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A condition for isotopic approximation

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Abstract

In this paper, we give a very simple and purely topological condition for two surfaces to be isotopic. This work is motivated by the problem of surface approximation. Applications to implicit surfaces are given, as well as connections with the well-known concepts of skeleton and local feature size.

Keywords: Surface approximation; Isotopy; Implicit surfaces; Medial axis; Local feature size

1. Introduction and related works

Finding approximations of given surfaces certainly is one of the core problems in the processing of 3-dimensional geometry. When seeking for an approximation S' of a surface S, in addition to geometric closeness, one usually requires that S' should be topologically equivalent to S. While much work has been done on homeomorphic approximation, in particular in the context of surface reconstruction [1], only a few recent articles tackle the more difficult problem of ensuring isotopic approximation [2,17,3]. Let us recall that two surfaces are isotopic whenever they can be continuously deformed one into the other without introducing self-intersections. Isotopy

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is thus a finer relation than homeomorphy, since for instance a knotted torus is not isotopic to an unknotted one, though both are homeomorphic. Rather than homeomorphy, isotopy is what one should look for, since it completely captures the topological aspects of surface approximation.

The main result of [17] is that S and S' are isotopic whenever the projection on S defines a homeomorphism from S' to S, the projection on S being defined as the map that associates to each point its nearest neighbour on S (when it is uniquely defined). In [2], it is shown that a specific piecewise linear approximation of S is isotopic to S, using indirectly the same condition as the one considered in [17]. Note that this condition involves not only the topology of the surfaces, but also their geometry, as the projection on S is involved. In particular, it cannot be met when S is not smoothly embedded, as the projection is then undefined in the vicinity of singular areas. Also, checking this condition usually requires to bound the angle between the normals to S and S' carefully, which is useful for other purposes, but may seem irrelevant for strictly topological purposes. In [3], some technical conditions are given to ensure isotopy between curvilinear objects in $\mathbf{R}^3$, i.e., geometric objects made up of properly joined patches defined in terms of control points.

In this work, we show that if S' and S are homeomorphic, then a simple and purely topological condition is sufficient to ensure the existence of an isotopy between them. When S is connected, the condition is merely that S' is contained in some topological thickening of S and separates the two boundary components of that thickening. We also show that if in addition S separates the boundary components of some topological thickening of S', then the homeomorphy condition can be dropped with the same conclusion.

Note that the smoothness of S is not required any more. Tedious analysis of the deviation between normals is also avoided. Finally, the condition is easy to check, and as we will see, various interesting corollaries can be obtained according to the kind of thickenings considered. The proof of our theorem is based on several results of 3-manifold topology. To begin with, we state the theorem precisely (Section 2), and give some applications (Section 3), including a quantitative version of an existential result proved in [17] about interval solids. Furthermore, an isotopy criterion involving skeleta is derived, and the case of implicitly defined surfaces is discussed. Before proving our result (Section 5), we give some mathematical preliminaries (Section 4).

2. Main results

Throughout the paper we use the following notations. For any set X, $\overline{X}$, X^c, and ∂X denote, respectively, the closure of X, the complement of X, and the boundary of X. Also, S and S' denote two compact orientable surfaces embedded in $\mathbf{R}^3$.

Definition 2.1 (*Isotopy and ambient isotopy*). An isotopy between S and S' is a continuous map $F: S \times [0,1] \to \mathbf{R}^3$ such that $F(.,0)$ is the identity of S, $F(S,1) = S'$, and for each $t \in [0,1]$, $F(.,t)$ is a homeomorphism onto its image. An ambient isotopy between S and S' is a continuous map $F: \mathbf{R}^3 \times [0,1] \to \mathbf{R}^3$ such that $F(.,0)$ is the identity of $\mathbf{R}^3$, $F(S,1) = S'$, and for each $t \in [0,1]$, $F(.,t)$ is a homeomorphism of $\mathbf{R}^3$.

Restricting an ambient isotopy between S and S' to $S \times [0,1]$ thus yields an isotopy between them. It is actually true that if there exists an isotopy between S and S', then there is an ambient isotopy between them [7], so that both notions are equivalent in our case. If $X \subset \mathbf{R}^3$, we will say that S and S' are *isotopic in X* if there exists an isotopy between S and S' whose image is included in X. Isotopies between sub-surfaces of other 3-manifolds than $\mathbf{R}^3$, which we will consider in the proof of the theorem, are defined in the same way.

Definition 2.2 (*Topological thickening*). A topological thickening of S is a set $M \subset \mathbf{R}^3$ such that there exists a homeomorphism $\Phi : S \times [0,1] \to M$ satisfying $\Phi(S \times \{1/2\}) = S \subset M$.

Our definition actually is a special case of what is usually called a thickening in the algebraic topology literature. The boundary of a topological thickening M of S thus is the union of $\Phi(\partial S \times [0,1])$ and two surfaces, $\Phi(S,0)$ and $\Phi(S,1)$, which will be referred to as the *sides* of M. Our main theorem is the following:

Theorem 2.1. *Suppose that*:

1. *S' is homeomorphic to S.*
2. *S' is included in a topological thickening M of S.*
3. *S' separates the sides of M.*

Then S' is isotopic to S in M.

