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Stability of Curvature Measures

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Abstract

We address the problem of curvature estimation from sampled compact sets. The main contribution is a stability result: we show that the Gaussian, mean or anisotropic curvature measures of the offset of a compact set K with positive μ-reach can be estimated by the same curvature measures of the offset of a compact set K' close to K in the Hausdorff sense. We show how these curvature measures can be computed for finite unions of balls. The curvature measures of the offset of a compact set with positive μ-reach can thus be approximated by the curvature measures of the offset of a point-cloud sample.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—

1. Introduction

1.1. Motivation

We present in this work a *stable* notion of curvature. A common definition of curvature considers quantities defined pointwisely on a twice differentiable manifold. However, the objects we have to deal with in practice are not twice differentiable: consider the situation where a physical object is known through a sufficiently dense point cloud measured on the object boundary. Intuitively, it seems feasible to infer some meaningful information on the curvature of the physical object itself. Let us assume that we know both the *accuracy* of the measure, which is an upper bound on the distances between the measured points and their closest points on the physical object boundary, and the *sampling density*, which is an upper bound on the distance between the points on the object boundary and their closest measured sample points. The fact that the measure accuracy and sampling density are below a known small value ε can be expressed by saying that the Hausdorff distance (see the definition in Section 2) between the assumed physical object and the measured point cloud is less than ε. Even with the guarantee of a small Hausdorff distance, the knowledge of the measured point cloud allows many possible shapes for the physical object boundary and it is hopeless, without additional assumptions, to infer the usual pointwise curvature quantities on the physical object.

Indeed, a first difficulty is that the geometrical and topological properties of a physical object have to be considered at some scale: for example, if one is interested in the shape of a ship hull, it may make sense to see it as a smooth surface at a large scale (10^{-1} meters). However, at a finer scale (10^{-4} meters) this same object may appear with many sharp features near the rivets and small gaps between assembled sheets of metals. At a still finer scale, one could even consider each atom as a separate connected component of the same physical object.

A second difficulty is related to the "pointwise" nature of usual curvature definition. If the physical object is known up to ε in Hausdorff distance, it seems impossible to distinguish between the pointwise curvature at two points whose distance is of the order of ε. Without very strong assumptions on the regularity of the unknown object, it is again meaningless to evaluate a pointwise curvature quantity.

In this work, we overcome the first difficulty by considering a scale dependent notion of curvature which consists merely, for a compact set K, in looking at the curvature of the r-offsets of K (*i.e.* the set of points at distance less than or equal to r from K). This offset, which can be seen as a kind of convolution, in the same spirit of similar operators in mathematical morphology or image processing, filters out high frequencies features of the object's boundary.

We overcome the second difficulty by considering curvature measures instead of pointwise curvature. In the case of smooths manifolds, curvature measures associate to a subset of the manifold the integral of pointwise curvatures over the subset. However the curvature measures are still defined on non smooth objects such as convex sets or more generally sets with positive reach [Fed59].

In practice, in order to state our stability theorem, one has to make some assumption on the unknown physical object, namely the *positive μ-reach* property defined below. To be more precise, our stability theorem still applies to objects whose offsets have positive μ-reaches.

Some other approaches assume stronger properties such as smoothness or positive reach. But is it legitimate to make such assumptions on an unknown object? In practice, the only available informations about the physical object appears through the physical measures. A distinctive character of our assumption on the μ-reach of offsets of physical object is that, thanks to the so-called critical values separation theorem [CCL06], it can be reliably checked from the measured point sample. In this situation our curvature estimations reflect reliably the intrinsic properties of the physical object.

1.2. Related previous work.

Due to its applications in geometry processing, many methods have been suggested that, given a triangulated surface, are able to estimate the curvature of an assumed underlying smooth surface (see [Pet02] for a survey). Several authors (for example [CP03]) compute the curvature of a smooth polynomial surface approximating locally the triangulated surface. Other authors [Hua06, PWY*07, PWHY09] consider integral invariants that allow to estimate curvatures of underlying smooth objects. Other authors use finite element based methods or laplacians [Dzi88, HPW06, LP05], statistical approaches [KSNS07] or tensor-fitting approaches [Rus04]. Cheeger and his coauthors also showed results of convergence for the Lipschitz-Killing curvatures [CMS86]. Closely related to our work is [Mor08,Coh04,CM03,CM07], which study a general definition of curvature measure that applies to both smooth surfaces and their approximation by triangulated surfaces, based on the so-called normal cycle (defined below). The proximity of curvature measure is proved using the powerful notion of *flat norm* [Mor87] between the corresponding normal cycles. These ideas are thoroughly reused in the present work.

In [CCL06], in order to address the question of topology determination through Hausdorff approximation, the authors have introduced the class of sets with positive μ-reach, which can be regarded as a mild regularity condition. In particular, this condition does not require smoothness. If a set has a positive μ-reach, or at least if it has offsets with positive μ-reach, it is possible to retrieve the topology of its offset from the topology of some offsets of a Hausdorff close point sample. A stable notion of normal cone has also been defined on this class of sets [CCL07]. More recently, the authors have proved [CCLT07] that the complement of offsets of sets with positive μ-reach have positive reach. Since the normal cycle and the associated curvature measures are defined for sets with positive reach [Fed59, Fu89] we may consider normal cycles of offsets of sets with positive μ-reach. This paper develops this idea and gives a stability result whose proof uses the fact that the boundary of the double offsets of sets with positive μ-reach are smooth surfaces.

1.3. Contributions.

Our main result states that if the Hausdorff distance between two compact sets with positive μ-reach is less than ε, then the curvature measures of their offsets differs by less that $O(\sqrt{\varepsilon})$, using an appropriate notion of distance between measures. This is then extended through the *critical values separation theorem* to the case where only one set has positive μ-reach, which allows to evaluate the curvature measure of an object from a noisy point cloud sample. These results improve on the stability results in [Mor08, Coh04, CM03, CM07], which were only limited to the approximation of smooth hypersurfaces by homeomorphic triangulated manifolds. In order to provide a concrete algorithm we give the formulas that express these curvatures measures on a union of balls (*i.e.* on an offset of the point cloud). Our experimental results are preliminary and the evaluation on ground truth surfaces under varying conditions of noise and sampling and the comparison to other methods are still left open.

Closest to our work is [CCM07], which also gives a stability result for curvature measures. The main differences are that our result also applies to anisotropic curvature measures, whereas [CCM07] is only limited to the usual curvature measures. On the other hand, the stability result for curvature measures in [CCM07] derives from a stability result for so-called boundary measures, which holds without any assumptions on the underlying compact set, whereas ours requires to assume a lower bound on the μ-reach. While the two results seem related at first sight, the proof techniques are drastically different.

1.4. Outline.

The paper first recalls definitions and properties related to the distance function to a compact set and its gradient, the critical function and the μ-reach. Then, we introduce the curvature measures without using the notion of normal cycle, whose formal definition is somewhat lengthy. In Section 3, we state the stability theorem for sets with positive μ-reaches. We then extend this theorem to the case where only one set has positive μ-reach. In Section 4, we give the expressions of curvature measures for unions of balls. The last sections gives the proof.

