < \epsilon$ (i.e. $\forall x \in X, \exists p \in F_\epsilon$ s.t. $\rho(x, p) < \epsilon$).

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A homomorphism of degree ϵ between two persistence modules $\mathbb{U}$ and $\mathbb{V}$ is a collection Φ of linear maps

$$(\phi_a : U_a \to V_{a+\epsilon} \mid a \in \mathbf{R})$$

such that $v_{a+\epsilon}^{b+\epsilon} \circ \phi_a = \phi_b \circ u_a^b$ for all $a \leq b$.

$$\begin{tikzcd}
U^a \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & U^b \arrow[d] \\
V^{a+\epsilon} \arrow[r] & V^{b+\epsilon}
\end{tikzcd}$$

An ε-interleaving between $\mathbb{U}$ and $\mathbb{V}$ is specified by two homomorphisms of degree ϵ $\Phi : \mathbb{U} \to \mathbb{V}$ and $\Psi : \mathbb{V} \to \mathbb{U}$ s.t. $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the "shifts" of degree 2ϵ between $\mathbb{U}$ and $\mathbb{V}$.

$$\begin{tikzcd}
\cdots \arrow[rr] & & U^a \arrow[rr, "u_a^{a+2\epsilon}"] \arrow[dr, "\phi_a"] & & U^{a+2\epsilon} \arrow[rr] \arrow[dr] & & \cdots \\
& \cdots \arrow[ur] \arrow[rr] & & V^{a+\epsilon} \arrow[ur, "\psi_{a+\epsilon}"] \arrow[rr, "v_{a+\epsilon}^{a+3\epsilon}"'] & & V^{a+3\epsilon} \arrow[rr] & & \cdots
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Stability Theorem [CCGGO'09-CdSGO'12]:
If $\mathbb{U}$ and $\mathbb{V}$ are q-tame and ϵ-interleaved for some $\epsilon \geq 0$ then

$$d_B(\mathsf{dgm}(\mathbb{U}), \mathsf{dgm}(\mathbb{V})) \leq \epsilon$$

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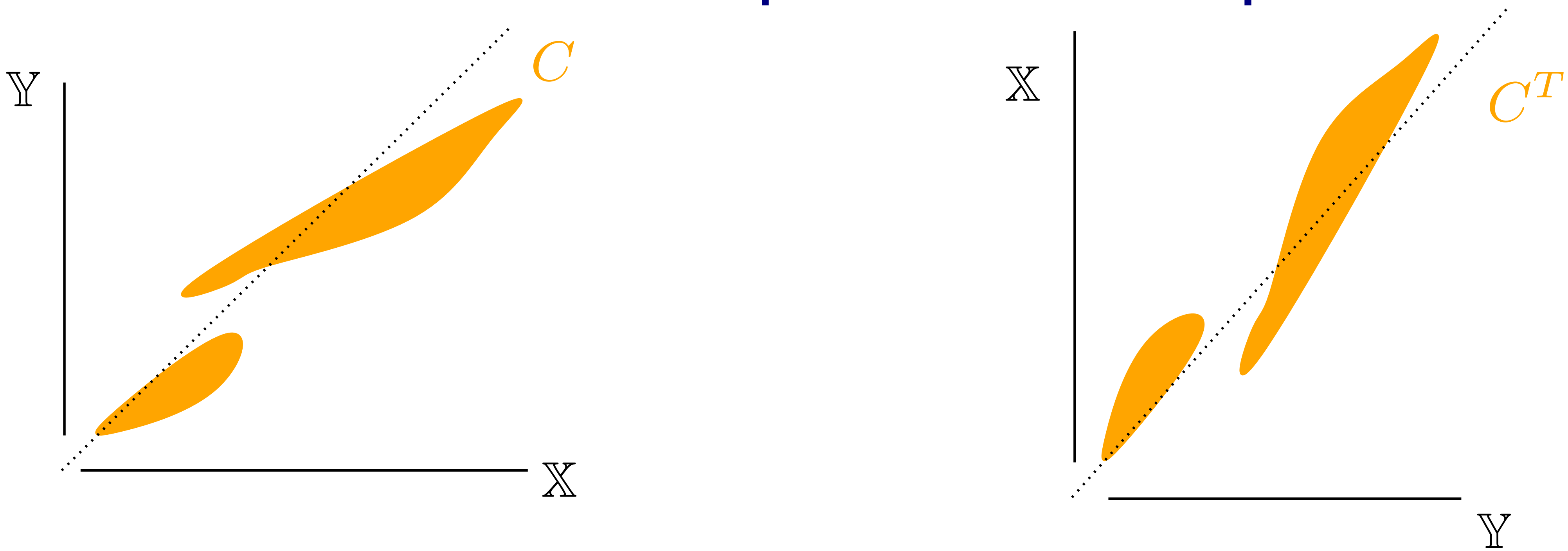
Strategy: build filtered complexes on top of metric spaces that induce **q-tame** homology persistence modules and that turn out to be **ϵ-interleaved** when the considered spaces are $O(\epsilon)$-*close*.

Need to be defined.