Here "separates" means that any continuous path in M from one side of M to the other one intersects S'. Proving that two surfaces are homeomorphic is not straightforward in general. The next theorem shows that if the assumptions 2. and 3. of Theorem 2.1 also hold when S and S' are exchanged, then homeomorphy is not needed:

Theorem 2.2. *Suppose that*:

1. *S' is included in a topological thickening M* of *S.*
2. *S is included in a topological thickening M'* of *S'.*
3. *S' separates the sides of M.*
4. *S separates the sides of M'.*

Then S and S' are isotopic in M and in M'.

3. Applications

This section gives several applications of Theorems 2.1 and 2.2.

3.1. Isotopy between implicit surfaces

For implicitly defined surfaces, dedicated topological thickenings are provided by Morse theory (we refer to [12] for some background on Morse theory). Recall that if f is a Morse function defined on $\mathbf{R}^3$, a real number c is said to be a *critical value* of f if there exists a point $p \in \mathbf{R}^3$ such that $\nabla f(p) = 0$ and $f(p) = c$. Such a point p is called a *critical point*. Recall that f is said to be *proper* if for any compact set $K \subset \mathbf{R}$, $f^{-1}(K)$ is a compact subset of $\mathbf{R}^3$. In particular, if f is proper, any level set $f^{-1}(a)$ of f is compact.

Theorem 3.1 (*Morse*). *Let f be a proper Morse function defined on $\mathbf{R}^3$ and I a closed interval containing no critical value of f. Then for any $a \in I$, $f^{-1}(I)$ is diffeomorphic to $f^{-1}(a) \times [0,1]$.*

Let us denote by m_f the magnitude of the critical value of f of minimum magnitude: $m_f = \min\{|f(c)| : c \text{ is a critical point of } f\}$. Together with Theorem 2.2, the previous theorem gives the following:

Theorem 3.2. *Let f and g be two proper Morse functions defined on $\mathbf{R}^3$. If $sup|f - g| < min(m_f, m_g)$, then the zero-sets of f and g are isotopic.*

Proof. Set $m = min(m_f, m_g)$ and take $S = f^{-1}(0)$, $M = f^{-1}([-m, m])$, $S' = g^{-1}(0)$, and $M' = g^{-1}([-m, m])$ in Theorem 2.2. □

To approximate the level-sets of a function f by the ones of a function g in a topologically correct way, it is thus sufficient to control the supremum norm of $f - g$ and the critical values of g.

3.2. Isotopy criteria involving skeleta

Let us first recall the definitions of tubular neighbourhood and skeleton. In this section we assume that S is $\mathscr{C}^2$-smooth and closed. The skeleton Sk of S is defined as the closure of the set of points in S^c, the complement of S, which have at least two closest points on S:

$$Sk = \text{closure}\{x \in S^c : \exists y,\ z \in S,\ y \neq z,\ d(x,y) = d(x,z) = d(x,S)\}.$$

For $\varepsilon > 0$, one denotes by $S^\varepsilon = \{x \in \mathbf{R}^3 : d(x,S) \leqslant \varepsilon\}$ the tubular neighbourhood of S, which is sometimes called the ε-offset of S. If Sk is the skeleton of S, $lfs(S)$ denotes the number $lfs(S) = \inf_{x \in S} d(x, Sk)$. S being $\mathscr{C}^2$, one has $lfs(S) > 0$ (see [20] or [2]). It can be shown that if ε is smaller than $lfs(S)$ then S^ε is diffeomorphic to $S \times [-\varepsilon; +\varepsilon]$, so that tubular neighborhoods are topological thickenings. Also, $\mathbf{R}^3 \setminus Sk$ is known to be homeomorphic to $S \times \mathbf{R}$.

3.3. Topological criteria

Corollary 3.1. *Suppose that S' is homeomorphic to S, S is connected, and that S' encloses the bounded connected component of Sk. Then S' is isotopic to S.*

Proof. This result follows almost immediately from Theorem 2.1. All we need to do is to shrink $\mathbf{R}^3 \setminus Sk$ slightly in order to get a topological thickening of S. More precisely, denote by $h: S \times \mathbf{R} \to \mathbf{R}^3 \setminus Sk$ a homeomorphism. Because S' is compact, the Hausdorff distance between S' and Sk is nonzero. There exists a real $K > 0$ such that $S' \subset h(S \times [-K, +K])$. Taking $M = h(S \times [-K, +K])$ gives the desired result. Indeed, S' separates the sides of M since the components of S' enclose the inner side of M but not the outer one. □

Note that it is sufficient to check that S' is connected and has the same Euler characteristic as S to decide whether it is homeomorphic to S. In particular, if S' is a triangulated surface, which is an important case in practice, these conditions are straightforward to check.

