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Normal cone approximation and offset shape isotopy

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ABSTRACT

This work addresses the problem of the approximation of the normals of the offsets of general compact sets in Euclidean spaces. It is proven that for general sampling conditions, it is possible to approximate the gradient vector field of the distance to general compact sets. These conditions involve the μ-reach of the compact set, a recently introduced notion of feature size. As a consequence, we provide a sampling condition that is sufficient to ensure the correctness up to isotopy of a reconstruction given by an offset of the sampling. We also provide a notion of normal cone to general compact sets that is stable under perturbation.

1. Introduction

1.1. Motivation

Let K' be a finite set of points measured, possibly with some noise, on a physical object K. Given K' as *input*, is it possible to infer some reliable information on first order properties such as tangent planes or sharp edges, of the boundary of K? We consider here the case when the approximation K' of K has an error bounded under the Hausdorff distance. In other words we only assume that $d_H(K, K') < \varepsilon$ which means that any point of K' lies within a distance ε of some point of K and symmetrically, any point of K lies within a distance ε of a point of K'. The question is of primary interest in surface reconstruction applications. More generally, in the context of geometric processing, we would like to be able to extrapolate to a large class of non-smooth compact sets, including finite points samples and meshes, the usual notions of tangent plane or normal cones. Our goal here is to explore the notion of tangent plane first through the generalized gradient of the distance function (Section 3) and second through the Clarke Gradient of the distance function (Section 5), which brings also informations on concaves sharp edges.

1.2. Previous work on smooth manifolds

When K' is sampled exactly: $K' \subset K$, on a smooth boundary, it has been proved [2,3], that the normals to K can be estimated from the *poles*: for each point sample $q \in K'$, its pole is the Voronoi vertex farthest from q on the boundary of the Voronoi cell of q. In [12] this Voronoi based approach has been extended to the approximation of normals and feature lines from *noisy* sampling of a smooth manifold by considering only the poles corresponding to sufficiently large Delaunay balls.

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1.3. Reconstruction of "sufficiently regular" non-smooth objects from samples

In [5,7], the authors have considered the problem of recovering the topology of a compact set K given a sampling K' without any smoothness assumption on K.

In the same manner as the resolution of a microscope constrains the minimal size of observable details, any topological feature (such as a connected component or a tunnel for example) of a compact set K which would be small with respect to ε can certainly not be "reliably detected" from the knowledge of a sample K' with Hausdorff distance bounded by ε. A realistic measure of the topology should consider only the "topological information observable at the scale ε": in the context of [5,7], this has lead to consider topological features which are stable under sufficiently large offsets. Note that *topological persistence* [8] is an algebraic counterpart of this notion of stable topology.

The problem of the reconstruction, from a set of measured points, of a geometric numerical model carrying the same topology as the sampled object has been addressed previously for smooth manifolds [1,19], for which the sampling condition is related to the distance to the medial axis of K. The main contribution of [5] is to give a sampling condition for non-smooth objects, through the notion of *critical function* which encodes the regularity of the compact set boundary at different "scales".

When it is reasonable to assume some regularity conditions on the object's boundary, which can be formally expressed through lower bounds on the critical function, it is possible to recover the object's topology from a sufficiently dense and accurate sampling. By contrast, if we make no assumption about the regularity of the measured object K, it is still possible to decide some guaranteed topological information, not about the object K itself of course, but about offsets of K.

1.4. Contributions

The aim of the present work is to apply the previous approach, which has been successful for the retrieval of topological information, to the determination, beyond the topology, of first order information, which include tangents planes or the detection of sharp edges. Note that classical "exact" definitions of first order geometric informations such as tangent planes on surfaces, are not preserved in general by Hausdorff approximations. In other words, they are in general destroyed by arbitrary small perturbations (small for Hausdorff distance) of the object boundary. For example, a finite set sampled "near" the boundary of a smooth shape "contains" some information about the shape boundary tangent plane, but has no tangent plane in the usual sense. Still if one consider a d-offset of the point sample, that is a union of spheres of radius d centered on the points, the tangent planes to the offset boundary may bring some meaningful tangency informations about the initial shape. Following this simple idea and using properties of the distance function to compact sets developed in [5] we propose to introduce "stable" quantities that extend usual exact first order differential quantities. These quantities are preserved by small Hausdorff distance perturbation of the object: from this perspective, they can be "really observed" and carry more reality than their classical "exact and ideal" counterpart. These stable quantities are generalization of first order differential properties of surfaces. They apply to a large class of compact sets, which suggest applications for meshes and point clouds modeling. For smooth manifolds, our quantities coincide, in the limit, with usual definitions of first order tangent affine manifold.

The paper follows two complementary approaches. Both are necessary because they bring two distinct stability results, none being a corollary of the other. The first stability result given in Section 3 applies to the generalized gradient of the distance function and is necessary to get the isotopy result of Section 4. On the other hand, the second stability result (Section 5) applies to the Clarke gradient of the distance function which brings more information than the generalized gradient, in particular it may allow to recognize concave edges. In some sense, the first stability result uses weaker conditions (this is why the second stability result can not be used in the proof of isotopy in Section 4). On another hand the second stability result ensures a better convergence of the estimation of the gradient direction with respect to the Hausdorff distance (error bounded by $O(\epsilon^{\frac{1}{2}})$ instead of $O(\epsilon^{\frac{1}{4}})$) for the gradient estimation near a smooth surfaces).

1.5. Outline

Section 2 gives the necessary background notions on the distance function and its generalized gradient. Section 3 and in particular Corollary 3.2 gives a first stability property of the generalized gradient with respect to perturbations of the compact sets bounded in Hausdorff distance. This property bounds the maximal angular deviation between the gradient of the distance functions to two compact sets K and K'. An important consequence of this theorem is Theorem 4.2 which asserts the isotopy between the offsets of the compact set and its sampling with almost the same sampling conditions as in the main theorem in [5]. Section 5 introduces a stability theorem on the Clarke Gradient of the distance function. The stable quantity is a kind of "interval Clarke Gradient": to be more precise, it is the convex hull of the union of the values taken by the Clarke gradient in a ball. From this stability theorem, one introduces (Section 6), a *normal cone at a given scale*, which is a stable generalization of first order differential properties, defined at any point on or nearby a compact set.

2. Definitions and background on distance functions

We are using the following notations in the sequel of the paper. Given $X \subset \mathbb{R}^n$, one denotes by $\overline{X}$ the closure of X, by ∂X its boundary and by $co(X)$ its convex hull. For a point $x \in \mathbb{R}^n$ and a number $r \geqslant 0$, $\mathbb{B}(x,r) = \{y \in \mathbb{R}^n,\ d(y,x) \leqslant r\}$ denotes the closed ball with center x and radius r.

The *distance function* R_K of a compact subset K of $\mathbb{R}^n$ associates to each point $x \in \mathbb{R}^n$ its distance to K:

$$x \mapsto R_K(x) = \min_{y \in K} d(x, y)$$

where $d(x, y)$ denotes the Euclidean distance between x and y. Conversely, this function characterizes completely the compact set K since $K = \{x \in \mathbb{R}^n \mid R_K(x) = 0\}$. Note that R_K is 1-Lipschitz. The *Hausdorff distance* $d_H(K, K')$ between two compact sets K and K' in $\mathbb{R}^n$ is the minimum number r such that $K \subset K'^r$ and $K' \subset K^r$, where K^r denotes the r-offset of K, that is:

$$K^r = R_K^{-1}([0,r]) = \{x \in \mathbb{R}^n \mid R_K(x) \leqslant r\}$$

It is not difficult to check that the Hausdorff distance between two compact sets is the maximum difference between the distance functions associated with the compact sets:

$$d_H(K, K') = \sup_{x \in \mathbb{R}^n} \left| R_K(x) - R_{K'}(x) \right|$$

Given two homeomorphic compact subsets K and K' of $\mathbb{R}^n$, let $\mathcal{F} = \{f : K \to K' \colon f \text{ is an homeomorphism}\}$ be the set of all homeomorphisms between K and K'. Given such a homeomorphism f, $\sup_{x \in K} d(x, f(x))$ is the maximum displacement of the points of K by f. The *Fréchet distance* between K and K' is the infimum of this maximum displacement among all the homeomorphisms. It is defined by

$$d_F(K, K') = \inf_{f \in \mathcal{F}} \sup_{x \in K} d(x, f(x))$$

It is a classical exercise to check that the Fréchet distance satisfies the properties defining a distance and that one always has $d_H(S, S') \leqslant d_F(S, S')$.

Given a compact subset K of $\mathbb{R}^n$, the *medial axis* $\mathcal{M}(K)$ of K is the set of points in $\mathbb{R}^n \setminus K$ that have at least two closest points on K. The minimal distance between K and $\mathcal{M}(K)$ is called, according to Federer [14], the *reach* of K and is denoted $\mathrm{reach}(K)$.