Multivalued maps and correspondences



A multivalued map $C : \mathbb{X} \rightrightarrows \mathbb{Y}$ from a set $\mathbb{X}$ to a set $\mathbb{Y}$ is a subset of $\mathbb{X} \times \mathbb{Y}$, also denoted C, that projects surjectively onto $\mathbb{X}$ through the canonical projection $\pi_{\mathbb{X}} : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$. The image $C(\sigma)$ of a subset σ of $\mathbb{X}$ is the canonical projection onto $\mathbb{Y}$ of the preimage of σ through $\pi_{\mathbb{X}}$.

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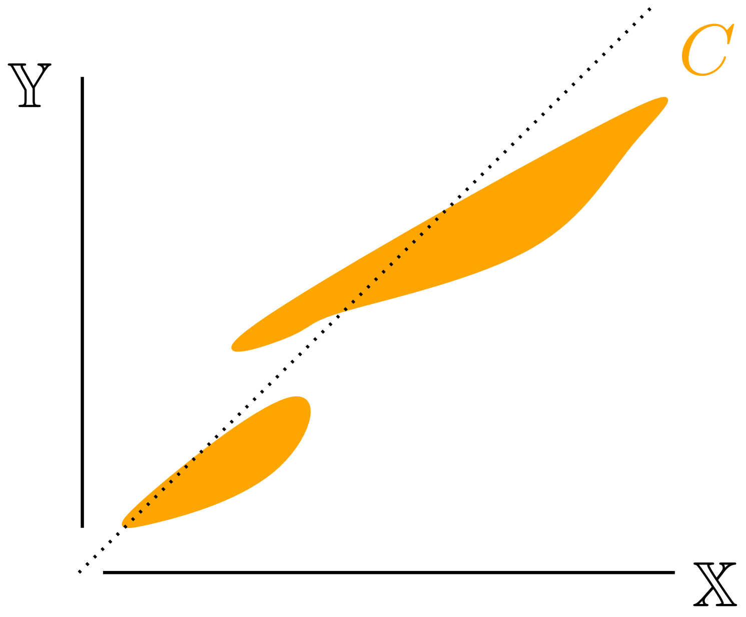