If S' is also C^2, closed and connected, and Sk' denotes the skeleton of S', the same argument as above used with Theorem 2.2 yields:

Corollary 3.2. *If S' encloses the bounded component of Sk and S encloses the bounded component of Sk', then S and S' are isotopic.*

3.4. Metric criteria

We denote by $d(X'|X)$ the "half Hausdorff distance" from a subset $X' \subset \mathbf{R}^3$ to another subset $X \subset \mathbf{R}^3$, that is:

$$d(X'|X) = \sup_{x \in X} \inf_{x' \in X'} d(x, x').$$

Note that $d(X'|X)$ is the minimum value of ε such that $X \subset X'\varepsilon$. Also, $d(X, X') = \max(d(X|X'), d(X'|X))$ denotes the Hausdorff distance between X and X'. By using offsets as topological thickenings, one obtains the following results:

Corollary 3.3. *If S' is homeomorphic to S and $d(S|S') < \min(lfs(S), lfs(S'))$, then S' is isotopic to S. Moreover, the isotopy F can be chosen in such a way that the half Hausdorff distance from S to $F(S', t)$ never exceeds the initial half Hausdorff distance.*

Proof. We apply Theorem 2.1 with $M = S^{\varepsilon}$, where $\varepsilon = \min(lfs(S), lfs(S'))$. The only condition that is not trivially satisfied is that S' separates the sides of M. We now prove it by contradiction, in the connected case. Let S_1 and S_2 be the sides of M. First remark that for any $x \in S_1$ there exists a unique point, $f(x) \in S_2$ such that the segment $[x, f(x)]$ is included in M and meets S perpendicularly (see Fig. 1). Suppose that S' does not separate S_1 and S_2. Then for any $x \in S_1$ if the segment $[x, f(x)]$ intersects S', then it intersects in at least two points (if it is not the case, one can construct a path from x to $f(x)$ which does not intersect S' and the union of this path with the segment $[x, f(x)]$ is a closed path which meets S' in only one point: a contradiction since S' has no boundary).

Now for any point $y \in S'$ there exists a unique point $\varphi(y) \in S_1$ such that $y \in [\varphi(y), f(\varphi(y))]$. Let $y \in S'$ be such that the distance between y and $\varphi(y)$ is the largest among all the points in S'. Thus the segment $[\varphi(y), f(\varphi(y))]$ is also normal to S' at point y. Let now $y' \neq y$ be another intersection point between $[\varphi(y), f(\varphi(y))]$ and

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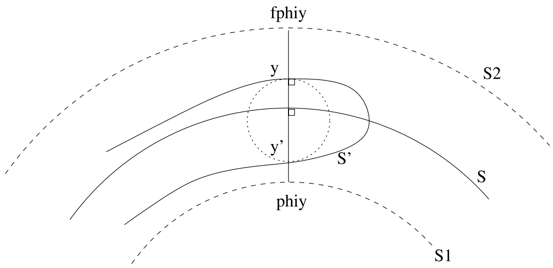

Fig. 1. Proof of Corollary 3.3.

S'. The ball with diameter $[y, y']$ is tangent to S' at y and meets S' in at least two points: the segment joining its center and y has to contain a point of Sk'. But such a point is at distance less than ε from S', which is a contradiction. □

Assuming S' is closed, the argument used in the preceding proof applied the other way around leads to:

Theorem 3.3. *If* $d(S,S') < \min(lfs(S), lfs(S'))$, *then* S' *is isotopic to* S. *Moreover, the isotopy* F *can be chosen in such a way that the Hausdorff distance between* $F(S',t)$ *and* S *never exceeds the initial Hausdorff distance.*

3.5. Interval solid models

Another consequence of Theorem 2.1 is related to the notion of Interval Solid Models studied in [18,17]. Roughly speaking, an interval solid $S^{\mathcal{B}}$ associated to a smooth $\mathcal{C}^2$ surface S embedded in $\mathbf{R}^3$ is a finite covering of S by rectangular boxes whose edges are parallel to the co-ordinate axes which satisfy some additional conditions (see [18] for precise definition). It is proven in [18] that the two boundary components S_1 and S_2 of this covering are homeomorphic to S. Moreover, [17] recalls the notion of ε-isotopy which is stronger than the notion of isotopy: points cannot move outside of an ε-neighbourhood of their initial position during the isotopy. T. Sakkalis and T.J. Peters prove in [17], Section 5, that if the boxes are small enough then S_1 and S_2 are ε-isotopic to S. Note that this result is existential, that is it does not provide any particular bound on the maximum box size allowed to guarantee that isotopy holds. In our setting, one can slightly generalize their result.

Corollary 3.4. *If* $S^{\mathcal{B}}$ *does not intersect the skeleton of* S, *then its two boundary components are isotopic to* S.

So one can relax the hypothesis about the size of the boxes in [17]: here, the diameter of the boxes should merely be smaller than $lfs(S)$. The major drawback is that one does not obtain that S_1 and S_2 are ε-isotopic to S any more. Indeed, one has that the boundary components of $S^{\mathcal{B}}$ can be isotoped to S within $S^{\mathcal{B}}$, so that the Hausdorff distance is controlled, but each particular point may move arbitrarily far from its initial position during the isotopy.