2.1. The gradient and its flow

The distance function R_K is not differentiable on $\mathcal{M}(K)$. However, it is possible [18] to define a *generalized gradient* vector field $\nabla_K : \mathbb{R}^n \to \mathbb{R}^n$ that coincides with the usual gradient of R_K at points where R_K is differentiable. For any point $x \in \mathbb{R}^n \setminus K$, we denote by $\Gamma_K(x)$ the set of points in K closest to x (Fig. 1):

$$\Gamma_K(x) = \{y \in K \mid d(x, y) = d(x, K)\}$$

Note that $\Gamma_K(x)$ is a non-empty compact set. The function $x \mapsto \Gamma_K(x)$ is upper semi-continuous (see [18] Lemma 4.6, also [10] 2.1.4 for the same definition of semi-continuity p. 29):

$$\forall x, \forall r > 0,\ \exists \alpha > 0, \quad \|y - x\| \leqslant \alpha \Rightarrow \Gamma_K(y) \subset \{z\colon\ d(z, \Gamma_K(x)) \leqslant r\} \tag{1}$$

There is a unique smallest closed ball $\sigma_K(x)$ enclosing $\Gamma_K(x)$ (cf. Fig. 1). We denote by $\theta_K(x)$ the center of $\sigma_K(x)$ and by $\mathcal{F}_K(x)$ its radius. $\theta_K(x)$ can equivalently be defined as the point on the convex hull of $\Gamma_K(x)$ nearest to x. For $x \in \mathbb{R}^n \setminus K$, the generalized gradient $\nabla_K(x)$ is defined as follows:

$$\nabla_K(x) = \frac{x - \theta_K(x)}{R_K(x)}$$

It is natural to set $\nabla_K(x) = 0$ for $x \in K$. For $x \in \mathbb{R}^n \setminus K$, one has the following relation [18]:

$$\|\nabla_K(x)\|^2 = 1 - \frac{\mathcal{F}_K(x)^2}{R_K(x)^2}$$

Equivalently, $||\nabla_K(x)||$ is the cosine of the (half) angle of the smallest cone with apex x that contains $\Gamma_K(x)$. As an immediate consequence, one has the following lemma.

Lemma 2.1. *Let $K \subset \mathbb{R}^n$ be a compact set. For any $x \in \mathbb{R}^n$,*

$$\|\nabla_K(x)\| \leqslant \inf_{y, y' \in \Gamma_K(x)} \cos \frac{(\vec{xy}, \vec{xy'})}{2}$$

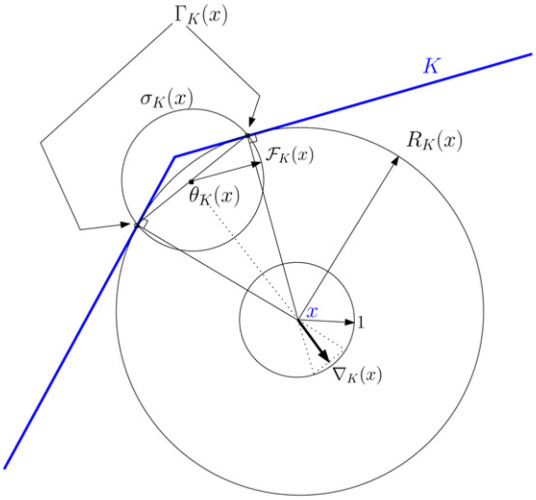

Fig. 1. A 2-dimensional example with 2 closest points.

The map $x \mapsto \|\nabla_K(x)\|$ is lower semi-continuous [18]. Although ∇_K is not continuous, it is shown in [18] that Euler schemes using ∇_K converge uniformly, when the integration step decreases, toward a continuous flow $\mathfrak{C}: \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$. The integral line of this flow starting at a point $x \in \mathbb{R}^n$ can be parameterized by arc length $s \mapsto \mathfrak{C}(t(s), x)$. It is possible to express the value of R_K at the point $\mathfrak{C}(t(l), x)$ by integration along the integral line with length l downstream the point x:

$$R_K\big(\mathfrak{C}\big(t(l), x\big)\big) = R_K(x) + \int_0^l \big\|\nabla_K\big(\mathfrak{C}\big(t(s), x\big)\big)\big\|\, ds \tag{2}$$

It is proved in [18] that the functions $\mathcal{F}_K$ and R_K are increasing along the trajectories of the flow. In the particular case where K is a finite set, various notions of flows related to this one have been independently introduced by H. Edelsbrunner [13], J. Giesen and al. [15] and R. Chaine [4] using Voronoi diagrams.

2.2. Critical point theory for distance functions

The *critical points* of R_K are defined as the points x for which $\nabla_K(x) = 0$. Equivalently, a point x is a critical point if and only if it lies in the convex hull of $\Gamma_K(x)$. When K is finite, this last definition means that critical points are precisely the intersections of Delaunay k-dimensional simplices with their dual $(n-k)$-dimensional Voronoi facets [15]. Note that this notion of critical point is the same as the one considered in the setting of non-smooth analysis [10] and Riemannian geometry [9,16].

The topology of the offsets $R_K^{-1}(a), a > 0$ of a compact set K are closely related to the critical values of R_K. The next proposition shows that it can change only at critical values.

Theorem 2.2 *(isotopy lemma). (See [16].) If $0 < a < b$ are such that $R_K^{-1}([a,b])$ does not contain any critical point of R_K, then all the level sets $R_K^{-1}(d)$, $d \in [a,b]$, are isotopic topological manifolds and $R_K^{-1}([a,b])$ is homeomorphic to $R_K^{-1}(a) \times [a,b]$.*

Recall that an *isotopy* between two manifolds S and S' is a continuous map $F: S \times [0,1] \to \mathbb{R}^n$ such that $F(.,0)$ is the identity of S, $F(S,1) = S'$, and for each $t \in [0,1]$, $F(.,t)$ is a homeomorphism onto its image. An *ambient isotopy* between S and S' is a continuous map $F: \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that $F(.,0)$ is the identity of $\mathbb{R}^n$, $F(S,1) = S'$, and for each $t \in [0,1]$, $F(.,t)$ is a homeomorphism of $\mathbb{R}^n$. Restricting an ambient isotopy between S and S' to $S \times [0,1]$ thus yields an isotopy between them. It is actually true that if there exists an isotopy between S and S', then there is an ambient isotopy between them [17].

The *weak feature size* of K, or $\mathrm{wfs}(K)$, is defined as the infimum of the positive critical values of R_K. Equivalently it is the minimum distance between K and the set of critical points of R_K. Notice that $\mathrm{wfs}(K)$ may be equal to 0. Nevertheless, $\mathrm{wfs}(K)$ is non-zero for a large class of compact sets including polyhedrons and piecewise analytic sets (see [6,7]). As an immediate consequence of the previous proposition, one deduces that the distance level sets $R_K^{-1}(d)$ are all isotopic for $0 < d < \mathrm{wfs}(K)$.

2.3. The critical function and the μ-reach

The results of this paper rely strongly on the notions of *μ-critical point*, *critical function* and *μ-reach*, introduced in [5].

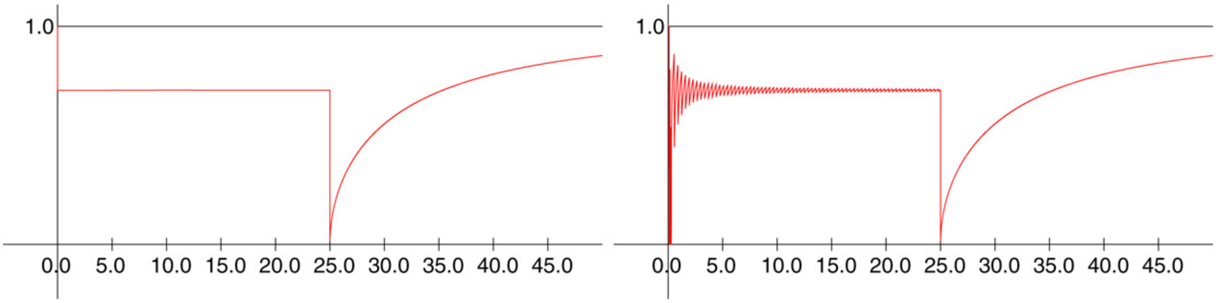

Fig. 2. Critical function of a square embedded in $\mathbb{R}^3$ with side length 50 (left), and of a sampling of that square (right).

Definition 2.3 *(μ-critical point).* A *μ-critical point x* of the compact set K is a point at which the norm of the gradient ∇_K does not exceed μ: $\|\nabla_K(x)\| \leqslant \mu$.

The most important property of μ-critical points is their stability with respect to Hausdorff perturbations of K proved in [5].