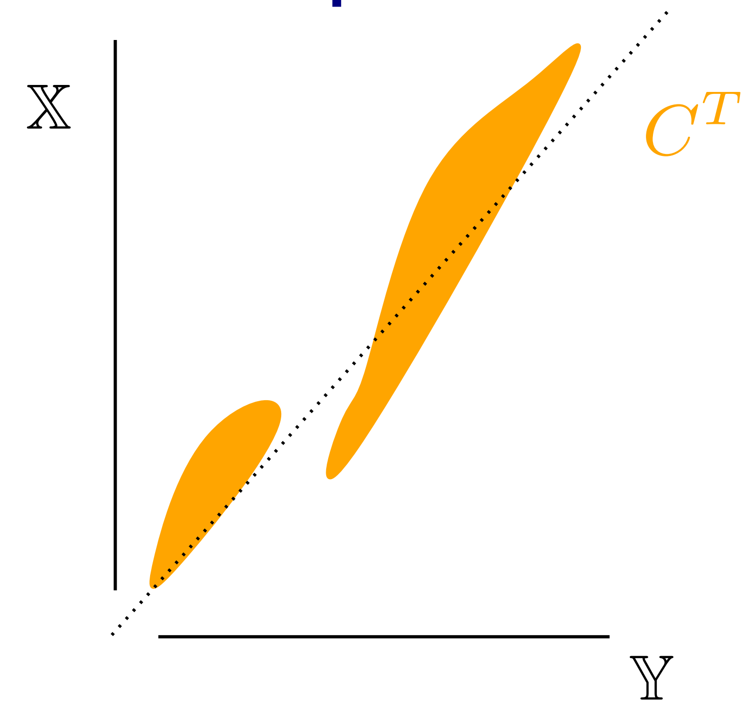

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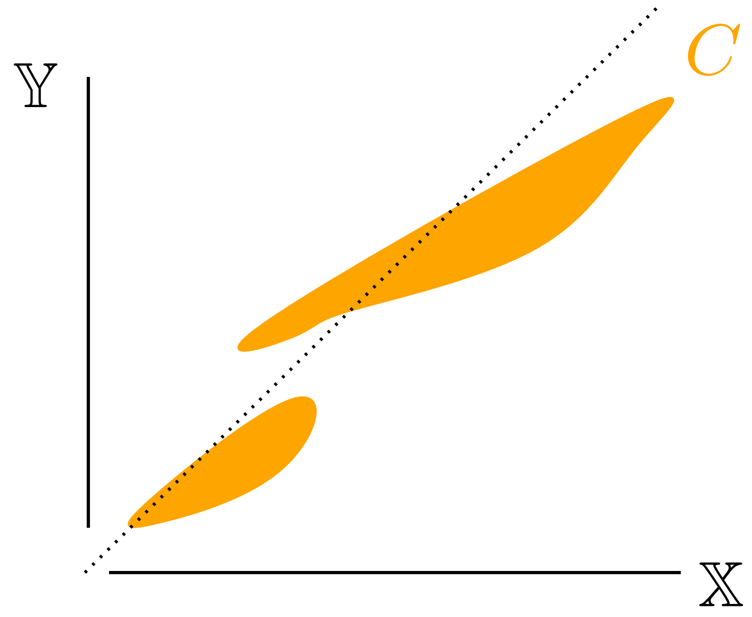
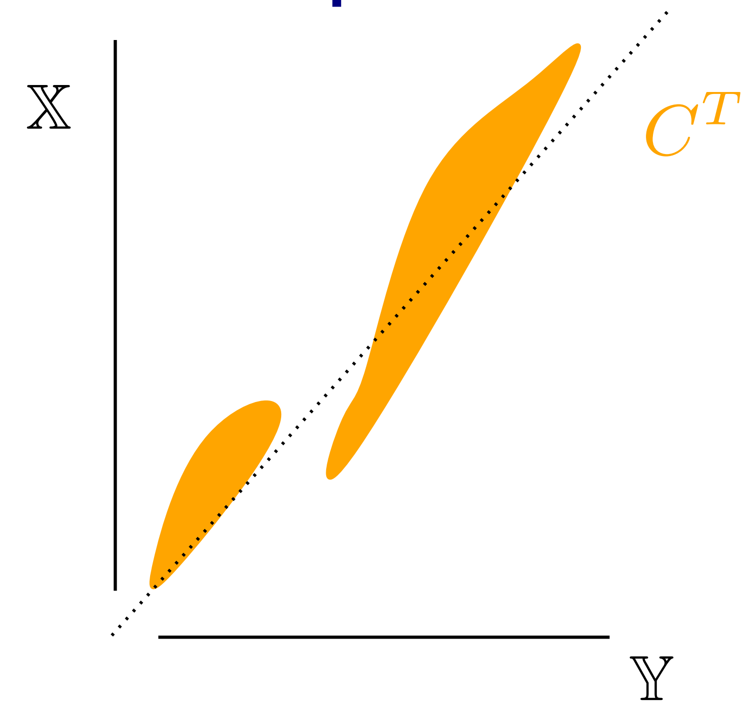
The transpose of C, denoted C^T, is the image of C through the symmetry map $(x, y) \mapsto (y, x)$.

A multivalued map $C : \mathbb{X} \rightrightarrows \mathbb{Y}$ is a correspondence if C^T is also a multivalued map.

Multivalued maps and correspondences





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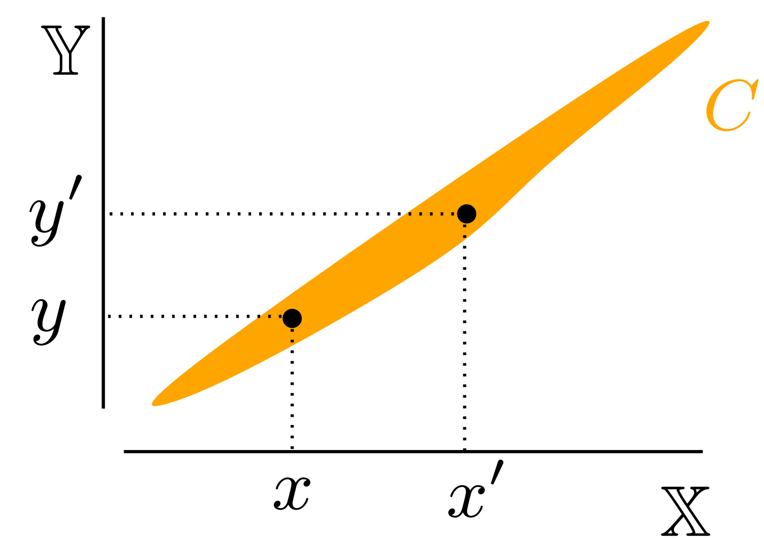
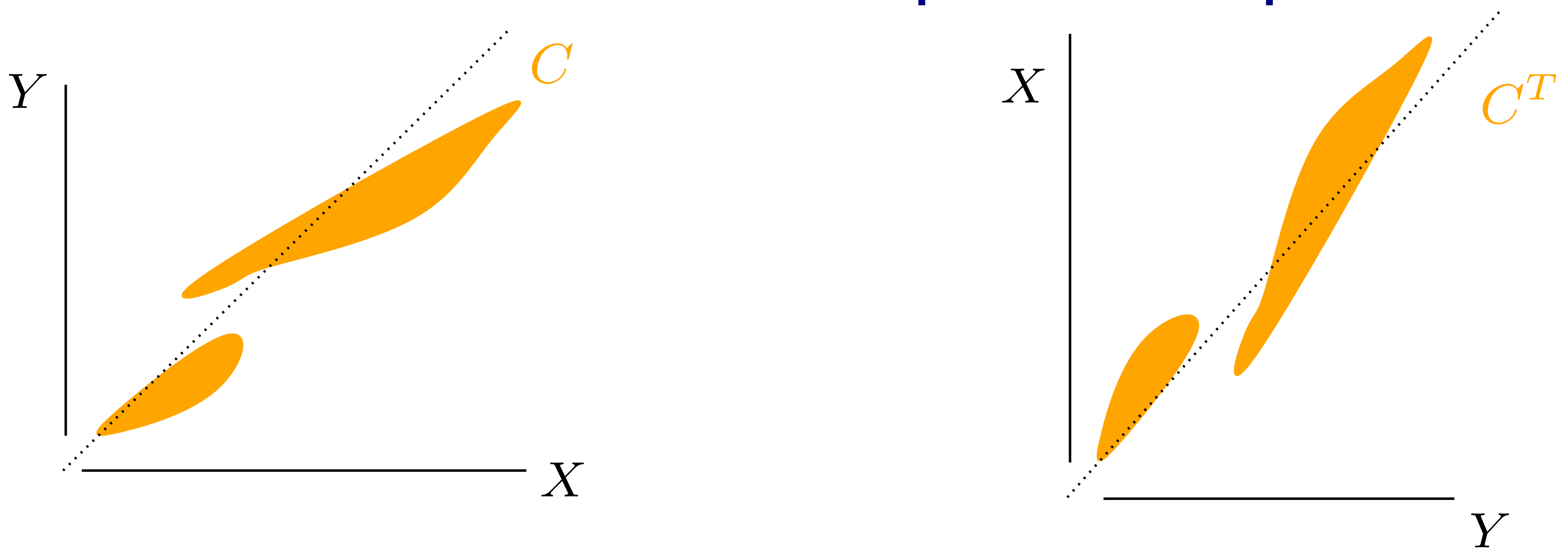
Example: ϵ-correspondence and Gromov-Hausdorff distance.