4. Mathematical preliminaries

4.1. Surface topology: Euler characteristic and coverings

This section is dedicated to some basic recall about topology of compact orientable surfaces which are widely used in the following. Let S be a compact orientable surface with possibly non empty boundary ∂S. Denote by b the number of connected components of ∂S. If $\mathcal{T}$ is a triangulation of S, denote by f the number of its faces, by e the number of its edges and by s the number of its vertices. The *Euler characteristic* $\chi(S)$ of S is defined as

$$\chi(S) = f - e + s.$$

It is well known that such a number does not depend on the choice of the triangulation $\mathcal{T}$ (see [11] for example). It is also well known that S always admits a triangulation (see [15] or [13]). So Euler characteristic is well defined for compact surfaces and two homeomorphic surfaces have the same Euler characteristic. The *genus*, $g(S)$ of S is defined as

$$g(S) = \frac{1}{2}(2 - \chi(S) - b).$$

The genus and the number of boundary components (or equivalently the Euler characteristic and the number of boundary components) are sufficient to classify compact connected orientable surfaces.

Theorem 4.1 (*see* [11] *for a proof*). *Two connected compact orientable surfaces are homeomorphic if and only if they have the same genus and the same number of boundary components.*

In the following of this paper, we will also use the notion of topological covering between surfaces (see [11] for example). A map $p: S' \to S$ is a *topological covering* of S if there exists a non empty discrete set F (finite or infinite denumerable) satisfying the following property: for any point $x \in S$, there exists a neighbourhood V of x and an homeomorpism Φ between $p^{-1}(V)$ and $V \times F$ such that $p_1 \circ \Phi = p$ where $p_1: V \times F \to V$ is the canonical projection. If F is finite, the cardinality of F is known as the number of sheets of the covering. In other words, a topological covering is a map $p: S' \to S$ such that every $x \in S$ has an open neighborhood V such that $p^{-1}(V)$ is a disjoint union of (countably many) open sets, each of which is mapped homeomorphically onto V by p. The simplest examples of topological coverings are canonical projections $p_1: V \times F \to V$; such coverings are called *trivial*. Let us now give a more interesting example: consider the map from the torus $S = S^1 \times S^1$ to itself defined by $p(\theta, \varphi) = (2\theta, \varphi)$. It is an easy exercise to prove that p is a 2-sheeted covering of torus S by itself. Important facts are, that a 1-sheeted covering between two compact surfaces is an homeomorphism and that if $p: S' \to S$ is a n-sheeted covering of S, then $\chi(S') = n\chi(S)$.

Finally, in the proofs of our main theorems, we will use an argument resorting to singular homology theory. This theory is beyond the scope of this paper and we refer the reader to [5] for an introduction to the subject.

4.2. 3-Manifold topology

The proof of Theorem 2.1 is based upon the following theorem ([9,19], see [6, p. 16 for a proof]), which we explain below.

Theorem 4.2. *Let $\tilde{M}$ be a connected compact irreducible Seifert-fibered manifold. Then any essential surface $\mathcal{S}'$ in $\tilde{M}$ is isotopic to a surface which is either vertical, i.e., a union of regular fibers, or horizontal, i.e., transverse to all fibers.*

Let us explain the various terms involved in this theorem. A 3-manifold $\tilde{M}$ is said to be *irreducible* if any 2-sphere embedded in $\tilde{M}$ bounds a 3-ball in $\tilde{M}$. A Seifert manifold is a 3-manifold that decomposes into a union of topological circles, the *fibers*, satisfying certain properties. The simplest example of Seifert manifold is the cartesian product of a surface $\mathcal{S}$ and a circle S^1, the fibers being the circles $\{x\} \times S^1$, $x \in \mathcal{S}$. In what follows, we shall only deal with Seifert manifolds of that kind. We will not explain what a *regular* fiber is because in our case all the fibers are regular. An oriented surface embedded in a 3-manifold $\tilde{M}$ is *incompressible* if none of its components is homeomorphic to a 2-dimensional sphere and if for any (topological) disk $D \subset \tilde{M}$ whose boundary is included in $\mathcal{S}$, there is a disk $D' \subset \mathcal{S}$ such that $\partial D = \partial D'$. Any disk D for which there is no such D' is called a *compressing disk* for $\mathcal{S}$ (see Fig. 2). Intuitively, $\mathcal{S}$ is incompressible when it has no extra handle with respect to $\tilde{M}$. An *essential surface* in a 3-manifold $\tilde{M}$ is an incompressible surface, satisfying certain additional conditions related to $\partial\tilde{M}$. In particular, when $\tilde{M}$ has no boundary, any incompressible surface is essential. We will actually see that all the incompressible surfaces considered in this paper are essential, even in the case with boundary. Finally, two sub-manifolds of $\tilde{M}$ are said to be *transverse* if in any point x where they intersect, the (vectorial) sum of their tangent space spans the tangent space of $\tilde{M}$ at x. The intersection of two transverse sub-manifolds $\mathcal{S}_1$ and $\mathcal{S}_2$ is again a submanifold, with codimension the sum of the codimensions of $\mathcal{S}_1$ and $\mathcal{S}_2$ (see [7]). In particular, a surface of a Seifert 3-manifold transverse to a fiber meets that fiber in a discrete set of points. Also, two surfaces in a 3-manifold are transverse if and only if they are not tangent at any point.