2. Definitions and background on distance functions

We are using the following notations in the sequel of this paper. Given $X \subset \mathbb{R}^d$, one denotes by X^c the complement of X, by $\overline{X}$ its closure and by ∂X the boundary of X. Given $A \subset \mathbb{R}^d$, $ch(A)$ denotes the convex hull of A.

The *distance function* d_K of a compact subset K of $\mathbb{R}^d$ associates to each point $x \in \mathbb{R}^d$ its distance to K:

$$x \mapsto d_K(x) = \min_{y \in K} d(x,y),$$

where $d(x,y)$ denotes the Euclidean distance between x and y. Conversely, this function characterizes completely the compact set K since $K = \{x \in \mathbb{R}^d \,|\, d_K(x) = 0\}$. Note that d_K is 1-Lipschitz. For a positive number r, we denote by K_r the r-offset of K, defined by $K_r = \{x \,|\, d_K(x) \leq r\}$. The *Hausdorff distance* $d_H(K,K')$ between two compact sets K and K' in $\mathbb{R}^d$ is the minimum number r such that $K \subset K'_r$ and $K' \subset K_r$. The triangular inequality implies that the following formulae:

$$d_H(K,K') = \sup_{x \in \mathbb{R}^n} |d_K(x) - d_{K'}(x)|$$

Given a compact subset K of $\mathbb{R}^d$, the *medial axis* $\mathcal{M}(K)$ of K is the set of points in $\mathbb{R}^d \setminus K$ that have at least two closest points on K. The infimum distance between K and $\mathcal{M}(K)$ is called, according to Federer, the *reach* of K and is denoted $\mathrm{reach}(K)$. The reach of K vanishes if K has concave sharp edges or corners. The projection map p_K that associates to a point x its closest point $p_K(x)$ on K is thus defined on $\mathbb{R}^d \setminus \mathcal{M}(K)$.

A $\mathcal{C}^{1,1}$ function is a $\mathcal{C}^1$ function whose first differential is Lipschitz. A $\mathcal{C}^{1,1}$ hypersurface S is a $(d-1)$-manifold embedded in $\mathbb{R}^d$ such that each point of S has a neighborhood which is the regular image (that is the image by a function whose differential has maximal rank) by an injective $\mathcal{C}^{1,1}$ function of a neighborhood of 0 in $\mathbb{R}^{d-1}$. Informally, one can say that a $\mathcal{C}^{1,1}$ surface is a surface with bounded curvature, which is strictly stronger than C^1 and strictly weaker than C^2. An embedded $\mathcal{C}^1$ compact manifold is $\mathcal{C}^{1,1}$ if and only if it has positive reach [Fed59]. Note that a non planar polyedral surface is not of class $\mathcal{C}^1$, thus does not have positive reach.

2.1. The gradient

The distance function d_K is not differentiable on $\mathcal{M}(K)$. However, it is possible [Lie04] to define a *generalized gradient* function $\nabla_K : \mathbb{R}^d \to \mathbb{R}^d$ that coincides with the usual gradient of d_K at points where d_K is differentiable. For any point $x \in \mathbb{R}^d \setminus K$, we denote by $\Gamma_K(x)$ the set of points in K closest to x (Figure 1):

$$\Gamma_K(x) = \{y \in K \,|\, d(x,y) = d_K(x)\}$$

Note that $\Gamma_K(x)$ is a non empty compact set. There is a unique smallest closed ball $\sigma_K(x)$ enclosing $\Gamma_K(x)$ (cf. Figure 1). We denote by $\theta_K(x)$ the center of $\sigma_K(x)$ and by $\mathcal{F}_K(x)$ its radius. $\theta_K(x)$ can equivalently be defined as the point on the convex hull of $\Gamma_K(x)$ nearest to x. For $x \in \mathbb{R}^d \setminus K$, the

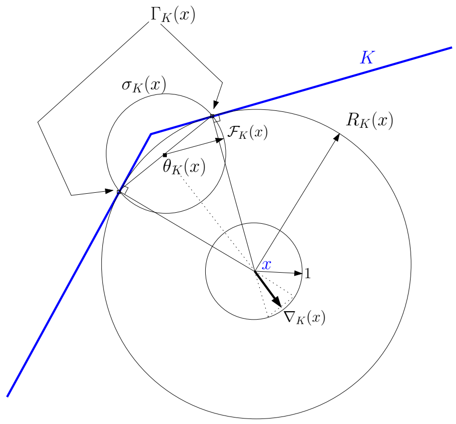

Figure 1: *A 2-dimensional example with 2 closest points.*

generalized gradient $\nabla_K(x)$ is defined as follows:

$$\nabla_K(x) = \frac{x - \theta_K(x)}{d_K(x)}$$

Note that the definition of ∇_K is chosen so that it can be integrated into a continuous flow [Lie04]. Furthermore, $||\nabla_K(x)||$ is the cosine of the (half) angle of the smallest cone with apex x that contains $\Gamma_K(x)$.

2.2. Critical points, critical function and the μ-reach

The *critical points* of d_K are defined as the points x for which $\nabla_K(x) = 0$. Equivalently, a point x is a critical point if and only if it lies in the convex hull of $\Gamma_K(x)$. When K is finite, this last definition means that critical points are precisely the intersections of Delaunay k-dimensional simplices with their dual $(d-k)$-dimensional Voronoi facets [GJ03]. Note that this notion of critical point is the same as the one considered in the setting of non smooth analysis [Cla83] and Riemannian geometry [Che90, Gro93].

The results of this paper rely strongly on the notions of *critical function* and *μ-reach*, introduced in [CCL06].

Definition 1 (critical function) Given a compact set $K \subset \mathbb{R}^d$, its *critical function* $\chi_K : (0,+\infty) \to \mathbb{R}_+$ is the real function defined by:

$$\chi_K(d) = \inf_{d_K^{-1}(d)} ||\nabla_K||$$

The function χ_K is lower semicontinuous [Lie04]. The μ-reach of a compact set K is the maximal offset value d for which $\chi_K(d') \geq \mu$ for $d' < d$.