Let $(\mathbb{X}, \rho_{\mathbb{X}})$ and $(\mathbb{Y}, \rho_{\mathbb{Y}})$ be compact metric spaces.
A correspondence $C : \mathbb{X} \rightrightarrows \mathbb{Y}$ is an ϵ-correspondence if $\forall (x, y), (x', y') \in C$, $|\rho_{\mathbb{X}}(x, x') - \rho_{\mathbb{Y}}(y, y')| \leq \varepsilon$.



$$d_{GH}(\mathbb{X}, \mathbb{Y}) = \frac{1}{2} \inf\{\varepsilon \geq 0 : \text{there exists an } \varepsilon\text{-correspondence between } \mathbb{X}\text{and } \mathbb{Y}\}$$

Multivalued simplicial maps



Let $\mathbb{S}$ and $\mathbb{T}$ be two filtered simplicial complexes with vertex sets $\mathbb{X}$ and $\mathbb{Y}$ respectively. A multivalued map $C : \mathbb{X} \rightrightarrows \mathbb{Y}$ is ε-simplicial from $\mathbb{S}$ to $\mathbb{T}$ if for any $a \in \mathbf{R}$ and any simplex $\sigma \in \mathbb{S}_a$, every finite subset of $C(\sigma)$ is a simplex of $\mathbb{T}_{a+\varepsilon}$.

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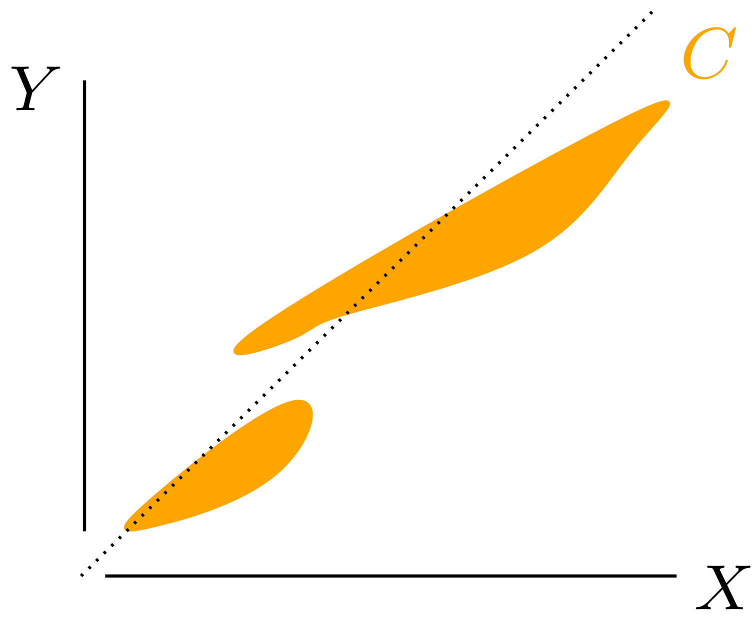