Theorem 2.4 *(Critical point stability theorem). Let K and K' be two compact subsets of $\mathbb{R}^n$ and $d_H(K, K') \leqslant \varepsilon$. For any μ-critical point x of K, there is a $(2\sqrt{\varepsilon/R_K(x)} + \mu)$-critical point of K' at distance at most $2\sqrt{\varepsilon R_K(x)}$ from x.*

Definition 2.5 *(Critical function).* Given a compact set $K \subset \mathbb{R}^n$, its *critical function* $\chi_K : (0, +\infty) \to \mathbb{R}_+$ is the real function defined by:

$$\chi_K(d) = \inf_{R_K^{-1}(d)} \|\nabla_K\|$$

Fig. 2 shows the respective critical functions of a square in 3-space and of a sampling of it. We note that the infimum can be replaced by a minimum since $\|\nabla_K\|$ is lower semi-continuous and $R_K^{-1}(d)$ is compact. It also results from the compactness of $R_K^{-1}(d)$ that $d \mapsto \chi_K(d)$ is lower semi-continuous. The critical function is in some sense "stable" with respect to small (measured by Hausdorff distance) perturbations of a compact set, precisely [5]:

Theorem 2.6 *(Critical function stability theorem). Let K and K' be two compact subsets of $\mathbb{R}^n$ and $d_H(K, K') \leqslant \varepsilon$. We define the interval $I(d, \varepsilon) = [d - \varepsilon, d + 2\chi_K(d)\sqrt{\varepsilon d} + 3\varepsilon]$. For all $d \geqslant 0$, we have:*

$$\inf\{\chi_{K'}(u) \mid u \in I(d, \varepsilon)\} \leqslant \chi_K(d) + 2\sqrt{\frac{\varepsilon}{d}}$$

Theorem 2.6 claim can be read as $\chi_K(d) \geqslant \inf\{\chi_{K'}(u) \mid u \in I(d, \varepsilon)\} - 2\sqrt{\frac{\varepsilon}{d}}$ and says that the knowledge of a lower bound on the critical function of a compact set K' gives a lower bound on the critical function of "nearby" (for Hausdorff distance) compact sets K. In particular, if a set K' of measured points is known to lie within some Hausdorff distance of a physical object represented by the unknown compact set K, the critical function of K' gives, by Theorem 2.6, a lower bound on the critical function of the partially known physical object K. Note that as explained in [5], starting from the Voronoi complex of the sample, the computation of the critical function of a finite sample is straightforward. This stability of the critical function with respect to small perturbations of the object in Hausdorff distance makes it particularly relevant in the context of approximate data.

The μ-reach of a compact set K is the maximal offset value d for which $\chi_K(d') \geqslant \mu$ for $d' < d$. More precisely, it is defined by:

$$r_\mu(K) = \inf\{d \mid \chi_K(d) < \mu\}$$

Closely related to the μ-reach and the critical point stability theorem is the following result [5] that will be used in Section 4.

Theorem 2.7 *(Critical values separation theorem). Let K and K' be two compact subsets of $\mathbb{R}^n$, ε be the Hausdorff distance between K and K', and μ be a non-negative number. The distance function $R_{K'}$ has no critical values in the interval $]4\varepsilon/\mu^2, r_\mu(K) - 3\varepsilon[$.*

These previous notions allow us to define a sampling condition for compact sets that lead to a reconstruction Theorem [5]. Given two non-negative real numbers κ and μ, we say that a compact set $K \subset \mathbb{R}^n$ is a *(κ, μ)-approximation* of a compact set $K' \subset \mathbb{R}^n$ if the Hausdorff distance between K and K' does not exceed κ times the μ-reach of K'.

Theorem 2.8 *(Reconstruction theorem). Let $K' \subset \mathbb{R}^n$ be a (κ, μ)-approximation of a compact set K. If*

$$\kappa < \frac{\mu^2}{5\mu^2 + 12}$$

then the complement of $R_{K'}^{-1}([0,\alpha])$ is homotopy equivalent to the complement of K, and $R_{K'}^{-1}([0,\alpha])$ is homotopy equivalent to $R_K^{-1}([0,\eta])$ for sufficiently small η, provided that

$$\frac{4d_H(K,K')}{\mu^2} \leqslant \alpha < r_\mu(K) - 3d_H(K,K')$$

In the remainder of the paper, we prove that under a similar condition, one can improve this result by comparing the topology of the level sets of R_K and $R_{K'}$ up to isotopy.

3. A first stability property of the gradient

In this section we deduce results on the stability of the gradient of distance functions from the stability theorem for μ-critical points. In the following, given two compact sets K and K', for any $x \in \mathbb{R}^n$, one denotes by $\tilde{\Gamma}_{K'}(x)$ the projection of $\Gamma_{K'}(x)$ on the sphere $\mathbb{S}(x, R_K(x))$: $\tilde{y} \in \tilde{\Gamma}_{K'}(x)$ if and only if there exists $y \in \Gamma_{K'}(x)$ such that $\tilde{y}$ is the intersection of the half-line $[xy)$ with the sphere $\mathbb{S}(x, R_K(x))$.

Theorem 3.1. *Let $K, K' \subset \mathbb{R}^n$ be two compact sets and let $\varepsilon > 0$ be such that $d_H(K,K') < \varepsilon$. If $x \in \mathbb{R}^n$ is a μ-critical point of $K''_x = K \cup \tilde{\Gamma}_{K'}(x)$ then there exists a $(\mu + 2\sqrt{\frac{2\varepsilon}{R_K(x)}})$-critical point of K at distance at most $2\sqrt{2\varepsilon R_K(x)}$ from x.*

Proof. Let $x \in \mathbb{R}^n$ and let $K'' := K''_x$. Since $d_H(K,K') < \varepsilon$, one has $d_H(K,K'') < 2\varepsilon$. One obtains immediately from the critical point stability theorem applied to K, K'' and x that there exists a $(\mu + 2\sqrt{\frac{2\varepsilon}{R_{K''}(x)}})$-critical point of K at distance at most $2\sqrt{2\varepsilon R_{K''}(x)}$ from x. It suffices to note that $R_K(x) = R_{K''}(x)$ to conclude the proof. □

As a consequence of Theorem 3.1, one obtains a bound on the angle between the vector fields ∇_K and $\nabla_{K'}$ of two nearby compact sets.

Corollary 3.2. *Let $K, K' \subset \mathbb{R}^n$ be two compact sets and let $\varepsilon > 0$ be such that $d_H(K,K') < \varepsilon$. Given $\mu > 0$, if $x \in \mathbb{R}^n$ is such that $\|\nabla_K(z)\| > \mu$ for any $z \in \mathbb{B}(x, 2\sqrt{2\varepsilon R_K(x)})$, then, if $K''_x = K \cup \tilde{\Gamma}_{K'}(x)$, then*

$$\|\nabla_{K''_x}(x)\| \geqslant \mu - 2\sqrt{\frac{2\varepsilon}{R_K(x)}} \tag{3}$$

Moreover, for any $y \in \Gamma_K(x)$ and any $y' \in \Gamma_{K'}(x)$,

$$\cos\frac{(\vec{xy}, \vec{xy'})}{2} \geqslant \mu - 2\sqrt{\frac{2\varepsilon}{R_K(x)}} \tag{4}$$

Proof. The first claim is just a contraposition of Theorem 3.1. The second inequality thus follows from Lemma 2.1 (Fig. 3). □

Recall that the direction of the vector $\nabla_K(x)$ (resp. $\nabla_{K'}(x)$) is contained in the convex hull of the directions defined by the segments joining x to the points of $\Gamma_K(x)$ (resp. $\Gamma_{K'}(x)$). So, the previous theorem immediately leads to the following result.