Definition 2 (μ-reach) The *μ-reach* $r_\mu(K)$ of a compact set $K \subset \mathbb{R}^d$ is defined by:

$$r_\mu(K) = \inf\{d \mid \chi_K(d) < \mu\}$$

We have that $r_1(K)$ coincides with the reach introduced by Federer [Fed59]. It can be shown that a polyedron always has positive μ-reach. For example, the union of two adjacent triangles in $\mathbb{R}^3$ has a positive μ-reach for any $\mu \leq \cos(\beta/2)$ (where β is the dihedral angle). The critical function is in some sense "stable" with respect to small (measured by Hausdorff distance) perturbations of a compact set [CCL06]. That implies the following theorem [CCL06]:

Theorem 1 (critical values separation theorem) Let K and K' be two compact subsets of $\mathbb{R}^d$, $d_H(K,K') \leq \varepsilon$ and μ be a non-negative number. The distance function d_K has no critical values in the interval $\left]4\varepsilon/\mu^2, r_\mu(K') - 3\varepsilon\right[$. Besides, for any $\mu' < \mu$, χ_K is larger than μ' on the interval

$$\left]\frac{4\varepsilon}{(\mu-\mu')^2}, r_\mu(K') - 3\sqrt{\varepsilon r_\mu(K')}\right[.$$

2.3. Complement of offsets

It has been proved in [CCLT07] that the complement $\overline{K_r^c}$ of the offset K_r has positive reach for any value $0 < r < r_\mu$. Moreover, one has a lower bound for the critical function of $\overline{K_r^c}$:

Theorem 2 For $r \in (0, r_\mu)$, one has $\mathrm{reach}(\overline{K_r^c}) \geq \mu r$. Moreover for any $t \in (\mu r, r)$,

$$\chi_{\overline{K_r^c}}(t) \geq \frac{2\mu r - t(1+\mu^2)}{t(1-\mu^2)}. \tag{1}$$

2.4. Curvature measures

Let us first recall some basic definitions and notations in the case where M is a smooth surface that is the boundary of a compact set V of $\mathbb{R}^d$. The unit normal vector at a point $p \in M$ pointing outward V will be refered as $n(p)$. Note that M is thereby oriented. Given a vector v in the tangent space T_pM to M at p, the derivative of n in the direction v at p is orthogonal to $n(p)$. The derivative D_pn of n at p thus defines an endomorphism of T_pM, known as the Weingarten endomorphism. The Weingarten endomorphism is symmetric. The associated quadratic form is called the second fundamental form. Eigenvectors and eigenvalues of the Weingarten endomorphism are respectively called principal directions and principal curvatures. In the 3-dimensional case, both principal curvatures can be recovered from the trace and determinant of D_pn, also called mean and Gaussian curvature.

It is possible to define the curvature measures of a set with positive reach as the limit of the curvature measures of its offsets. The advantage of these definitions is that they do not rely on the notion of normal cycle.

More precisely, let V be a set with positive reach $R > 0$ and let $t < R$. It is known that ∂V_t is a $\mathcal{C}^{1,1}$ hypersurface of $\mathbb{R}^d$ [Fed59]. The second fundamental form and the principal curvatures of ∂V_t are thus defined almost everywhere. There is of course no pointwise convergence of the principal curvatures when t tends to 0. However, the integrals of the curvatures of ∂V_t converge to the integrals of the curvatures of V when t tends to 0 (as can be seen for instance using the tube formula [Fed59]). This allows us to define the isotropic curvature measures of V for every Borel subset B of $\mathbb{R}^d$ as follows:

$$\Phi_V^k(B) = \lim_{t\to 0} \int_{\partial V_t \cap B'} s^k(p)dp$$

where p_V is the projection onto V, $B' = \{p \in \mathbb{R}^d,\ p_V(p) \in B\}$, s^k is the k-th elementary symmetric polynomial of the principal curvatures $\lambda_1,\ldots,\lambda_{d-1}$ of ∂V_t, the integral being taken with respect to the uniform measure on ∂V_t. In other words, s^k satisfies for every $x \in \mathbb{R}$: $(x+\lambda_1)\ldots(x+\lambda_{d-1}) = s^0 + s^1x + \ldots + s^{d-1}x^{d-1}$. Now, remark that we have:

$$\Phi_V^k(B) = \lim_{t\to 0} \int_{\partial V_t} \mathbf{1}_B(p_V(p))\, s^k(p)dp,$$

where $\mathbf{1}_B$ is the indicator function of B. Therefore, we can extend this notion of curvature measure to any integrable real function. This point of view is crucial and will allow us to state simple results of stability in this paper. More precisely, one defines the isotropic curvature measure for every integrable function f on $\mathbb{R}^d$ by:

$$\Phi_V^k(f) = \lim_{t\to 0} \int_{\partial V_t} f(p_V(p))\, s^k(p)dp.$$

Similarly, one extends the notion of anisotropic curvature measure of [Coh04, CM03]: the anisotropic curvature measure of V associates to any integrable function f the $d \times d$ symmetric matrix defined by:

$$\overline{H}_V(f) = \lim_{t\to 0} \int_{\partial V_t} f(p_V(p))\, H_{\partial V_t}(p)dp,$$

where $H_{\partial V_t}$ is a matrix-valued function defined on $\mathbb{R}^d$ that coincides with the second fundamental form of ∂V_t on the tangent space, and vanishes on the orthogonal component.

Now, let K be a compact set whose μ-reach is greater than $r > 0$. Then $V = \overline{K_r^c}$ has a reach greater than μr. It is then possible to define the extended notions of curvature measures of K_r (see [RZ01, RZ03] for more details) by:

$$\Phi_{K_r}^k(f) = (-1)^k \Phi_V^k(f) \quad \text{and} \quad \overline{H}_{K_r}(f) = -\overline{H}_V(f).$$

In the 3-dimensional case, it is also possible to define a second anisotropic measure curvature $\overline{\widetilde{H}}$ by:

$$\overline{\widetilde{H}}_V(f) = \lim_{t\to 0} \int_{\partial V_t} f(p_V(p))\, \widetilde{H}_{\partial V_t}(B)$$

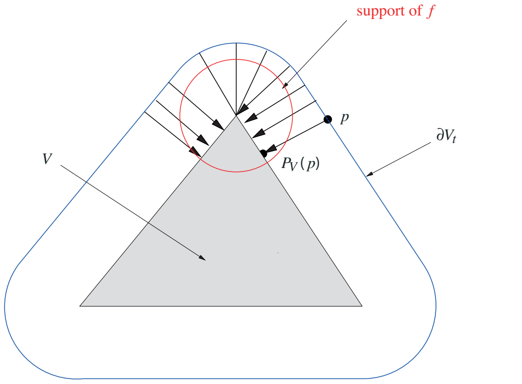

Figure 2: *Here the support of* f *is a ball and* $\Phi^k_V(f)$ *captures the curvature of* ∂V *inside the support of* f

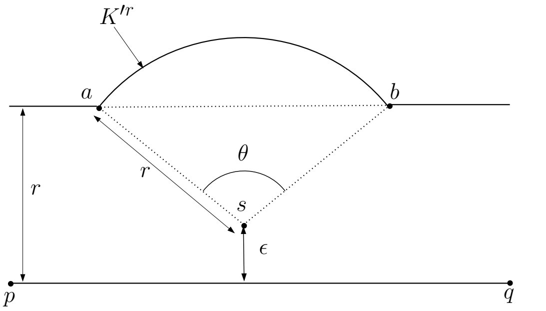

Figure 3: *Tightness of the bound: we take* $K=[p,q]$ *and* $K'=[p,q]\cup\{s\}$, *where* s *is at a distance* ε *from* K. *We have* $d_H(K,K')=\varepsilon$ *and the total curvature* θ *of* K'_r *between* a *and* b *satisfies* $\theta = 2\arccos\left(\frac{r-\varepsilon}{r}\right) = O\left(\sqrt{\varepsilon}\right)$.

and

$$\overline{\widetilde{H}}_{K_r}(f) = -\overline{\widetilde{H}}_V(f),$$

where $\widetilde{H}_{\partial V_t}$ is defined as having the same eigenvectors as $H_{\partial V_t}$, but with swapped eigenvalues on the tangent plane, and vanishes on the orthogonal component of ∂V_t.