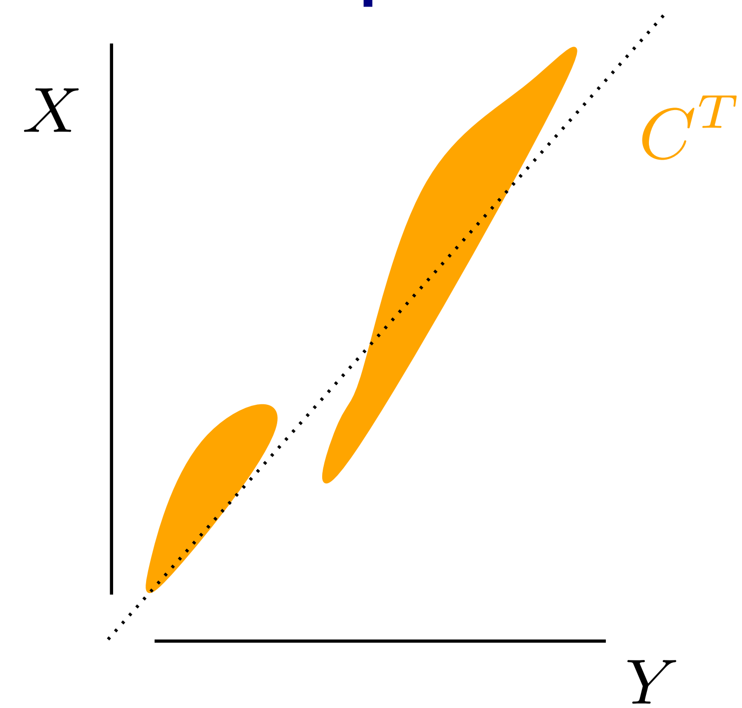

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Proposition: Let $\mathbb{S}$, $\mathbb{T}$ be filtered complexes with vertex sets $\mathbb{X}$, $\mathbb{Y}$ respectively. If $C : \mathbb{X} \rightrightarrows \mathbb{Y}$ is a correspondence such that C and C^T are both ε-simplicial, then together they induce a canonical ε-interleaving between $\mathrm{H}(\mathbb{S})$ and $\mathrm{H}(\mathbb{T})$.

The example of the Rips and Čech filtrations

Proposition: Let $(\mathbb{X}, \rho_{\mathbb{X}})$, $(\mathbb{Y}, \rho_{\mathbb{Y}})$ be metric spaces. For any $\epsilon > 2\mathrm{d}_{\mathrm{GH}}(\mathbb{X}, \mathbb{Y})$ the persistence modules $\mathrm{H}(\mathrm{Rips}(\mathbb{X}))$ and $\mathrm{H}(\mathrm{Rips}(\mathbb{Y}))$ are ϵ-interleaved.

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Proof: Let $C : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a correspondence with distortion at most ϵ.
If $\sigma \in \mathrm{Rips}(\mathbb{X}, a)$ then $\rho_{\mathbb{X}}(x, x') \leq a$ for all $x, x' \in \sigma$.
Let $\tau \subseteq C(\sigma)$ be any finite subset.
For any $y, y' \in \tau$ there exist $x, x' \in \sigma$ s. t. $y \in C(x)$, $y' \in C(x')$ so

$$\rho_{\mathbb{Y}}(y, y') \leq \rho_{\mathbb{X}}(x, x') + \epsilon \leq a + \epsilon \text{ and } \tau \in \mathrm{Rips}(\mathbb{Y}, a + \epsilon)$$

$\Rightarrow C$ is ϵ-simplicial from $\mathrm{Rips}(\mathbb{X})$ to $\mathrm{Rips}(\mathbb{Y})$.
Symetrically, C^T is ϵ-simplicial from $\mathrm{Rips}(\mathbb{Y})$ to $\mathrm{Rips}(\mathbb{X})$.

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Remark: Similar results for witness complexes (fixed landmarks)

Tameness of the Rips and Čech filtrations

Theorem: Let $\mathbb{X}$ be a compact metric space. Then $\mathrm{H}(\mathrm{Rips}(\mathbb{X}))$ and $\mathrm{H}(\check{\mathrm{C}}\mathrm{ech}(\mathbb{X}))$ are q-tame.