Corollary 3.3. *Let $K, K' \subset \mathbb{R}^n$ be two compact sets and let $\varepsilon > 0$ be such that $d_H(K,K') < \varepsilon$. Given $\mu > 0$, if $x \in \mathbb{R}^n$ is such that $\|\nabla_K(y)\| > \mu$ for any $y \in \mathbb{B}(x, 2\sqrt{2\varepsilon R_K(x)})$, then*

$$\cos\frac{(\nabla_K(x), \nabla_{K'}(x))}{2} \geqslant \mu - 2\sqrt{\frac{2\varepsilon}{R_K(x)}} \tag{5}$$

The bound of the corollary is tight: there are some examples where the cosine of angle between $\nabla_K(x)$ and $\nabla_{K'}(x)$ is of order $1 - O(\sqrt{\varepsilon})$. Let K be the circle of center $O \in \mathbb{R}^2$ and radius 1 and let O' be a point such that $d(O,O') = 2\sqrt{\varepsilon}$. The circle with center O' and radius $(1 - 2\sqrt{\varepsilon} + \varepsilon)$ meets K in two points A and B. Let K' be the boundary of the union of the disc of center O and radius 1 with angular area of radius $1 + \varepsilon$ and delimited by the half-lines OA and OB (see Fig. 4). The vector field ∇_K is continuous in a neighborhood of O' and $\nabla_K(O')$ is collinear to $O'O$. Along the segment $[O'A]$, $\nabla_{K'}$ is

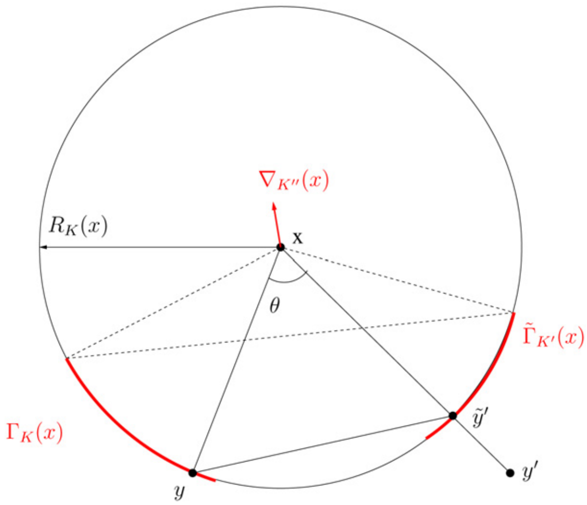

Fig. 3. Proof of Corollary 3.2.

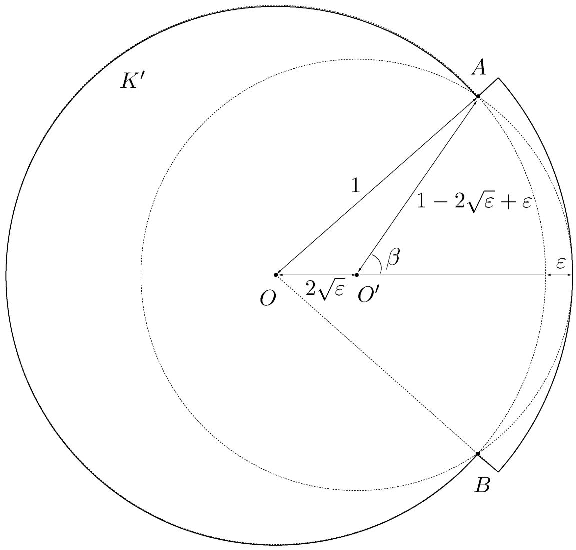

Fig. 4. An example showing the tightness of the bound of Corollary 3.3.

collinear to AO' and makes an angle β with OO'. An easy computation leads to $\cos\beta = 1 - \frac{1}{2}\sqrt{\varepsilon} + O(\varepsilon)$. As a consequence, if $x \in [O'A]$ is chosen sufficiently close to O', then it satisfies the hypothesis of the previous theorem and

$$\cos\frac{(\nabla_K(x), \nabla_{K'}(x))}{2} = 1 - O\left(\sqrt{\frac{\varepsilon}{R_K(x)}}\right)$$

4. Isotopy between offsets

We are now able to use the stability properties of the gradient established in the previous section to compare the topology of distance level sets of two nearby compact sets. Let $K, K' \subset \mathbb{R}^n$ be two compact sets and let $\varepsilon > 0$ be such that $d_H(K, K') < \varepsilon$.

Lemma 4.1. *Let $a > 0$ be such that for any $x \in R_K^{-1}([a-\varepsilon, a+\varepsilon])$, $\|\nabla_{K''_x}(x)\| \neq 0$ where $K''_x = K \cup \tilde{\Gamma}_{K'}(x)$. Then $R_K^{-1}(a)$ and $R_{K'}^{-1}(a)$ are isotopic hypersurfaces. Moreover, if*

$$\nu = \inf\left\{\left\|\nabla_{K''_x}(x)\right\|:\ x \in R_K^{-1}\big([a-\varepsilon, a+\varepsilon]\big)\right\} > 0$$

then the Fréchet distance between $R_K^{-1}(a)$ and $R_{K'}^{-1}(a)$ is bounded by $\frac{\varepsilon}{\nu}$.

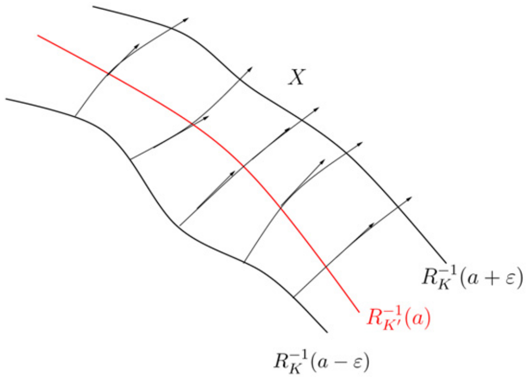

Fig. 5. Proof of Lemma 4.1.

Proof. The proof of the lemma is based upon a classical technique in differential geometry: we construct a $\mathcal{C}^\infty$ vector field which is "transverse" to the level sets of R_K and $R_{K'}$ in $A := R_K^{-1}([a-\varepsilon, a+\varepsilon])$ and that allows to realize an isotopy between $R_K^{-1}(a)$ and $R_{K'}^{-1}(a)$.

Let $x \in A$ and let $v(x) = \nabla_{K''_x}(x) \neq 0$. Since $y \to \Gamma_K(y)$ and $y \to \Gamma_{K'}(y)$ are upper semi-continuous (see [10] 2.1.4 for a definition), there exist $\delta_0 > 0$ such that

$$R_K\left(y + t\frac{v(x)}{\|v(x)\|}\right) \geqslant R_K(y) + t\frac{\|v(x)\|}{2}$$
$$R_{K'}\left(y + t\frac{v(x)}{\|v(x)\|}\right) \geqslant R_{K'}(y) + t\frac{\|v(x)\|}{2}$$

for any $y \in \mathbb{B}(x, \delta_0)$ and any $t \in (-\delta_0, \delta_0)$. Since A is compact, it is covered by a finite set of balls $\mathbb{B}(x_i, \delta_0(x_i))$, $i = 1, \ldots, p$. Using a $\mathcal{C}^\infty$ partition of unity associated to this covering and the constant vector field $v(x_i)$ on each $\mathbb{B}(x_i, \delta_0(x_i))$, one constructs a $\mathcal{C}^\infty$ vector field X on A such that for any trajectory $\phi(x,t)$ of X in A, one has

$$R_K\big(\phi(x,t)\big) \geqslant R_K(x) + t \tag{6}$$
$$R_{K'}\big(\phi(x,t)\big) \geqslant R_{K'}(x) + t \tag{7}$$

with $\nu = \min_{i=1,\ldots,p} \frac{\|v(x_i)\|}{2}$. It follows immediately that any trajectory $t \to \phi(x,t)$ issued from $R_K^{-1}(a-\varepsilon)$ meets $R_K^{-1}(a+\varepsilon)$ and $R_{K'}$ is strictly increasing along this trajectory. Moreover, since $\|v(x_i)\| < 1$ for all $i = 1, \ldots, p$, $\|X\| < 1$ and the length of the trajectory between x and $\phi(x,t)$ is bounded by $|t|$. It follows from inequality (7) that the length of any trajectory between $R_K^{-1}(a)$ and $R_K^{-1}(a-\varepsilon)$ or $R_K^{-1}(a+\varepsilon)$ is bounded by $\frac{\varepsilon}{\nu}$.

Now, since $\|R_K - R_{K'}\| < \varepsilon$, $R_{K'}$ is smaller than a on $R_K^{-1}(a-\varepsilon)$ and bigger than a on $R_K^{-1}(a+\varepsilon)$ (see Fig. 5). So, $R_{K'}^{-1}(a)$ is contained in A and it separates the two boundary components $R_K^{-1}(a-\varepsilon)$ and $R_K^{-1}(a+\varepsilon)$ of A. As a consequence for each $x \in R_K^{-1}(a)$, the trajectory $t \to \phi(x,t)$ intersects $R_{K'}^{-1}(a)$ in exactly one point $f(x) = \phi(x, t_x)$ (note that t_x may be negative). The map $x \to f(x)$ defines a continuous bijection between $R_K^{-1}(a)$ and $R_{K'}^{-1}(a)$ and the flow of X allows us to define an isotopy between these two hypersurfaces. The distance between x and $f(x)$ is bounded by the length of the trajectory between x and $f(x)$. So, $d(x, f(x)) < \frac{\varepsilon}{\nu}$ and the Fréchet distance between $R_K^{-1}(a)$ and $R_{K'}^{-1}(a)$ is bounded by $\frac{\varepsilon}{\nu}$. $\square$

The following results provide a sufficient condition involving the critical function for two compact sets to have isotopic offsets.