3. Stability results

3.1. Curvature measures of the offsets

The main contribution of this paper is Theorem 3. Thanks to the formulation of the curvature measures with measurable functions, the statement is simple. This theorem states that if two compact sets K and K' with positive μ-reaches are close in the Hausdorff sense, then the curvature measures of their offsets are close. We recall that the covering number $\mathcal{N}(A,t)$ of a compact set A is the minimal number of closed balls of radius t needed to cover A.

Theorem 3 Let K and K' be two compact sets of $\mathbb{R}^d$ whose μ-reaches are greater than r. We suppose that the Hausdorff distance $\varepsilon = d_H(K,K')$ between K and K' is less than $\frac{r\mu\,(2-\sqrt{2})}{2}\min(\mu,\frac{1}{2})$. If $f:\mathbb{R}^d\to\mathbb{R}$ is a Lipschitz function satisfying $|f|\le 1$, then:

$$|\Phi^i_{K_r}(f) - \Phi^i_{K'_r}(f)| \le k(r,\mu,d,f)\ \sup(Lip(f),1)\ \sqrt{\varepsilon},$$

and

$$\|\overline{H}_{K_r}(f) - \overline{H}_{K'_r}(f)\| \le k(r,\mu,d,f)\ \sup(Lip(f),1)\ \sqrt{\varepsilon},$$

where $k(r,\mu,d,f)$ only depends on f through the covering number $\mathcal{N}(spt(f)_{O(\sqrt{\varepsilon})},\mu r/2)$; $Lip(f)$ is the Lipschitz-constant of f; $spt(f) = \overline{\{x\in\mathbb{R}^d,\ f(x)\ne 0\}}$.

We recall that $spt(f)_{O(\sqrt{\varepsilon})}$ denotes an offset of $spt(f)$ of parameter $O(\sqrt{\varepsilon})$.The proof of this theorem is given in Section 5. We show in Figure 3 that this bound is tight. Furthermore, in the 3-dimensional case, this result also holds for the anisotropic curvature measure $\overline{\widetilde{H}}$.

Now, if we take the function $f(x) = \max(1-\|x-c\|/r, 0)$ equal to 1 at a point $c\in\partial K'_r$ that radially decreases in a "small" ball $\mathbb{B}$ of radius r and vanishes out of $\mathbb{B}$, then we can get local information about the curvature of K'_r from the curvature of K_r in the neighborhood of c.

We also note that the conclusion of the theorem may be rephrased by saying that the *bounded Lipschitz distance* between the curvature measures of K_r and K'_r is bounded by $O(\sqrt{\varepsilon})$. The bounded Lipschitz distance between measures is similar to the Wasserstein distance (also called earth's mover distance), except that it applies to general signed measures whereas Wasserstein distance is limited to probability measures. We refer to [CCM07] for precise definitions.

3.2. General result

The result of the previous section ensuring the stability of the curvature measures assume that both the compact sets K and K' have sufficiently large μ-reach. Nevertheless, in practical settings, particularly when dealing with point clouds, such a hypothesis is rarely satisfied. Using Theorem 1, it is still possible to approximate the curvature measures of the offsets of a compact set with positive μ-reach from any sufficiently close approximation of it.

Theorem 4 Let K and K' be two compact subsets of $\mathbb{R}^d$ such that $r_\mu(K') > r$. Assume that the Hausdorff distance $\varepsilon = d_H(K,K')$ between K and K' is such that $\varepsilon < \frac{\mu^2}{60+9\mu^2}r$. Then the conclusions of Theorem 3 also hold.

Of course, if for $s>0$, the compact set K' satisfies $r_\mu(K'_s) > c > 0$, the same theorem applies to K'_s. Furthermore, thanks to Theorem 1, the value of $r_\mu(K'_s)$ can be read on the critical function of the sample K.

Proof It follows from Theorem 1 (the critical values separation theorem of [CCL06]) that the critical function of K

is greater than $\frac{\mu}{2}$ on the interval $\left(\frac{16\varepsilon}{\mu^2}, r - 3\sqrt{\varepsilon r}\right)$. Note that since $\mu > \frac{\mu}{2}$, the critical function of K' is also greater than $\frac{\mu}{2}$ on the same interval. As a consequence the two compact sets $\tilde{K} = K_{\frac{16\varepsilon}{\mu^2}}$ and $\tilde{K}' = K'_{\frac{16\varepsilon}{\mu^2}}$ have their $\frac{\mu}{2}$-reach greater than $r - 3\sqrt{\varepsilon r} - \frac{16\varepsilon}{\mu^2}$. Notice that for any $\delta > 0$, $\tilde{K}_\delta = K_{\frac{16\varepsilon}{\mu^2}+\delta}$ and since $d_H(K, K') = \varepsilon$ then $d_H(\tilde{K}, \tilde{K}') \leq \varepsilon$. To apply Theorem 3 to $\tilde{K}$ and $\tilde{K}'$, $\frac{\mu}{2}$ and r, the Hausdorff distance ε between $\tilde{K}$ and $\tilde{K}'$ must satisfy $0 < \varepsilon < \frac{2-\sqrt{2}}{8}\mu^2(r - 3\sqrt{\varepsilon r} - \frac{16\varepsilon}{\mu^2})$ (note that since $\frac{\mu}{2} \leq \frac{1}{2}$, $\min(\mu, \frac{1}{2}) = \mu$). One easily checks (by computing the solutions of the inequality and using that for all $0 \leq x \leq 1$, $\sqrt{1-x} \geq 1 - \frac{x}{2}$) that this inequality is implied by the assumption made on ε. The theorem now follows immediately from Theorem 3. $\square$

4. Computation of the curvature measures of 3D point clouds

When the compact K is a finite set of points in $\mathbb{R}^3$ it is possible to provide explicit formula for the curvature measures. Although the description and the analysis of a robust and efficient algorithm is beyond the scope of this paper, we show how curvature measures of a 3D point cloud can be computed in practice. We also provide some illustrating experimental results obtained by implementing the method described in this section.