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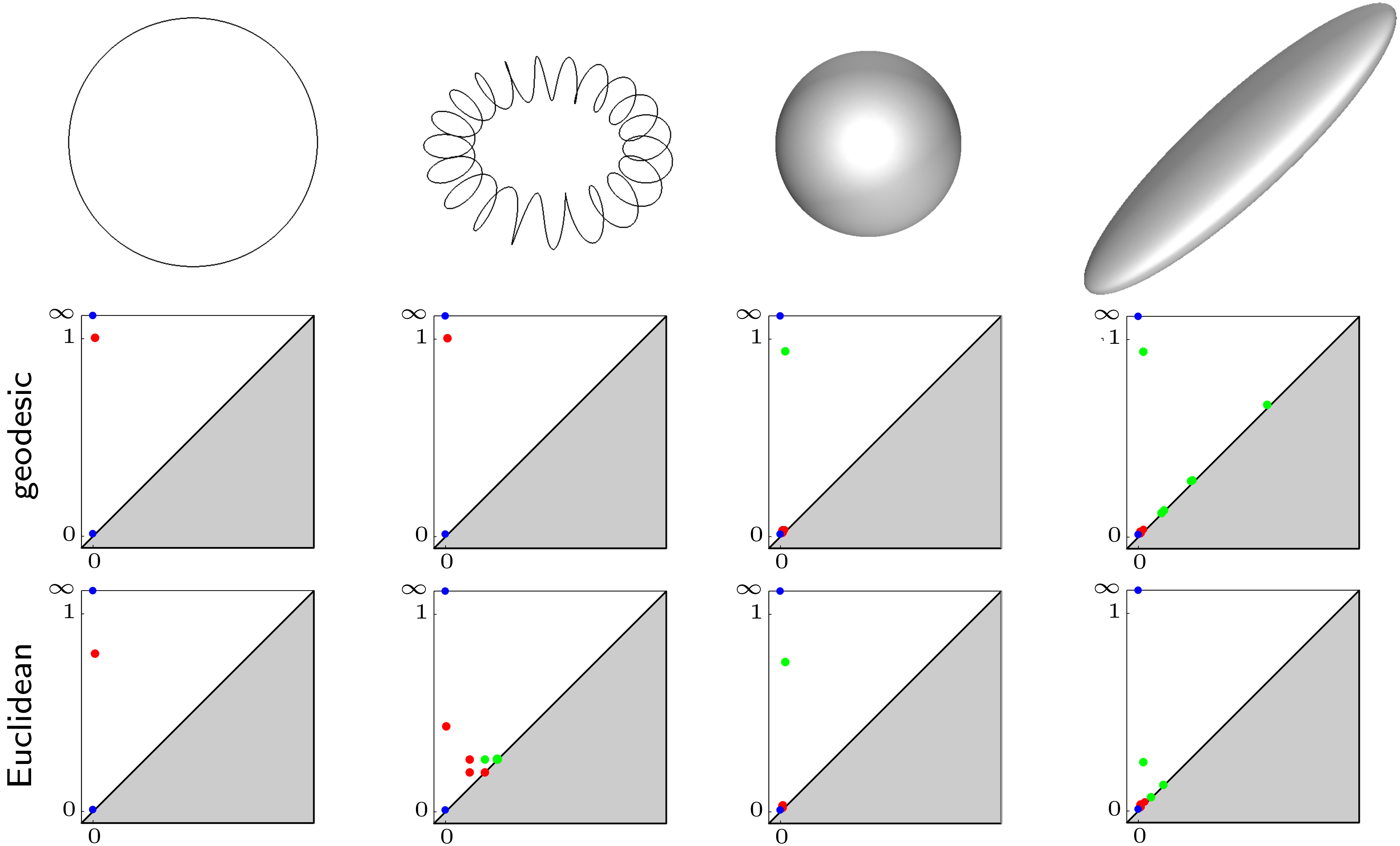
$$\mathrm{d_b}(\mathsf{dgm}(\mathrm{H}(\check{\mathrm{C}}\mathrm{ech}(\mathbb{X}))), \mathsf{dgm}(\mathrm{H}(\check{\mathrm{C}}\mathrm{ech}(\mathbb{Y})))) \leq 2\mathrm{d_{GH}}(\mathbb{X}, \mathbb{Y}),$$

$$\mathrm{d_b}(\mathsf{dgm}(\mathrm{H}(\mathrm{Rips}(\mathbb{X}))), \mathsf{dgm}(\mathrm{H}(\mathrm{Rips}(\mathbb{Y})))) \leq 2\mathrm{d_{GH}}(\mathbb{X}, \mathbb{Y}).$$

Remark: The proofs never use the triangle inequality! The previous approch and results easily extend to other settings like, e.g. spaces endowed with a similarity measure.