Theorem 4.2 *(Level sets isotopy theorem). Let $K, K' \subset \mathbb{R}^n$ be two compact sets such that $d_H(K, K') < \varepsilon$ for some $\varepsilon > 0$. If $a > 0$ is such that $\chi_K > \gamma + 2\sqrt{\frac{2\varepsilon}{a-\varepsilon}}$ on the interval $[a - \varepsilon - 2\sqrt{2\varepsilon(a+\varepsilon)}, a + \varepsilon + 2\sqrt{2\varepsilon(a+\varepsilon)}]$ for some constant $\gamma > 0$ then $R_K^{-1}(a)$ and $R_{K'}^{-1}(a)$ are isotopic hypersurfaces. Moreover the Fréchet distance between these two hypersurfaces is bounded by $\frac{\varepsilon}{\gamma}$.*

Proof. From Lemma 4.1, one just has to show that $x \notin co(\Gamma_K(x) \cup \tilde{\Gamma}_{K'}(x))$ for any $x \in A = R_K^{-1}([a-\varepsilon, a+\varepsilon])$. Suppose this is not the case for some $x \in A$. It follows from Theorem 3.1 that there exists a $(2\sqrt{\frac{2\varepsilon}{R_K(x)}})$-critical point y of K at distance at most $2\sqrt{2\varepsilon R_K(x)}$ from x. Since $a - \varepsilon \leqslant R_K(x) \leqslant a + \varepsilon$, y is a $(2\sqrt{\frac{2\varepsilon}{a-\varepsilon}})$-critical point of K at distance at most $2\sqrt{2\varepsilon(a+\varepsilon)}$ from x. Moreover

$$\begin{aligned} R_K(y) &\leqslant R_K(x) + 2\sqrt{2\varepsilon(a+\varepsilon)} \\ &\leqslant a+\varepsilon+2\sqrt{2\varepsilon(a+\varepsilon)} \end{aligned}$$

and in the same way $R_K(y) \geqslant a-\varepsilon-2\sqrt{2\varepsilon(a+\varepsilon)}$. These two last inequalities contradict the hypothesis of the theorem. The second part of the theorem follows from the second part of Lemma 4.1 and the first part of the Corollary 3.2. □

The previous theorem can be restated in terms of (κ,μ)-approximations to give the following result.

Theorem 4.3 *(Isotopic reconstruction theorem). Let $K \subset \mathbb{R}^n$ be a compact set such that $r_\mu(K) > 0$ for some $\mu > 0$. Let K' be a (κ,μ)-approximation of K where*

$$\kappa < \min\left(\frac{4\sqrt{2}-5}{14}, \frac{\mu^2}{16+2\mu^2}\right)$$

and let d, d' be such that

$$0 < d < \mathrm{wfs}(K) \quad \textit{and} \quad \frac{4\kappa r_\mu}{\mu^2} \leqslant d' < r_\mu(K) - 3\kappa r_\mu$$

Then the level set $R_{K'}^{-1}(d')$ is isotopic to the level set $R_K^{-1}(d)$.

Proof. Let $r_\mu = r_\mu(K)$, $a = r_\mu/2$ and $\varepsilon = \kappa r_\mu$. It follows from the isotopy Lemma 2.2 that $R_K^{-1}(d)$ is isotopic to $R_K^{-1}(a)$. It follows from the separation of the critical values theorem and from the isotopy Lemma 2.2 that $R_{K'}^{-1}(d')$ is isotopic to $R_{K'}^{-1}(a)$. Using that $\chi_K > \mu$ on $(0, r_\mu)$ and $\kappa < \frac{\mu^2}{16+2\mu^2}$ one easily checks that

$$\chi_K > 2\sqrt{\frac{2\varepsilon}{a-\varepsilon}}$$

on the interval $(0, r_\mu)$. Theorem 4.2 allows to conclude the proof provided that the interval with center a and half-length $\varepsilon + 2\sqrt{2\varepsilon(a+\varepsilon)}$ is included in the interval $(0, r_\mu)$. This last condition is equivalent to

$$\kappa r_\mu + 2\sqrt{2\kappa r_\mu\left(\frac{r_\mu}{2}+\kappa r_\mu\right)} < \frac{r_\mu}{2}$$

or, after division by r_μ,

$$2\kappa + 4\sqrt{\kappa(1+2\kappa)} < 1$$

This is satisfied as soon as $\kappa < \frac{4\sqrt{2}-5}{14}$. □

5. A second stability property of the gradient

In this section we consider the Clarke generalized gradient ∂R_K of the distance function [10] and prove a stability theorem of ∂R_K with respect to Hausdorff distance perturbation of the compact set K. ∂R_K includes more information than the generalized gradient ∇_K. Indeed, consider for example a concave sharp edge on a polyedron in 3-space. The concave edge will induce a sheet on the medial axis and the value of the Clarke gradient of the distance function on this sheet of the medial axis is a line segment orthogonal to the sharp edge and its length is related by an elementary expression to the sharp edge angle. More generally, the value of the Clarke gradient at medial axis points located near the set gives information about the geometry of concave sharp features. For this reason, we expect the new stability property to allow us to "extract" more geometric information about a compact set K from a Hausdorff approximation of it.

For a set E and a number $r \geqslant 0$, we recall that E^r denotes the set $E^r = \{z{:}\ d(z,E) \leqslant r\}$.

5.1. Clarke gradient of the distance function

The aim of this section is to prove Lemma 5.2 that provides a very simple relation between $\Gamma_K(x)$ and the Clarke Gradient of R_K at point x.

Instead of the usual definition of Clarke gradient we use the following characterization. For $f: \mathbb{R}^n \to \mathbb{R}$ we denote by Ω_f the set of point where f fails to be differentiable and, for $x \notin \Omega_f$, we denote $\frac{\partial f}{\partial X}(x)$ the usual gradient of the function f at x.

Definition 5.1 *(F.H. Clarke, adapted from [10], Section 2.5.1).* Let f be Lipschitz near x, then:

$$\partial f(x) = co\left\{\lim_{x_i \to x} \frac{\partial f}{\partial X}(x_i),\ \ x_i \notin \Omega_f\right\}$$

Rephrasing [10], the above characterization means the following. Consider any sequence x_i converging to x with f differentiable at each x_i and such that the usual gradient $\frac{\partial f}{\partial X}(x_i)$ converges; then $\partial f(x)$ is the convex hull of such limit points. A characterization of $\partial R_K(x)$ for $x \in K$ is given in ([10], Section 2.5.6). However, because our stability property is meaningful for $x \notin K$ only, we first prove the following characterization of ∂R_K for $x \in \mathbb{R}^n \setminus K$.

For $x \in \mathbb{R}^n \setminus K$ and $\rho > 0$ we introduce the notations $\tilde{G}_K(x)$, $G_K(x)$ and $G_K(x, \rho)$:

$$\tilde{G}_K(x) = \left\{\frac{x - z}{R_K(x)},\ z \in \Gamma_K(x)\right\}$$

$$G_K(x) = co\big(\tilde{G}_K(x)\big)$$

$$G_K(x, \rho) = co\left(\bigcup_{\|y - x\| \leqslant \rho} G_K(y)\right)$$

Lemma 5.2. *If $x \in \mathbb{R}^n \setminus K$, one has*:

$$\partial R_K(x) = G_K(x)$$

Proof. We first prove $G_K(x) \subset \partial R_K(x)$. For that we use Lemma 5.3 below. If a function f is differentiable at point x, we denote by $\frac{\partial f}{\partial X}(x)$ the value of the usual gradient of f at point x.