Let $K \subset \mathbb{R}^3$ be a finite set of points and let $r > 0$. To avoid technicalities, we assume that the set of balls of radius r and centered in K are in general position (as in [EM94]): no 4 centers lie on a common plane; no 5 centers lie on a common sphere; and the smallest sphere through any 2, 3 or 4 centers of K has a radius different from r. In the 3-dimensional case, we denote by $\Phi^H_{K_r} = \Phi^1_{K_r}$ the mean curvature measure and by $\Phi^G_{K_r} = \Phi^2_{K_r}$ the Gaussian curvature measure. The computation of $\Phi^G_{K_r}$ and $\Phi^H_{K_r}$ is done in three main steps described in the following paragraphs.

4.1. Computation of the boundary of a union of balls

The boundary of K_r is a spherical polyhedron: its faces are spherical polygons; its edges are circle arcs contained in the intersection of pairs of spheres of radius r with centers in K; its vertices belong to the intersection of three spheres of radius r with centers in K. Moreover, the combinatorial structure of ∂K_r is easily deduced from an α-shape of K. Indeed, it follows from Lemma 2.2 in [Ede93] that it is in one-to-one correspondence with the boundary of the α-shape of K for $\alpha = r$: faces (that may not be connected nor simply connected) of the former are in one-to-one correspondence with vertices of the latter; edges and vertices are in correspondence with edges and faces respectively. As a consequence, the combinatorial structure of ∂K_r is easily deduced from the computation of the α-shape of K. Getting the geometric structure of ∂K_r is more tricky since it requires to compute intersections of spheres and arragements of circles on sphere. To get the experimental results below we used a half-edge data structure for boundaries of union of balls designed and implemented by S. Loriot (INRIA Sophia-Antipolis) that is based upon the α-shape data structure of the library CGAL [Cga].

4.2. Computation of the curvature measures of the cells of ∂K_r

Let C be a cell of ∂K_r (*i.e.* a face, an edge or a vertex). Under the general position assumption, to compute $\Phi^G_{K_r}(C)$ and $\Phi^H_{K_r}(C)$, it is sufficient to compute the curvature measures $\Phi^G_{\mathcal{S}}(C)$ and $\Phi^H_{\mathcal{S}}(C)$, where $\mathcal{S}$ is the union of one, two or three balls of radius r. The result of such computations, is summarized in Proposition 1 below where the orientation of the boundary of the union of balls is taken so that the normal is pointing outside. Notice that $\Phi^G_{K_r}(C)$ and $\Phi^H_{K_r}(C)$ are proportional to either the area, or the length, or the Dirac measure of C (depending on if C is a face or an edge or a vertex). As a consequence, once computed on each cell C of ∂K_r, the values $\varphi^G_{K_r}(C)$ and $\varphi^H_{K_r}(C)$ can simply be stored by adding an extra information to the elements of the data structure representing ∂K_r. These values will then be used for the integration (in step 3.).

Proposition 1
i) Let $\mathbb{B}$ be a ball of radius r of $\mathbb{R}^3$ and let B be a Borel set of $\mathbb{R}^3$. Then the curvature measures of $\mathbb{B}$ above B are given by

$$\begin{aligned} & \Phi^H_{\mathbb{B}}(B) = \varphi^H_{\mathbb{B}}\, \mathit{Area}(B \cap \partial\mathbb{B}) \\ \text{and} \quad & \Phi^G_{\mathbb{B}}(B) = \varphi^G_{\mathbb{B}}\, \mathit{Area}(B \cap \partial\mathbb{B}) \end{aligned}$$

where

$$\varphi^H_{\mathbb{B}} = \frac{2}{r} \quad \text{and} \quad \varphi^G_{\mathbb{B}} = \frac{1}{r^2}.$$

ii) Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be two intersecting balls of $\mathbb{R}^3$, of same radius $r > 0$ and of centers A_1 and A_2, and C be the circle $\partial\mathbb{B}_1 \cap \partial\mathbb{B}_2$. Let B be a ball of $\mathbb{R}^3$. Then the curvature measures of $\mathbb{B}_1 \cup \mathbb{B}_2$ above $B \cap C$ are given by

$$\begin{aligned} & \Phi^H_{\mathbb{B}_1\cup\mathbb{B}_2}(B \cap C) = \varphi^H_{\mathbb{B}_1\cup\mathbb{B}_2}\, \mathit{length}(B \cap C) \\ \text{and} \quad & \Phi^G_{\mathbb{B}_1\cup\mathbb{B}_2}(B \cap C) = \varphi^G_{\mathbb{B}_1\cup\mathbb{B}_2}\, \mathit{length}(B \cap C), \end{aligned}$$

where

$$\begin{aligned} \varphi^H_{\mathbb{B}_1\cup\mathbb{B}_2} &= -\frac{4\pi r}{\mathit{length}(C)}\, \mathit{arcsin}\left(\frac{A_1A_2}{2r}\right)\sqrt{1 - \left(\frac{A_1A_2}{2r}\right)^2} \\ \varphi^G_{\mathbb{B}_1\cup\mathbb{B}_2} &= -\frac{2\pi A_1A_2}{r\ \mathit{length}(C)}. \end{aligned}$$

iii) Let $\mathbb{B}_1$, $\mathbb{B}_2$ and $\mathbb{B}_3$ be three intersecting balls of $\mathbb{R}^3$, of radius r, of centers A_1, A_2 and A_3, and $p \in \partial\mathbb{B}_1 \cap \partial\mathbb{B}_2 \cap \partial\mathbb{B}_3$. Then the curvature measures of $\mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3$ above p are given by

$$\Phi^H_{\mathbb{B}_1\cup\mathbb{B}_2\cup\mathbb{B}_3}(\{p\}) = 0$$

and $\Phi^G_{\mathbb{B}_1\cup\mathbb{B}_2\cup\mathbb{B}_3}(\{p\}) = \varphi^G_{\mathbb{B}_1\cup\mathbb{B}_2\cup\mathbb{B}_3}$ is equal to

$$4\ \arctan\sqrt{\tan\left(\frac{\sigma}{2}\right)\tan\left(\frac{\sigma-\alpha_{1,2}}{2}\right)\tan\left(\frac{\sigma-\alpha_{2,3}}{2}\right)\tan\left(\frac{\sigma-\alpha_{1,3}}{2}\right)},$$

where

$$\alpha_{i,j} = \angle\left(\overrightarrow{pA_i}, \overrightarrow{pA_j}\right) = 2\ arcsin\left(\frac{A_iA_j}{2r}\right)$$
$$\text{and}\quad \sigma = \frac{\alpha_{1,2}+\alpha_{2,3}+\alpha_{3,1}}{2}.$$

The proof of Proposition 1 is given in Section 6.