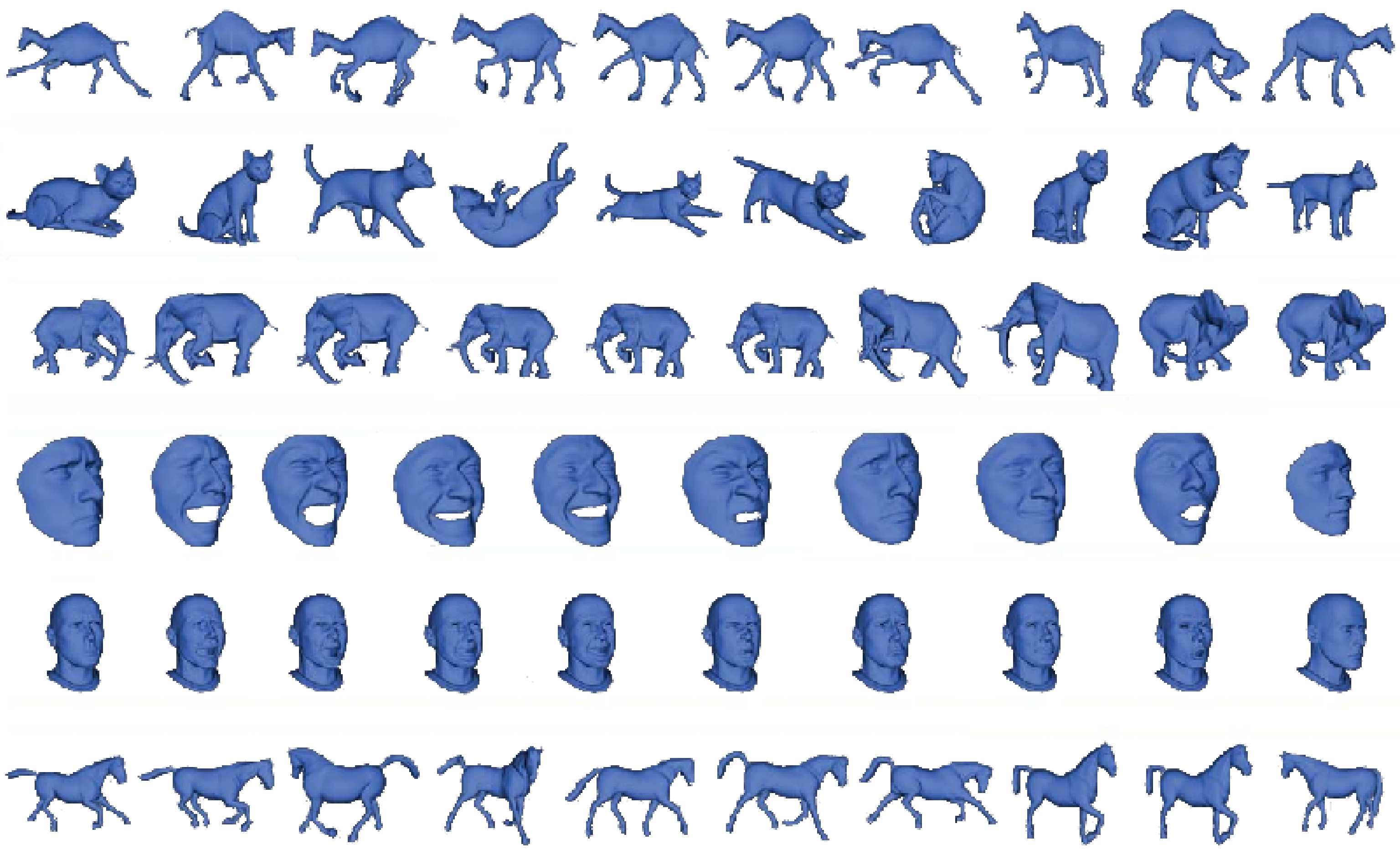
Persistence-based signatures

Signatures of some elementary shapes (approximated from finite samples):