Lemma 5.3. *If $x \in \mathbb{R}^n \setminus K$ and $v \in \tilde{G}_K(x)$ then for any z on the open line segment $(x, x - R_K(x)v)$, R_K is differentiable at z and*:

$$\frac{\partial R_K}{\partial X}(z) = v$$

Proof of Lemma 5.3. From the definition of $\tilde{G}_K(x)$, one has $v \in \tilde{G}_K(x) \Rightarrow x_v = x - R_K(x)v \in \Gamma_K(x)$. We denote by $\mathbb{B}_{(x,r)}$ and $\mathbb{B}^\circ_{(x,r)}$ respectively the closed and open balls centered at x with radius r and let $R^{\max}$ be such that $K \subset \mathbb{B}_{(x,R^{\max})}$. We consider the two compact sets $K^+ = \{x_v\}$ and $K^- = \mathbb{B}_{(x,R^{\max})} \setminus \mathbb{B}^\circ_{(x,R_K(x))}$ one has:

$$K^+ \subset K \subset K^-$$

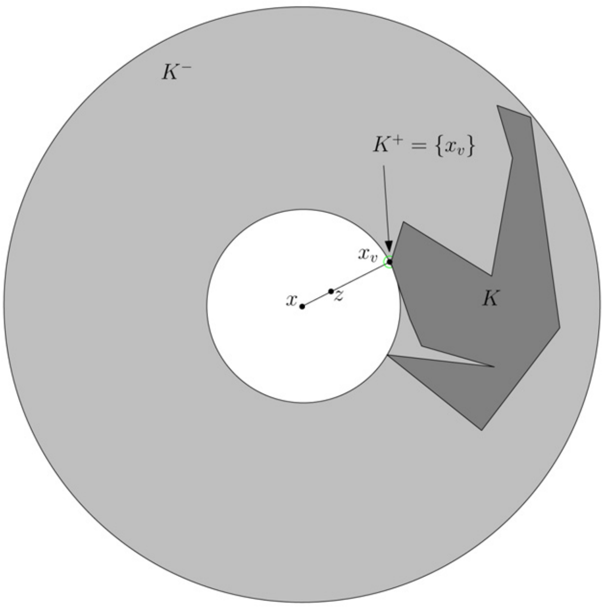

Fig. 6. Proof of Lemma 5.3.

which entails:

$$R_{K^-} \leqslant R_K \leqslant R_{K^+} \tag{8}$$

On another hand, R_{K^-} and R_{K^+} have simple radial expressions (see Fig. 6) which gives us that:

$$R_{K^-}(z) = R_K(z) = R_{K^+}(z) \tag{9}$$

R_{K^-} and R_{K^+} are differentiable in z and an easy computation shows that

$$\frac{\partial R_{K^-}}{\partial X}(z) = \frac{\partial R_{K^+}}{\partial X}(z) = v$$

this together with Eqs. (8) and (9) entails that the first order expansion of R_K is enclosed between the respective first order expansions of R_{K^-} and R_{K^+} and therefore R_K is differentiable at z and:

$$\frac{\partial R_K}{\partial X}(z) = v \qquad \square$$

Now let $v \in \tilde{G}_K(x)$. Lemma 5.3 entails that, for any positive integer n, there exists $x_n \in B_{(x,\frac{1}{n})}$ such that:

$$\frac{\partial R_K}{\partial X}(x_n) = v$$

this together with the Definition 5.1 implies that $v \in \partial R_K(x)$. We have proved that $\tilde{G}_K(x) \subset \partial R_K(x)$ and, because $\partial R_K(x)$ is convex, it entails $G_K(x) \subset \partial R_K(x)$.

We prove now $\partial R_K(x) \subset G_K(x)$. As seen in Section 2.1, Eq. (1), the function $x \mapsto \Gamma_K(x)$ is upper semi-continuous. When $R_K(x) > 0$, $G_K(x)$ is the image of $\Gamma_K(x)$ by a simple continuous transformation which allows easily to derive the following lemma that expresses that if a point y is taken sufficiently close to a point x, then $G_K(y)$ is close to $G_K(x)$ for a "half", or "one-sided", Hausdorff distance.

Lemma 5.4. *G_K is upper semi-continuous in $\mathbb{R}^n \setminus K$, in other words*:

$$\forall x \in \mathbb{R}^n \setminus K,\ \forall r > 0,\ \exists \alpha > 0, \quad \|y - x\| \leqslant \alpha \Rightarrow G_K(y) \subset G_K(x)^r$$

Let us consider a vector v such that there exists a sequence of points x_n which as in Definition 5.1, are such that $\lim_{n\to\infty} x_n = x$, R_K is differentiable at each x_n and

$$\lim_{n\to\infty} \frac{\partial R_K}{\partial X}(x_n) = v \tag{10}$$

Let us consider $\varepsilon > 0$. From Lemma 5.4, there is $\alpha > 0$ such that:

$$\|y - x\| \leqslant \alpha \Rightarrow G_K(y) \subset G_K(x)^{\frac{\varepsilon}{2}} \tag{11}$$

From (10), there is x_k such that:

$$\|x_k - x\| \leqslant \alpha \quad \text{and} \quad \left\| \frac{\partial R_K}{\partial X}(x_k) - v \right\| < \frac{\varepsilon}{2} \tag{12}$$

From ([10] 2.5.4) R_K differentiable at x_k entails that $\Gamma_K(x_k)$ is a single point and, if we denote $\{y_k\} = \Gamma_K(x_k)$ and $v_k = \frac{x_k - y_k}{R_K(x_k)}$ one has $v_k = \frac{\partial R_K}{\partial X}(x_k)$, which gives, with (12):

$$\|v_k - v\| < \frac{\varepsilon}{2}$$

On the other hand one has from (11):

$$\{v_k\} = G_K(x_k) \subset G_K(x)^{\frac{\varepsilon}{2}}$$

which entails:

$$\{v\} \subset G_K(x)^{\varepsilon}$$

Because this inclusion holds for any $\varepsilon > 0$ and $G_K(x)$ is closed, it entails

$$v \in G_K(x)$$

Because $G_K(x)$ is convex and $\partial R_K(x)$ is defined in Definition 5.1 as the convex hull of all such v, we get $\partial R_K(x) \subset G_K(x)$. $\square$

5.2. Stability of ∂R_K

The aim of this section is to prove Theorem 5.6 which is the main stability result for the Clarke gradient of the distance function. The idea is that, by confronting equation (13) below and the mean value theorem of Lebourg (Theorem 5.5 below) we get a relation between $G_{K'}(x)$ and $G_K(x,\rho)$ for two sets K and K' nearby for the Hausdorff distance.

We consider again two compact subsets of $\mathbb{R}^n$, K and K' which are "close" to each other for the Hausdorff distance: $d_H(K,K') \leqslant \varepsilon$.

Let x be a point in $\mathbb{R}^n \setminus K'$. For any $w' \in \tilde{G}_{K'}(x)$, the point $z' = x - R_{K'}(x)\,w'$ is in $\Gamma_{K'}(x)$ and therefore in K'. One has then, for any $y \in \mathbb{R}^n$:

$$\begin{aligned} R_{K'}(y)^2 &\leqslant (y-z')^2 \\ &= (x-z')^2 + 2\langle x-z', y-x\rangle + (y-x)^2 \\ &\leqslant R_{K'}(x)^2 + 2\langle w', y-x\rangle R_{K'}(x) + (y-x)^2 \end{aligned}$$

which gives, for any $w' \in \tilde{G}_{K'}(x)$

$$R_{K'}(y) - R_{K'}(x) \leqslant R_{K'}(x)\left(\sqrt{1 + \frac{2}{R_{K'}(x)}\langle w', y-x\rangle + \frac{(y-x)^2}{R_{K'}(x)^2}} - 1\right)$$

And, from $\sqrt{1+\alpha} \leqslant 1 + \frac{\alpha}{2}$:

$$R_{K'}(y) - R_{K'}(x) \leqslant \langle w', y-x\rangle + \frac{(x-y)^2}{2R_{K'}(x)} \tag{13}$$

Notice that this last equation can be read as the membership of w' to a half-space. Therefore, if holds for any $w' \in \tilde{G}_{K'}(x)$, it holds also for any w' in the convex hull $G_{K'}(x)$ of $\tilde{G}_{K'}(x)$.

Lemma 5.2 says that $\partial R_K(x) = G_K(x)$ which allows to use the following *mean value theorem* theorem which holds in general for Clarke gradients. We denote by (x, y) the relatively open segment between points x and y.

Theorem 5.5 *(Lebourg [10] 2.3.7). Let x and y be points in X, and suppose that f is Lipschitz in an open set containing the line segment* $[x, y]$. *Then there exists a point* $w \in (x, y)$ *such that*:

$$f(y) - f(x) \in \big\langle \partial f(w), y-x \big\rangle$$

Let $\rho > 0$ and x such that $R_K(x) \geqslant \rho$, applying Theorem 5.5 to the function R_K gives: $\forall y \in B(x,\rho)$, $\exists w \in G_K(x,\rho)$ such that:

$$R_K(y) - R_K(x) = \langle w, y-x\rangle$$

Using $R_K(y) - R_K(x) \leqslant R_{K'}(y) - R_{K'}(x) + 2\varepsilon$ and Eq. (13) we get: $\forall y \in B(x,\rho)$ there is $w \in G_K(x,\rho)$ such that for any $w' \in G_{K'}(x)$:

$$\langle w, y-x\rangle \leqslant \langle w', y-x\rangle + \frac{(x-y)^2}{2R_{K'}(x)} + 2\varepsilon$$

or:

$$\langle w'-w, x-y\rangle \leqslant \frac{(x-y)^2}{2R_{K'}(x)} + 2\varepsilon$$

Assuming now $\rho = \|y-x\|$, we consider the unit vector $u = -\frac{y-x}{\rho}$, which gives the following property: $\forall u$, $\|u\| = 1$, $\forall w' \in G_{K'}(x)$ there is $w \in G_K(x,\rho)$ such that:

$$\langle w'-w, u\rangle \leqslant \frac{\rho}{2R_{K'}(x)} + \frac{2\varepsilon}{\rho} \tag{14}$$

This property, which holds for any unit vector u gives in fact a relation between the support functions of the compact sets $G_{K'}(x)$ and $G_K(x,\rho)$.