4.3. Computing the curvatures for Lipschitz functions

Let now f be a Lipschitz function on $\mathbb{R}^d$. To compute the curvature measures $\Phi^G_{K_r}(f)$ and $\Phi^H_{K_r}(f)$ of the union of balls K_r, we triangulate ∂K_r. More precisely, we triangulate every face C of ∂K_r into "small" spherical patches of diameter less than η. We then approximate f by a function $\tilde{f}$ that is constant equal to k_i on each "small" patch Δ_i and that coincides with f in at least one point of each patch. We then calculate the isotropic curvature measures with $\tilde{f}$ "above the face" C by using Proposition 1:

$$\Phi^G_{K_r}(\tilde{f}\mathbf{1}_C) = \sum_i k_i \Phi^G_{K_r}(\Delta_i) \quad \textit{and} \quad \Phi^H_{K_r}(\tilde{f}\mathbf{1}_C) = \sum_i k_i \Phi^H_{K_r}(\Delta_i)$$

Similarly, by using Proposition 1, we calculate the isotropic curvature measures above the "spherical edges" with a function $\tilde{f}$ piecewise constant on "small" edges of lengths less than η. Finally, we calculate exactly the isotropic curvature measures "above" the vertices by using Proposition 1. A simple calculation shows that the numerical error done by approximating f by a piecewise constant function $\tilde{f}$ is then given by:

$$\max\left(|\Phi^G_{K_r}(f) - \Phi^G_{K_r}(\tilde{f})|, |\Phi^H_{K_r}(f) - \Phi^H_{K_r}(\tilde{f})|\right) \le Lip(f)\, k\, \eta,$$

where k is a constant depending only on the "size" of the support of f and on K_r.

4.4. Experimental results

In the figures 4, 5 and 6 below, the curvatures have been computed using the following algorithm:

Input: a 3D point cloud K, a radius r and two values $0 < r_1 < r_2$
Output: an estimated curvature value on each vertex of ∂K_r

1. Compute ∂K_r
2. For each cell C (faces, edges, vertices) of ∂K_r compute $\Phi^G_{K_r}(C)$ and $\Phi^H_{K_r}(C)$
3. For each vertex V of ∂K_r, the estimated Gauss and mean curvature values at C are $\Phi^G_{K_r}(f_V)$ and $\Phi^H_{K_r}(f_V)$ where $f_V(x) = 1$ if $\|x-V\| \le r_1$, $f_V(x) = 0$ if $\|x-V\| > r_2$ and $f_V(x) = 1 - \frac{\|x-V\|-r_1}{r_2-r_1}$ otherwise.

The curvatures are then represented on the boundary of the α-shape (for $\alpha = r$) of the point clouds where each triangle is colored according to the curvature value of its corresponding vertex in ∂K_r and to the colorbar on the right of Figure 4. Note that the color values are different for the different examples (since the extrema values are different).

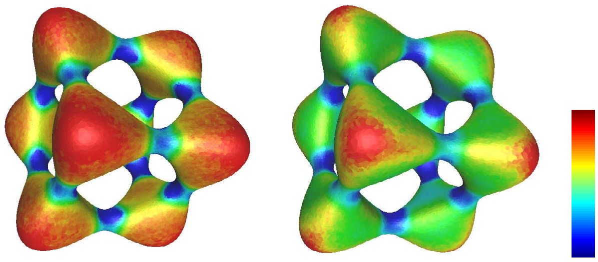

Figure 4: *The Gauss (left) and mean (right) curvatures computed on the offset of a point cloud sampled around a smooth surface. The colors are related to the values of the curvature according to the colorbar on the right, the blue color corresponding to the lowest values.*

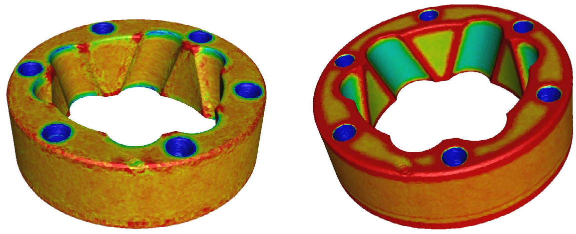

Figure 5: *The Gauss (left) and mean (right) curvatures computed on the offset of a point cloud sampled around a non smooth mechanical part (model is provided courtesy of INRIA and ISIT by the AIM@SHAPE Shape Repository). One clearly sees that the mean curvature detects all the sharp edges as highly curved parts while the Gauss curvatures only detects the curved sharp edges. Notice that the positively curved corners are well-detected by the Gauss curvature.*

Remark 1 This algorithm can be easily adapted to calculate the anisotropic curvature measures for a finite set of points (see [CCLT08]). In particular, this allows to estimate the principal curvatures and principal directions from a cloud of points.

Our program has not been optimized and the following details are just given for an indication (on a laptop with 3.5 GiB Memory, Processor Intel(R) Core(TM) 2 Duo CPU P8600 @ 2.4 GHz). We indicate the number of points, the time for constructing the half-edge data structure for the boundary of the union of balls and the time to compute the Gaussian and mean curvature measures.

	♯ Points	structure	Curvatures
Tangle Cube	82036	242s	63 s
Horse	59550	174 s	79s
Rolling stage	132230	370 s	150 s

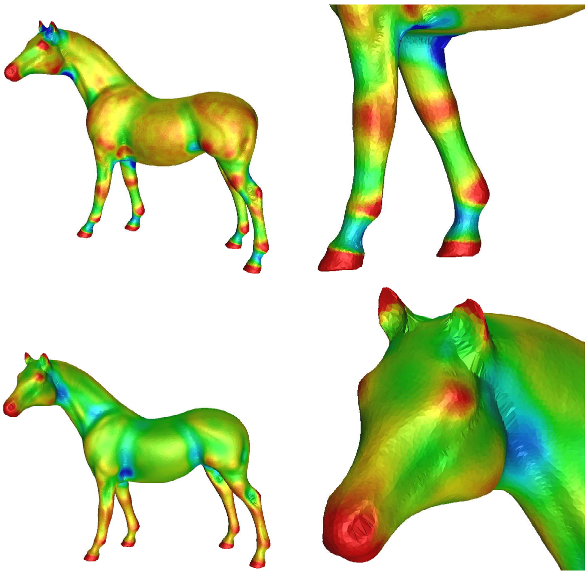

Figure 6: *The Gauss (upper row) and mean (lower row) curvatures computed on the offset of a point cloud sampled around a horse model.*

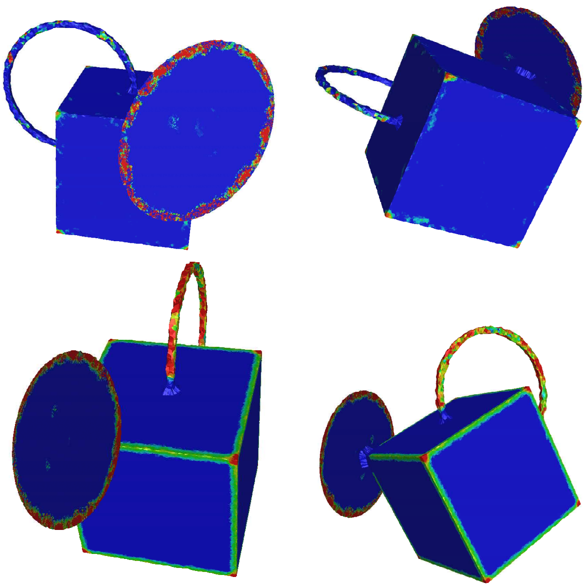

Figure 7: *The Gauss (upper row) and mean (lower row) curvatures computed on the offset of a point cloud sampled around a non-manifold set union of a cube with a disc and a circle. As expected, the vertices and the boundary of the disc have a large Gaussian curvature.*

5. Sketch of proof of Theorem 3

Due to space limitation, we give here the proof of a slightly weaker result: the constant $k(r,\mu,d,f)$ of Theorem 3 is replaced by a constant $k(r,\mu,d,\mathcal{N})$, where $\mathcal{N}$ is the covering number of the offset K_r.