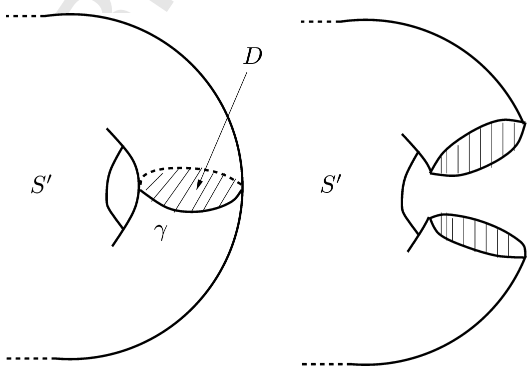

Fig. 2. Surgery along a compressing disk.

5. Proofs

In Sections 5.1 and 5.2, we prove Theorem 2.1 in the case where S is connected. Section 5.3 completes the proof of Theorems 2.1 and 2.2 in the case where S has several connected components. Let M be a topological thickening of S, and suppose that S, S', and M fulfill the assumptions of Theorem 2.1. From now on, we identify M with $S \times [0,1]$, using the map Φ associated with M (see Definition 2.2). Let $\tilde{M}$ be the Seifert 3-manifold $S \times S^1$ obtained from M by identification of its sides $S \times \{0\}$ and $S \times \{1\}$. We denote by $\mathscr{S}$ the surface corresponding to the sides of M in $\tilde{M}$, and by $\mathscr{S}'$ the surface corresponding to S' in $\tilde{M}$. Note that in $\tilde{M}$, S corresponds to the surface $S \times \{1/2\}$. As $S \times \{1/2\}$ and $\mathscr{S} = S \times \{0\} = S \times \{1\}$ are obviously isotopic in $\tilde{M}$, it will be sufficient to prove that $\mathscr{S}'$ is isotopic to $\mathscr{S}$ in $\tilde{M}$ to prove our result.

By the assumptions of Theorem 2.1, $\mathscr{S}$ and $\mathscr{S}'$ are homeomorphic and disjoint. Also:

Lemma 5.1. *$\tilde{M} \setminus \mathscr{S}'$ is connected.*

Proof. By assumption, the two sides of M lie in two different components of $M \setminus S'$, say C_1 and C_2. To prove that $\tilde{M} \setminus \mathscr{S}'$ is connected, it is sufficient to prove that $M \setminus S'$ has no other component than C_1 and C_2, since these two components are merged upon identification of the two sides of M. The boundary of say C_1 intersects S' along a closed non empty subset of S'. This subset is also an open subset of S' for the induced topology. Since S' is connected, we get that S' is included in the boundary of C_1. The same is true for C_2. Now suppose that $M \setminus S'$ has another component C_3. By a similar argument, the boundary of C_3 would contain S', so that a point $x \in S'$ would lie in the closure of C_1, C_2, and C_3. But this is not possible since x has arbitrarily small neighborhoods that S' separates in only two components. □

Note that since we do not assume that S is closed (a closed surface is a surface without boundary component), $\mathscr{S}$, and thus $\mathscr{S}'$ and $\tilde{M}$ may have non-empty boundaries. Although it is possible to prove directly the proposition in the general case, one first gives the proof in the case where S is closed to avoid some technical difficulties. The additional technicalities occuring in the case with boundary are detailed in Section 5.2. Any compact topological surface which admits a thickening is isotopic to a $\mathscr{C}^\infty$ smooth surface. So from now on, we suppose (without loss of generality) that $\mathscr{S}$ and $\mathscr{S}'$ are $\mathscr{C}^\infty$ smooth surfaces.

5.1. The case of a surface without boundary

Note that the case where $\mathscr{S} = S^2$ is a 2-dimensional sphere, $\tilde{M} = S^2 \times S^1$ is not irreducible ([6] prop 1.12, p.18), so it has to be considered separately. Fortunately, isotopy holds when $\mathscr{S} = S^2$ is a sphere, since it follows from Schoenflies theorem (see [16] P.34 for a statement of it and [4] for a proof) that there is no smooth knotted 2-sphere in $\mathbf{R}^3$. From now on, we assume that $\mathscr{S}$ is not a sphere.