Let $w' \in G_{K'}(x)$ such that $w' \notin G_K(x,\rho)$ and let $w'' \in G_K(x,\rho)$ be its unique nearest point in the convex set $G_K(x,\rho)$:

$$d(w', w'') = d\big(w', G_K(x,\rho)\big)$$

let us consider the unit vector $u^\star = \frac{1}{\|w'-w''\|}(w'-w'')$. Because $G_K(x,\rho)$ is convex, $\forall w \in G_K(x,\rho)$, one has:

$$\langle w, u^\star\rangle \leqslant \langle w'', u^\star\rangle$$

or equivalently:

$$\langle w - w'', u^\star \rangle \leqslant 0$$

Combining this with (14), for $u = u^\star$, gives:

$$\langle w' - w'', u^\star \rangle \leqslant \frac{\rho}{2R_{K'}(x)} + \frac{2\varepsilon}{\rho}$$

that is, for any $w' \in G_{K'}(x) \setminus G_K(x,\rho)$, there is $w'' \in G_K(x,\rho)$ such that:

$$\|w' - w''\| \leqslant \frac{\rho}{2R_{K'}(x)} + \frac{2\varepsilon}{\rho}$$

which proves the following.

Theorem 5.6. *For any x such that $R_K(x) \geqslant \rho$, one has*:

$$G_{K'}(x) \subset G_K(x,\rho)^{\frac{\rho}{2R_{K'}(x)} + \frac{2\varepsilon}{\rho}}$$

6. Application to normal approximation

Based on the results from the previous section, we now introduce a scale-dependent notion of normal cone that allows to infer first order information from finite approximations of compact sets, even in the non-smooth case.

6.1. A stable notion of normal cone

The main object of study of this section is defined as follows (see Fig. 7):

Definition 6.1. Given two non-negative numbers $l \leqslant r$, the normal cone at scale (r,l) of a compact set K at the point $p \in \mathbb{R}^n$ is defined as:

$$N_K^{r,l}(p) = co\left\{ \frac{x - q}{R_K(x)} \;\middle|\; d(x,p) \leqslant r,\ R_K(x) \geqslant l,\ q \in \Gamma_K(x) \right\}$$

This notion of normal cone may be viewed as a generalization of the normal cone in the sense of Clarke (see [10] p. 51) since the latter coincides with the limit $\lim_{r\to 0} N_K^{r,0}(p)$ when p belongs to K ([10] Proposition 2.5.7 p. 68). While the original definition of Clarke's normal cone is outside the scope of the paper, the reader not acquainted with it can take the former limit as a definition. As we will see, our notion of normal cone allows in many cases to estimate Clarke's normal cone knowing only approximations of the compact set K.

First, we note that Theorem 5.6 gives the following result:

Lemma 6.2. *Let K and K' be two compact sets with Hausdorff distance ε and let $\eta \geqslant \varepsilon + 2\sqrt{\varepsilon l}$. We have for all $r \geqslant l \geqslant 6\varepsilon$*:

$$N_{K'}^{r,l}(p) \subset N_K^{r+\eta, l-\eta}(p)^{2\sqrt{\varepsilon/l}}$$

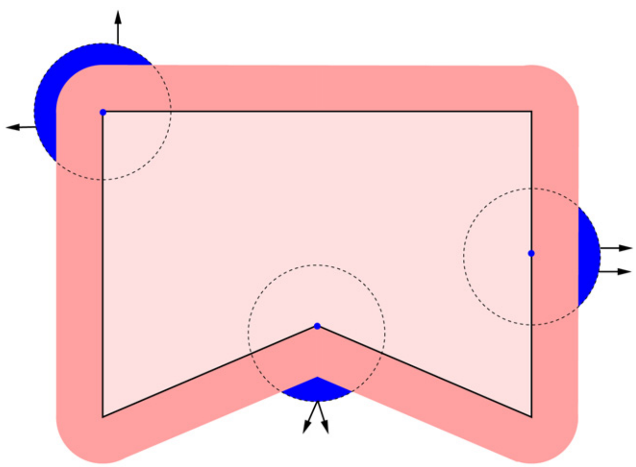

Fig. 7. Normal cones at three points of the boundary of a solid polygon.

Proof. Let x be such that $d(x,p) \leqslant r$ and $d(x,K') \geqslant l$. By Theorem 5.6, for any $\rho \leqslant l-\varepsilon$, we have:

$$G_{K'}(x) \subset G_K(x,\rho)^{\frac{\rho}{2l}+\frac{2\varepsilon}{\rho}}$$

taking $\rho = 2\sqrt{\varepsilon l}$ as in [5]:

$$G_{K'}(x) \subset G_K(x, 2\sqrt{\varepsilon l})^{2\sqrt{\varepsilon/l}}$$

provided that $2\sqrt{\varepsilon l} \leqslant l-\varepsilon$, which is satisfied as soon as $l \geqslant 6\varepsilon$. The conclusion follows. □

Endowing the set of compact subsets of $\mathbb{R}^n$ with the Hausdorff distance, the map $N_K^{\cdot,\cdot}(p)$ becomes a map between two metric spaces. From the previous lemma, one easily gets the following one:

Lemma 6.3. *Let $p \in \mathbb{R}^n$ and $0<l<r$ be such that $N_K^{\cdot,\cdot}(p)$ is κ-lipschitz on $[r-\nu, r+\nu] \times [l-\nu, l+\nu]$ for some positive constants κ and ν. Let K' be a compact set at Hausdorff distance at most ε from K. If*

$$\varepsilon \leqslant \min\left(\frac{\nu^2}{36l}, \frac{l}{6}\right)$$

then

$$d_H\left(N_{K'}^{r,l}(p), N_K^{r,l}(p)\right) \leqslant 12\kappa\sqrt{\varepsilon l} + 6\sqrt{\frac{\varepsilon}{l}}$$

In particular, if $N_K^{\cdot,\cdot}(p)$ is locally lipschitz at (r,l), then $N_{\cdot}^{r,l}(p)$ is locally $1/2$-Hölder at K.

Proof. Applying Lemma 6.2 twice, we get:

$$N_K^{r-\eta,l+\eta}(p) \subset N_{K'}^{r,l}(p)^{2\sqrt{\varepsilon/l}} \subset N_K^{r+\eta,l-\eta}(p)^{4\sqrt{\varepsilon/l}}$$

for suitable $\eta > 0$. The conditions on ε are chosen so that we may take $\eta = 6\sqrt{\varepsilon l}$. The lemma then follows from the double inclusion above. □

Since our notion of normal cone is stable under Hausdorff approximation, it can be inferred from finite approximations of compact sets. It now remains to pick suitable values for the parameters r and l. We consider two particular cases below: submanifolds with positive reach and piecewise linear complexes.

6.2. Normal cone estimation for submanifolds with positive reach

Consider the case where the compact set K is a submanifold of $\mathbb{R}^n$ with positive reach. Assuming that p belongs to K, it is not difficult to show that for $0<r<r_1(K)$, $N_K^{r,r}(p)$ does not depend on r and coincides with Clarke's normal cone $N_K(p)$. Unfortunately, we cannot apply Lemma 6.3 for estimating $N_K^{r,r}(p) = N_K(p)$ directly since it only applies when the two parameters of the normal cone are different. To apply Lemma 6.3, we need to estimate the lipschitz constant of the function $N_K^{\cdot,\cdot}(p)$ at points where the two parameters are different. Given three numbers λ, λ' and λ'' in $(0,1)$ we let $D = \{(r,l) \mid \lambda'' r \leqslant l \leqslant \lambda r,\ r \leqslant \lambda' r_1(K)\}$.

Lemma 6.4. *Let K be a compact submanifold of $\mathbb{R}^n$. If p belongs to K, the function $N_K^{\cdot,\cdot}(p)$ is κ-lipschitz on D, with*

$$\kappa = \left(1 - \frac{(1+\lambda)^2}{4}\right)^{-1/2} \max\left\{(1-\lambda')^{-1} r_1(K)^{-1}, (\lambda'' t)^{-1}\right\}$$

Proof. Function $N_K^{\cdot,\cdot}(p)$ is the composition $f_3 \circ f_2 \circ f_1$ where f_1 the set-valued function sending (r,l) to the closure of $B(p,r) \setminus K^l$, f_2 is the set-valued function sending any closed subset to its image by ∇_K, and f_3 is the set-valued function sending a compact set to its convex hull. The function f_3 being 1-Lipschitz for the Hausdorff distance it is sufficient to estimate the Lipschitz constants of f_1 and f_2. Let us first estimate the lipschitz constant of f_1. Function $r \mapsto B(p,r)$ is clearly 1-lipschitz. For $l < r_1(K)$, function $l \mapsto K^l$ is also 1-lipschitz. However the lipschitz constant of f_1, which is the set difference of the latter two functions, depends on the minimum angle α at which the boundaries of $B(p,r)$ and of K^l meet, for $(r,l) \in D$. More precisely, this constant is equal to $1/\sin(\alpha)$. We now show that the angle α satisfies $\cos\alpha \leqslant (1+\lambda)/2$.