In the following, we use classical notions concerning Hausdorff measures and rectifiable currents (see for instance [Mor87] for more details). The proof also strongly relies on the notion of double offset and on the notion of normal cycle of a set of positive reach that we define now. Let V be a set with positive reach. We define the set:

$$S(V) = \{(p,n) \in \mathbb{R}^d \times \mathbb{S}^{d-1},\ p \in \partial V \text{ and } n \in NC(p)\},$$

where $NC(p) = \{n \in \mathbb{S}^{d-1}, \forall x \in V\ n.\overrightarrow{px} \leq 0\}$ is the normal cone of V at p. One can show that $S(V)$ is a Lipschitz (d-1)-manifold. The normal cycle $N(V)$ of V is then by definition [Fu89] the *(d-1)-current* on $\mathbb{R}^d \times \mathbb{R}^d$ associated with the manifold $S(V)$. We recall that currents are linear forms on the space of differential forms. In our case, this linear form is defined for every (d-1)-differential form ω by:

$$N(V)(\omega) = \int_{S(V)} \omega.$$

The normal cycle contains in fact all the curvature information and allows to define the curvature measures (see [CM03, CM07, Fu89]). For example, the mean curvature measure $\phi_V^H(f)$ in dimension 3 is given by $N(V)(\overline{f}\omega^H)$ where $\overline{f}(p,n) = f(p)$ and ω^H is a particular 2-differential form on $\mathbb{R}^3 \times \mathbb{R}^3$, that does not depend on V.

Our proof consists in comparing the normal cycles of the complements of the offsets of compact sets K and K'. These are well-defined thanks to the following result obtained in [CCLT07]. For $0 < t < r$, the (r,t)-double offset $K_{r,t}$ of K is the set defined by $K_{r,t} = \overline{(K_r)^c}_t$.

Theorem 5 (Double offset theorem) Let K be a compact set with μ-reach greater than r. Then, $\overline{K_r^c}$ has reach at least μr and if $t < \mu r$, $\partial K_{r,t}$ is a smooth $\mathcal{C}^{1,1}$-hypersurface. Moreover the reach of $\partial K_{r,t}$ is greater than $\min(t, \mu r - t)$.

The proof can now be divided into three steps: in the first step, we show that the problem can be carried onto the double offsets (the advantage being that they are smooth); in a second step, we compare the normal cycles of the double offsets; in the last step, we combine Step 1 and Step 2 to show that the curvature measures of the two offsets are close. Let K and K' be two compact sets with positive μ-reach that satisfy all the assumptions of Theorem 3.

Step 1: Carrying the problem to the double offsets

First note that $\overline{K_r^c}$ and $\overline{K_r'^c}$ have positive reach. We introduce the map:

$$\begin{array}{rccc} F_{-t}: & \mathbb{R}^d \times \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \times \mathbb{R}^d \\ & (p,n) & \longmapsto & (p-tn,n) \end{array}.$$

If V is any compact set with positive reach, the map F_{-t} induces naturally a one-to-one correspondence between the support of the normal cycle of the offset V_t and the support of the normal cycle of V. In particular, this map allows to send simultaneously the normal cycles of $K_{r,t}$ and $K'_{r,t}$ to respectively the normal cycles of K_r and K'_r. More precisely, one has:

$$N(\overline{K_r{}^c}) - N(\overline{K_r'^c}) = F_{-t\sharp}(N(K_{r,t}) - N(K_{r,t})),$$

where $F_{-t\sharp}$ is the map induced by F_{-t} at the level of currents. Therefore, as we will see later (Step 3), in order to compare the normal cycles of $\overline{K_r{}^c}$ and $\overline{K_r'^c}$, it is sufficient to compare the normal cycles of the double offsets $K_{r,t}$ and $K'_{r,t}$.

Step 2: Comparison of the normal cycles of the double offsets

In order to compare the normal cycles of $K_{r,t}$ and $K'_{r,t}$, we first need to compare their supports in $\mathbb{R}^d \times \mathbb{R}^d$ (that is, the associated manifolds), which is the purpose of the following lemmas.

Lemma 1 Let K and K' be two compact sets of $\mathbb{R}^d$ whose μ-reaches are greater than r. Then for every $t \in (0, r\mu)$, we have:

$$d_H(K_{r,t}, K'_{r,t}) \leq \frac{\varepsilon}{\mu} \quad \text{and} \quad d_H(\partial K_{r,t}, \partial K'_{r,t}) \leq \frac{\varepsilon}{\mu}.$$

Proof First remark that if we take two compact sets A and B with $\tilde{\mu}$-reach greater than r, such that $d_H(A,B) \leq \tilde{\varepsilon}$, one has:

$$d_H\left(\overline{(A_r)^c}, \overline{(B_r)^c}\right) \leq \frac{\tilde{\varepsilon}}{\tilde{\mu}} \tag{2}$$

Indeed, let $x \in \overline{A_r}^c$. Then $d(x,A) \geq r$ and $d(x,B) \geq r - \tilde{\varepsilon}$. Let $s \mapsto \sigma(s)$, $\sigma(0) = x$ be the trajectory of ∇_B issued from x and parametrized by arc-length. While $\sigma(s) \in B_r$, we have [Lie04]:

$$d_B(\sigma(s)) = d_B(x) + \int_0^s \|\nabla_B(\sigma(s))\| ds \geq r - \tilde{\varepsilon} + s\tilde{\mu}.$$

We then have $\sigma(s) \in \overline{(B_r)^c}$ for $s \geq \frac{\tilde{\varepsilon}}{\tilde{\mu}}$. As a consequence, there exists $x' \in \overline{(B_r)^c}$ such that $d(x,x') \leq \frac{\tilde{\varepsilon}}{\tilde{\mu}}$. We apply Equation (2) with $A = K$ and $B = K'$ and we get:

$$d_H\left(\overline{(K_r)^c}, \overline{(K'_r)^c}\right) \leq \frac{\varepsilon}{\mu} \tag{3}$$

We apply again Equation (2) with $A = \overline{(K_r)^c}$ and $B = \overline{(K'_r)^c}$ with $\tilde{\varepsilon} = \frac{\varepsilon}{\mu}$ and $\tilde{\mu} = 1$:

$$d_H(\overline{K_{r,t}{}^c}, \overline{K'_{r,t}{}^c}) \leq \frac{\varepsilon}{\mu}.$$