We first prove that $\tilde{M}$ and $\mathcal{S}'$ fulfill the hypothesis of Theorem 4.2 and then deduce that $\mathcal{S}'$ is isotopic to $\mathcal{S}$. Since $\mathcal{S}$ is not a sphere, M is an irreducible manifold ([6] prop 1.12, p.18). Hence, we just have to prove the following

Proposition 5.1. *$\mathcal{S}'$ is an essential surface in $\tilde{M}$.*

Proof. Since $\tilde{M}$ has no boundary, it is sufficient to prove that $\mathcal{S}'$ is incompressible. Suppose $\mathcal{S}'$ is compressible. So one can find a simple curve γ on $\mathcal{S}'$ which is not null homotopic in $\mathcal{S}'$ and which bounds an embedded disc D in $\tilde{M}$. Do the following surgery: cut $\mathcal{S}'$ along γ and glue a disk homotopic to D along each of the two boundary components of $\mathcal{S}' \setminus \gamma$ (see Fig. 2). In this way, one obtains a new surface with Euler characteristic greater than $\chi(\mathcal{S}') = \chi(\mathcal{S})$. The previous surgery does not change the homology class: the new surface is homologous to $\mathcal{S}'$. The surface $\mathcal{S}'$ (with well-chosen orientation) is homologous to $\mathcal{S}$ ($\mathcal{S}$ and $\mathcal{S}'$ form the boundary of an open subset in $\tilde{M}$), and it follows from Künneth formula (see [5], p.198 for example) that the homology class of $\mathcal{S}$ in $\tilde{M}$ is not zero. So one of the connected components $\tilde{\mathcal{S}}'$ of the new surface in $\tilde{M}$ is not homologous to zero. Moreover, $\tilde{\mathcal{S}}'$ has a smaller genus than the one of $\mathcal{S}$. Indeed, suppose it is not the case. As the new surface has a larger Euler characteristic than $\chi(\mathcal{S}')$ and has at most 2 connected components, the only possibility is that this surface is the disjoint union of $\tilde{\mathcal{S}}'$ and a sphere. Indeed, the sphere is the only closed orientable connected surface with positive Euler characteristic. Considering the complement of the compressing disk in the sphere component shows that ∂D bounds a disk in $\mathcal{S}'$, which is a contradiction. □

Lemma 5.2. *It is possible to choose D such that $D \cap \mathcal{S} = \emptyset$.*

Proof. Consider the embedded disks having γ as boundary and which meet $\mathcal{S}$ transversally. Each of these disks meets $\mathcal{S}$ in a union of n closed loops. Take as D the disk such that this number n is minimum. Suppose that n is not zero. Among all these curves there is one, denoted by α, which bounds a disk in $D \setminus (\mathcal{S} \cap D)$ (when the curves are nested, consider any innermost curve on D, see Fig. 3 on the right). The surface $\mathcal{S}$ is incompressible: indeed, the injection of $\mathcal{S}$ in $\tilde{M}$ induces an injection between corresponding fundamental groups (see [6, p. 10]). So α also bounds a disk in $\mathcal{S}$. The 3-manifold $\tilde{M}$ being irreducible, the sphere defined by these 2 disks bounds a 3-ball. One can then make an isotopy to obtain a disk D' such that $D' \cap \mathcal{S} = (D \cap \mathcal{S}) \setminus \alpha$. This contradicts the minimality of n (see Fig. 3). □

The previous surgery cannot be iterated an infinite number of times, since the genus of $\tilde{\mathcal{S}}'$ decreases each time. Upon termination, one obtains a surface, called $\tilde{\mathcal{S}}'$ again, which is incompressible or the sphere S^2, and which does not intersect the surface $\mathcal{S}$ because we chose compressing disks that do not meet $\mathcal{S}$. If $\tilde{\mathcal{S}}'$ is a 2-sphere, it does not bound a 3-ball because its homology class in $H_2(\tilde{M})$ is not zero. This implies that $\tilde{M}$ is not irreducible: a contradiction. So $\tilde{\mathcal{S}}'$ is an incompressible surface. Applying Theorem 4.2, one deduces that $\tilde{\mathcal{S}}'$ is isotopic to either a horizontal or a vertical surface.

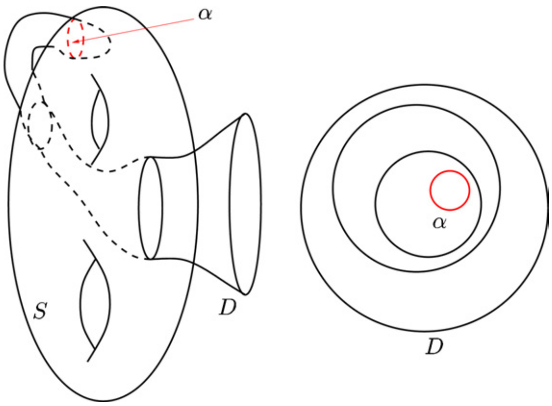

Fig. 3. Decreasing the number of components of $D \cap \mathscr{S}$.

Claim 5.1. $\tilde{\mathscr{S}}'$ is not isotopic to a vertical surface.