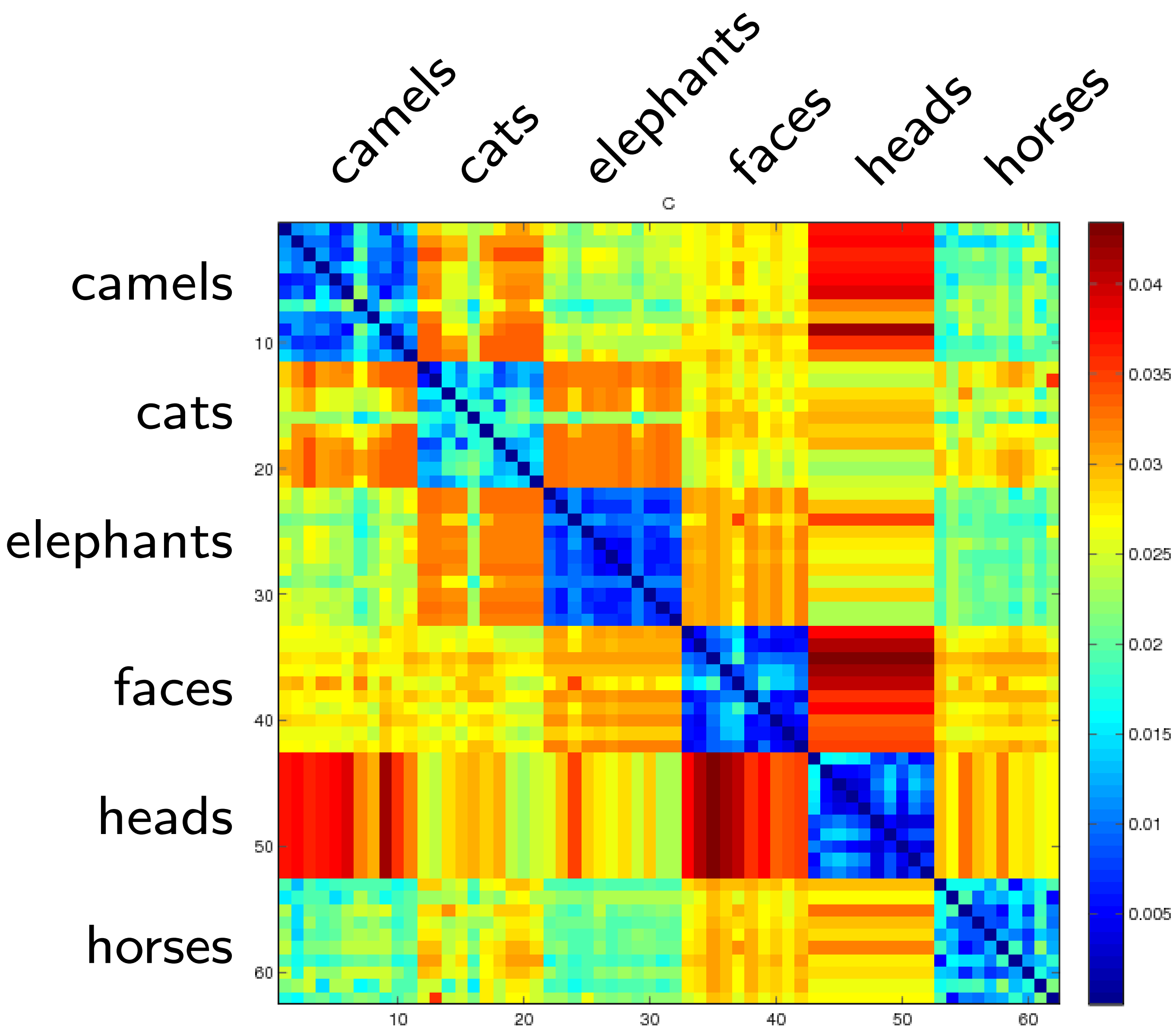
Example of application

Experimental results:



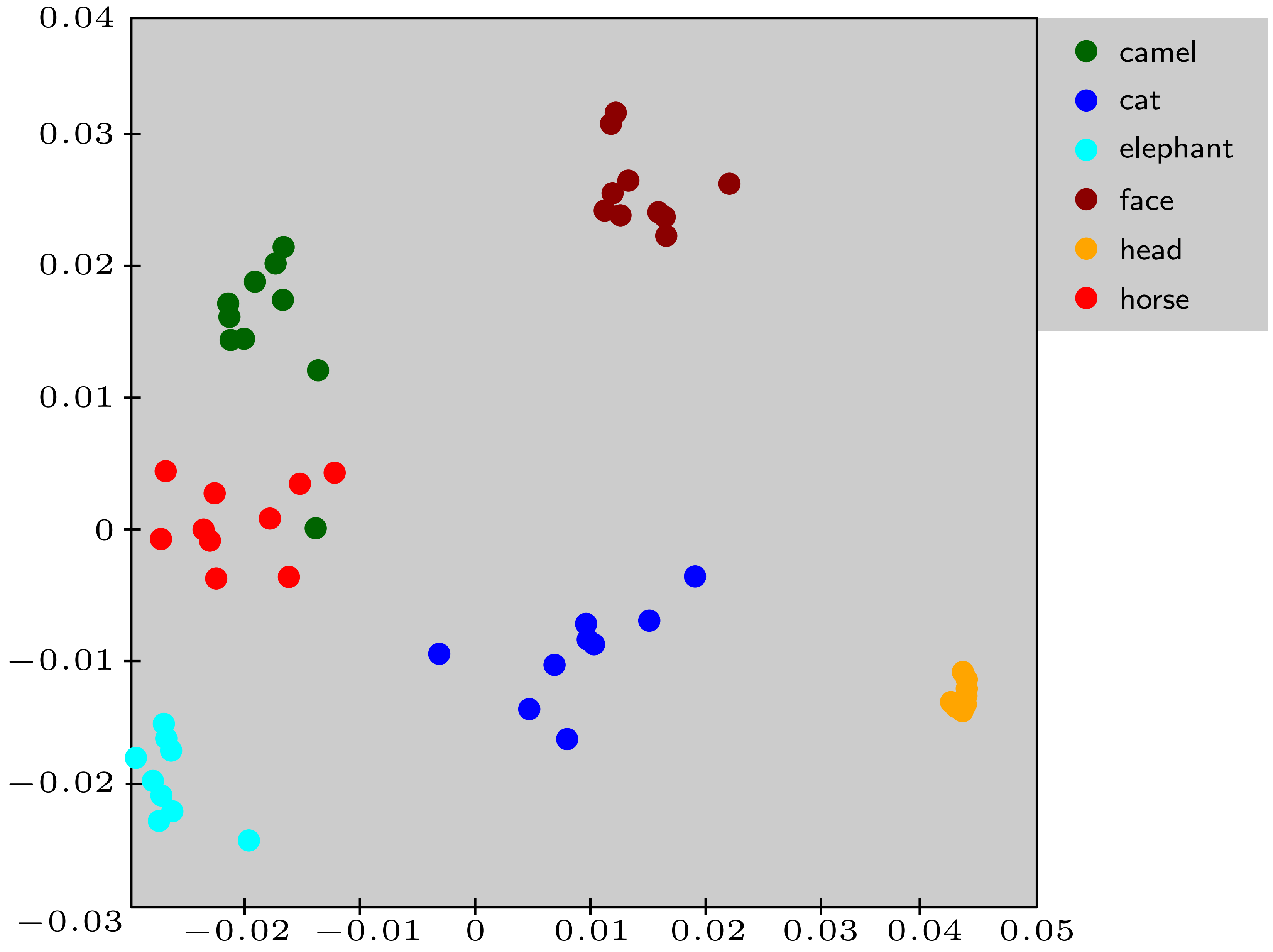
Example of application

Experimental results:



Example of application

Experimental results:



References

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- F. Chazal, D. Cohen-Steiner, L. J. Guibas, F. Mémoli, S. Oudot. Gromov-Hausdorff Stable Signatures for Shapes using Persistence. In Computer Graphics Forum, pp. 1393-1403, 2009.
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