Assume the boundaries of $B(p,r)$ and of K^l intersect at say $x \in \mathbb{R}^n$. Letting y be the point closest to x on K, we have that α is the angle between the lines (px) and (xy). Let c be the unique point on the line (xy) such that the ball centered at c and passing through y also passes through p. Since this ball is tangent to K and meets K at a point different from its tangency point, its radius R must exceed the reach of K. We distinguish two cases. If the triangle xyp has an obtuse angle at x, then the angle β between (yp) and (xy) is at most α. Since c is the intersection of the medial hyperplane of

$[yp]$ with (xy), we get that $R = \|cy\| = \|yp\|/(2\cos\beta) \leqslant (r+l)/(2\cos\alpha)$. Since $R > r_1(K) \geqslant r$, we get the desired bound on $\cos\alpha$. Assume now that xyp has an acute angle at x. For the sake of contradiction assume that $\cos\alpha > (1+\lambda)/2$. In particular, $\cos\alpha > \lambda$ hence $r\cos\alpha > l$, which implies that $y \in [xc]$. The triangle xcp thus has edge lengths $\|xp\| = r$, $\|cp\| = R$ and $\|xc\| = R + l$. Applying the generalized Pythagorean theorem at x gives:

$$(R+l)^2 + r^2 - 2(\cos\alpha)r(R+l) - R^2 = 0$$

The derivative with respect to R of the l.h.s. of the above equation is $2l - 2r\cos\alpha$ which is negative by assumption. Hence the l.h.s. becomes positive if we replace R by r. The desired bound on $\cos\alpha$ follows, a contradiction. As a consequence, the desired bound follows, from which we deduce that the lipschitz constant of f_1 is at most $(1-(1+\lambda)^2/4)^{-1/2}$.

To finish the proof of the lemma, we note that the lipschitz constant of f_2 equals the lipschitz constant of ∇_K, which for $r < r_1(K)$ is bounded by

$$\max\{(r_1(K)-r)^{-1}, l^{-1}\} \leqslant \max\{r_1(K)^{-1}(1-\lambda')^{-1}, (\lambda'' r)^{-1}\}$$

(see [14] for example). □

The previous lemma together with Lemma 6.3 show that for any $r > l$, $N_{K'}^{r,l}(p)$ allows to estimate $N_K^{r,l}(p)$ with precision $O(\sqrt{\varepsilon})$. It also gives the rate of convergence of $N_K^{r,l}(p)$ to $N_K^{r,r}(p) = N_K(p)$ as l tends to r. Indeed, we have that

$$d_H\big(N_K^{r,l}(p), N_K(p)\big) \leqslant \int_{x=l}^{r} \kappa(r,x)\,dx$$

where $\kappa(r,x)$ is the bound on the lipschitz constant of $N_K^{\cdot,\cdot}$ derived in the previous lemma. From the expression of κ, it can be checked that as x tends to r, we have $\kappa(r,x) \simeq c(r, r_1(K))(1-x/r)^{-1/2}$ for some constant $c(r, r_1(K))$. Hence

$$d_H\big(N_K^{r,l}(p), N_K(p)\big) \simeq \int_{x=l}^{r} \frac{c(r, r_1(K))}{\sqrt{1-x/r}}\,dx \simeq c'\big(r, r_1(K)\big)\sqrt{1-l/r}.$$

As a consequence, we get that $d_H(N_{K'}^{r,\lambda r}(p), N_K(p)) = O(\sqrt{\varepsilon} + \sqrt{1-\lambda})$ when λ tends to 1 and ε tends to 0. Remark however that our bounds on the Hausdorff distance between normal cones are only valid when ε is smaller than a certain function of $(\lambda, r, r_1(K))$ that we do not make explicit in the present paper to avoid additional calculations. Finally, as a comment, we note that the estimators we propose are related to the ones introduced by Dey et al. [11,12] for normal estimation in noisy smooth surfaces. One difference, though, is that our method can deal with submanifolds with arbitrary dimension and codimension, rather than hypersurfaces in $\mathbb{R}^3$.

6.3. Normal cone estimation for piecewise linear complexes

Let K be a (non-necessarily convex) piecewise linear complex. For any p in K, we define $r(p)$ as the distance from p to the closest (closed) face of K not containing p. The intersection of K with a sphere of radius less than $r(p)$, scaled so that it includes in the unit sphere, is a spherical complex called the link of p, and denoted by $\mathrm{Lk}(p)$. Let $\theta(p) \leqslant \pi$ be the maximal radius such that for any face $F \subset \mathrm{Lk}(p)$, there exists $y \in F$ such that the open geodesic ball $B(y, \theta(p))$ only meets faces of $\mathrm{Lk}(p)$ that are incident to F.

Lemma 6.5. *For $l \leqslant \sin(\theta(p)/2)r$ and $r < r(p)/2$, $N_K^{r,l}(p)$ coincides with Clarke's normal cone $N_K(p)$.*

Proof. Assume for simplicity that $r(p) > 2$, so that the link of p is the intersection of K with the unit sphere. For $r \leqslant 1$, $N_K^{r,0}(p)$ only depends on the link of p. Hence, we have that $N_K^{r,0}(p) = N_K^{1,0}(p)$ for any $r \in (0,1]$, which implies that $N_K(p) = N_K^{1,0}(p)$. In particular, we have the inclusion $N_K^{r,l}(p) \subset N_K(p)$. To prove the other inclusion, it is sufficient to consider the case $r = 1$ by homogeneity. First, let v be an extreme point in $N_K(p) = N_K^{1,0}(p)$. By definition, there is a point x at distance 1 from p and $q \in \Gamma_K(x)$ such that $x - q$ points in the same direction as v. Let F be the minimal face containing q and $y \in F \cap \mathrm{Lk}(p)$ be such that the open geodesic ball $B(y, \theta(p))$ in the unit sphere only meets faces of $\mathrm{Lk}(p)$ that are incident to F. Let now $s(t)$ be the geodesic on the unit sphere issued from y and with initial speed v. We have that the closest point (for the geodesic distance) on $\mathrm{Lk}(p)$ of the point $z = s(\theta(p)/2)$ lies in a face that is incident to F. Since $s'(0) = v$, y must be a local minimum of the distance to z. Because all faces are convex, y is also the absolute minimum in each face incident to F, hence also the absolute minimum in the union of all faces incident to F and, by the above remark, we get that y is the point closest to z in $\mathrm{Lk}(p)$. Elementary geometric considerations show that the closest point q' (for the ambient distance) to z on K lies on the segment py and is such that $z - q'$ points in the same direction as v. Also, the distance between q' and z is equal to $\sin(\theta(p)/2)$. Since z is at unit distance from p, we get that $v \in N_K^{1,\sin(\theta/2)}(p)$, which concludes the proof. □

Hence $N_{K'}^{r,l}(p)$ is a good estimate of Clarke's normal cone for such a choice of r and l. More precisely, since $N_K^{r,l}(p)$ does not depend on r and l in the prescribed range, it follows from lemma 6.3 that this estimator has precision $O(\sqrt{\varepsilon})$.

7. Discussion

The main purpose of this paper was to study the possibility of estimating "first order properties" of general compact sets from Hausdorff approximations. In Section 3, we have shown that tangent planes of offsets can be reliably estimated when the offset parameter lies in a sufficiently long interval where the critical function of the underlying shape is close to 1. When this is not the case, a one-sided stability result can still be obtained by introducing an interval version of Clarke gradient for distance functions (Section 5). This result suggests that normal cones of the compact set (rather than the ones of its offsets) may be estimated by a certain scale dependent notion of normal cone, as described in Section 6.

Several questions remain open. First, how can we choose suitable parameters for estimating the scale dependent normal cone introduced in Section 6? In the case where the underlying shape is either a smooth submanifold or a convex set, or a non-necessarily convex polyhedron, it is possible to find such parameters. Is there a way to infer these parameters from data? Also, can anything be said in the general case? Second, how can we design an efficient algorithm implementing the ideas of this paper? Exact computation of normal cones involves non-trivial geometric operations, such as intersecting balls with convex polyhedra. Can we compute approximations of these normal cones in an efficient way?

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