Proof. Suppose it is. Then there exists a surface $\tilde{\mathscr{S}}''$ which is an union of fibers of $\tilde{M}$ and which is isotopic to $\tilde{\mathscr{S}}'$. Choose one fiber φ included in $\tilde{\mathscr{S}}''$. Its intersection number with $\mathscr{S}$ is equal to 1 and has to remain constant during the isotopy. So $\tilde{\mathscr{S}}'$ contains a simple closed curve whose intersection number with $\mathscr{S}$ is equal to 1, namely the image of φ under the isotopy. But $\tilde{\mathscr{S}}'$ does not intersect $\mathscr{S}$: contradiction. Hence $\tilde{\mathscr{S}}'$ is isotopic to a horizontal surface, which is a covering of $\mathscr{S}$ under the canonical projection of $\tilde{M}$. But this is not possible since $genus(\tilde{\mathscr{S}}') < \text{genus}(\mathscr{S})$. So, $\mathscr{S}'$ is incompressible, which concludes the proof of Proposition 5.1. □

Now, it follows from Theorem 4.2 that $\mathscr{S}'$ is isotopic to either a horizontal or a vertical surface. $\mathscr{S}'$ does not intersect $\mathscr{S}$, so it cannot be isotopic to a vertical surface, by the same argument as above. So $\mathscr{S}'$ is isotopic to a horizontal surface. This surface is a covering of $\mathscr{S}$ under the canonical projection of $\tilde{M}$. Because $\tilde{M} \setminus \mathscr{S}'$ is connected, it follows from [6, pp. 17–18] that the covering is trivial. Hence, $\mathscr{S}'$ is isotopic to a horizontal surface $\mathscr{S}''$ which meets each fiber in one point. It is now a classical fact that this horizontal surface can be "pushed along the fibers" to construct an isotopy to $\mathscr{S}$ (see Fig. 4). Note that, using the same argument as the one used previously to prove that one can construct $\tilde{\mathscr{S}}'$ such that it does not intersect $\mathscr{S}$, the isotopy $F_t, t \in [0,1]$ between $\mathscr{S}''$ and $\mathscr{S}$ can be chosen so that $F_t(\mathscr{S}''), t \in]0,1]$ never intersects $\mathscr{S}$. So $\mathscr{S}'$ is isotopic to $\mathscr{S}$ in M.

5.2. The case of surfaces with boundary

The proof of Theorem 2.1 for a surface S with nonempty boundary is almost the same as the previous one. The few changes are outlined in this section. As in the case where $\mathscr{S} = S^2$, there is no smooth knotted disk in $\mathbf{R}^3$ and Theorem 2.1 holds if $\mathscr{S}$ is a disk. So consider the case where $\mathscr{S}$ is not a topological disk. Let us begin with a few remarks. First, note that if $\partial \mathscr{S} \neq \emptyset$, then $\tilde{M}$ is irreducible (see [6, p. 18] or [9, p. 13]).

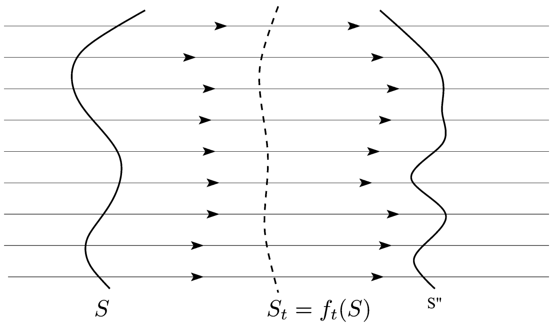

Fig. 4. Pushing $\mathscr{S}''$ to $\mathscr{S}$ along the fibers of $\tilde{M}$.

Second, since the boundary components of $\mathscr{S}$ are simple closed curves, the boundary of $\tilde{M}$ is a finite union of tori $T_1, \ldots, T_k$. Moreover, the boundary components of $\mathscr{S}$ are meridians of $T_1, \ldots, T_k$, respectively. Let $\gamma_1 \in T_1, \ldots, \gamma_k \in T_k$ be these meridians.

Each torus T_i contains exactly one boundary component γ_i' of $\mathscr{S}'$ and $\gamma_i \cap \gamma_i' = \emptyset$. Since $\tilde{M} \setminus \mathscr{S}'$ is connected, γ_i' is not null homotopic in T_i. So, γ_i' is also a meridian of T_i and it is then isotopic to γ_i (see Fig. 5).

So, since $\mathscr{S}$ is not a topological disk, the boundary components of $\mathscr{S}'$ are not null-homotopic in $\tilde{M}$. Now, Proposition 5.1 remains valid.

Proposition 5.2. *$\mathscr{S}'$ is an essential surface in $\tilde{M}$.*

Proof. The framework of the proof is the same as in Proposition 5.1. Each boundary component of $\tilde{M}$ being a torus, it follows from Lemma 1.10 p. 15 in [6] that if $\mathscr{S}'$ is incompressible, then $\mathscr{S}'$ is essential. So it is sufficient to prove that $\mathscr{S}'$ is incompressible.

To deal with the boundary of $\tilde{M}$, one has to consider the relative homology of $\tilde{M}$ mod $\partial\tilde{M}$ instead of the homology of $\tilde{M}$.

Suppose that $\mathscr{S}'$ is compressible. One can do the same surgery along a compressing disk D as in the proof of Proposition 5.1. Such a surgery does not change the homology class relative to $\partial\tilde{M}$: the surface obtained after the surgery is homologous (mod $\partial\tilde{M}$) to $\mathscr{S}'$ which is itself homologous to $\mathscr{S}$ (mod $\partial\tilde{M}$). Thus, one

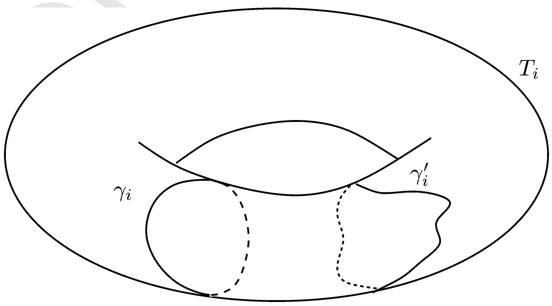

Fig. 5. Torus on the boundary of $\tilde{M}$.

of the connected component $\tilde{\mathscr{S}}'$ of the new surface is non homologous to 0. Unlike in the case without boundary, the surgery on $\mathscr{S}'$ may have two different consequences on the topology of $\tilde{\mathscr{S}}'$. The genus of $\tilde{\mathscr{S}}'$ either decreases or its number of boundary components decreases (see Fig. 6). So one has to consider the genus plus the number of boundary components of $\tilde{\mathscr{S}}'$ as the decreasing quantity during the surgery. As in above section, the compressing disk D may be chosen so that it does not intersect $\mathscr{S}$.

By iteration one obtains a surface, denoted $\tilde{\mathscr{S}}'$ again, which is incompressible or the sphere S^2 or a disc with boundary on the boundary of $\tilde{M}$. As in previous section, because $\tilde{M}$ is irreducible, $\tilde{\mathscr{S}}'$ cannot be a sphere. The boundary components of $\tilde{\mathscr{S}}'$ are boundary components of $\mathscr{S}'$ so they are not null-homotopic in $\tilde{M}$. It follows that $\tilde{\mathscr{S}}'$ cannot be a disk and then it is incompressible and hence it is isotopic to either a vertical or an horizontal surface. As in previous section, this surface cannot be vertical so it is horizontal. It follows that $\tilde{\mathscr{S}}'$ is a topological covering of $\mathscr{S}$: its genus and its number of boundary components must be at least as large as the one of $\mathscr{S}$. This is not the case. So, $\mathscr{S}'$ is incompressible and it is then isotopic to an horizontal or vertical surface. The proof of proposition then concludes in the same way as in the case of a surface without boundary.

The proof of Theorem 2.1 now ends as in previous section. □

5.3. Case of several connected components

Once we showed Theorem 2.1 in the connected case, the general case follows easily by repeated application of the pigeonhole principle. Indeed, since S and S' are homeomorphic, they have the same number of connected components. Moreover, as S' is included in M and separates its sides, each component C of M contains at least one component of S'. As a consequence, $C \cap S'$ is a connected surface. Similarly, S and S' have the same number of boundary components. Also, for each boundary component B of S, $B \times [0,1]$ has to contain at least one boundary component of S', otherwise S' would not separate the sides of M. Thus, $B \times [0,1]$ contains exactly one boundary component of S', that is $C \cap S'$ and $C \cap S$ have the same number of boundary

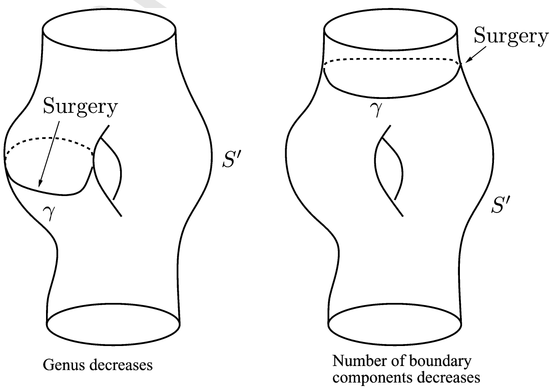

Fig. 6. The effects of a surgery on $\mathscr{S}'$.

components. They also have the same genus. Indeed, the proof of Theorem 2.1 in the connected case shows that the genus of a surface separating the sides of a topological thickening of a connected surface has to be larger or equal than the one of the surface. If equality would fail for any component of M, then the genus of S' would be larger than the one of S, a contradiction. We thus deduce that $C \cap S'$ and $C \cap S$ are homeomorphic by the classification of compact connected orientable surfaces, and conclude by applying the connected case separately to each component of S.

The proof of Theorem 2.2 follows similar lines: for each component C of M, $C \cap S'$ has at least as many components, boundary component, and handles as $C \cap S'$. Since the same holds for M', we deduce that all these inequalities are equalities: S and S' are thus homeomorphic, and the conclusion follows by Theorem 2.1.

6. Conclusion

We have presented two general conditions ensuring the existence of an isotopy between two surfaces embedded in $\mathbf{R}^3$, and given several applications of them in some widely considered particular situations. These conditions are a versatile and easy to use tool for proving that two surfaces are topologically equivalent, and we hope that they will prove useful in other applications than the ones mentioned in this paper. Though the formulation of our conditions directly extend to hypersurfaces of any dimension, the proof techniques used in this paper are typically 3-dimensional, and there is little hope that they extend in higher dimensions. It would be interesting to know which part of our results still hold in arbitrary dimension.

7. Uncited references

[8,10,14].

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