Remark also that for any compact sets A and B, one has $d_H(A_t, B_t) \leq d_H(A,B)$. Therefore, by Equation (3) one has:

$$d_H(K_{r,t}, K'_{r,t}) \leq \frac{\varepsilon}{\mu}.$$

The two last equations imply that

$$d_H(\partial K_{r,t}, \partial K'_{r,t}) \leq \frac{\varepsilon}{\mu}.$$

Indeed, let $x \in \partial K_{r,t}$. Then x is at a distance less than $\frac{\varepsilon}{\mu}$ from $K'_{r,t}$ and $\overline{K'_{r,t}{}^c}$. Then, there exists $y \in K'_{r,t}$ and $z \in \overline{K'_{r,t}{}^c}$ such that $xy \leq \frac{\varepsilon}{\mu}$ and $xz \leq \frac{\varepsilon}{\mu}$. Since the line-segment $[yz]$ intersects $\partial K_{r,t}$, there exists $x' \in \partial K'_{r,t}$ such that $xx' \leq \frac{\varepsilon}{\mu}$. $\square$

Lemma 2 Let X and X' be two compact sets of $\mathbb{R}^d$ with reaches greater than $R > 0$. Let $t \leq \frac{R}{2}$ and $\varepsilon = d_H(X, X')$. If $\varepsilon \leq \frac{t}{2}$, then for any x at a distance t from X, we have:

$$2 \sin \frac{\angle(\nabla_X(x), \nabla_{X'}(x))}{2} \leq 30\sqrt{\frac{\varepsilon}{t}}.$$

Proof For ρ such that $0 < \rho < t$, we denote by $G_X(x,\rho)$, as in section 5 of [CCL07], the convex hull for every $y \in B(x,\rho)$ of all the "classical" gradients $\nabla_X(y)$ of the distance function d_X. By using Theorem 5.6 in [CCL07], we have that:

$$\nabla_{X'}(x) \in G_X(x,\rho)_{\frac{\rho}{2d(x,X')} + \frac{2\varepsilon}{\rho}}. \tag{4}$$

On another hand we know ([Fed59] page 435) that the projection map π_X on X is $\frac{R}{R-(t+\rho)}$-Lipschitz for points at distance less than $(t+\rho)$ from X. Then, for $y \in B(x,\rho)$ one has $d(x,y) \leq \rho$ and $d(\pi_X(x), \pi_X(y)) \leq \frac{R\rho}{R-(t+\rho)}$ and:

$$\|\overrightarrow{y\pi_X(y)} - \overrightarrow{x\pi_X(x)}\| \leq \rho + \frac{R\rho}{R-(t+\rho)}.$$

Using the fact that $\nabla_X(z) = \frac{-1}{\|\overrightarrow{z\pi_X(z)}\|}\overrightarrow{z\pi_X(z)}$ one get, for $y \in B(x,\rho)$:

$$\|\nabla_X(y) - \nabla_X(x)\| \leq \frac{1}{t-\rho}\left(\rho + \frac{R\rho}{R-(t+\rho)}\right).$$

This and Equation(4) gives:

$$2 \sin \frac{\angle \nabla_X(x), \nabla_{X'}(x)}{2} = \|\nabla_X(x) - \nabla_{X'}(x)\|$$

$$\leq \frac{\rho}{t-\rho}\left(1 + \frac{R}{R-(t+\rho)}\right) + \frac{\rho}{2(t-\varepsilon)} + \frac{2\varepsilon}{\rho}.$$

Taking $\rho = \sqrt{\varepsilon t}$ and $\varepsilon \leq \frac{t}{2}$ one gets:

$$2 \sin \frac{\angle \nabla_X(x), \nabla_{X'}(x)}{2} \leq$$

$$\left(\frac{1}{1-\frac{\sqrt{2}}{2}}\left(1 + \frac{1}{1-\frac{1}{2}\left(1+\frac{\sqrt{2}}{2}\right)}\right) + 3\right)\sqrt{\frac{\varepsilon}{t}} \leq 30\sqrt{\frac{\varepsilon}{t}}.$$

$\square$

Applying the previous lemma with $X = \overline{K_r^c}$ and $X' = \overline{K'^c_r}$ shows that the difference between the normals of $\partial K_{r,t}$ and $\partial K'_{r,t}$ is bounded by $30\sqrt{\varepsilon/(\mu t)}$. Combining the last two results, we get that the supports of $N(K_{r,t})$ and $N(K'_{r,t})$ are close to each other. This allows to show that $N(K_{r,t})$ and $N(K'_{r,t})$ are close. More precisely:

Lemma 3 We can write

$$N(K_{r,t}) - N(K'_{r,t}) = \partial E, \tag{5}$$

where ∂E is the boundary of a particular d-current E whose mass $M(E)$ satisfies:

$$M(E) \leq \mathcal{H}^{d-1}(\partial K'_{r,t}) \left(1+\frac{1}{t^2}\right)^{\frac{d-1}{2}} \left[1+\left(\frac{1+t^2}{t\left(t-\frac{\varepsilon}{\mu}\right)^2}\right)^{\frac{d-1}{2}}\right] \sqrt{\left(\frac{\varepsilon}{\mu}\right)^2+900\frac{\varepsilon}{\mu t}}.$$

Here $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure (*i.e.* the k-volume), and the mass is the corresponding concept for currents [Mor87].

Proof The current E is built as follows. First, the closest point projection defines a homeomorphism from $\partial K'_{r,t}$ to $\partial K_{r,t}$, because the Hausdorff distance between these two manifolds is less than their reach. This homeomorphism can be lifted to a homeomorphism ψ between the supports of the corresponding normal cycles. Current E is then defined as the volume swept by the linear interpolation between the identity of the support of $N(K'_{r,t})$ and the latter homeomorphism. Formally, we define h as the affine homotopy between ψ and the identity

$$h: \begin{array}{ccc} [0,1]\times spt(N(K'_{r,t})) & \rightarrow & \mathbb{R}^d\times\mathbb{R}^d \\ (t,x) & \mapsto & (1-t)x+t\psi(x) \end{array}.$$

and let $E = h_\sharp([0,1]\times N(K'_{r,t}))$. By Federer ([Fed59], 4.1.9 page 364) or Fanghua ([FX02] page 187), we get:

$$M\left(h_\sharp\left([0,1]\times N(K'_{r,t})\right)\right) \leq$$

$$M\left(N(K'_{r,t})\right) \sup_{spt(N(K'_{r,t}))} |\psi - id| \sup_{spt(N(K'_{r,t}))} |1+J_{d-1}(\psi)|$$

where $J_{d-1}(\psi)$ stands for the (d-1)-dimensional jacobian (as defined in [Mor87], page 24-25). By Lemma 1, the space component of $\psi - id$ is less than $\frac{\varepsilon}{\mu}$. By Lemma 2, the normal component of $\psi - id$ is less than $30\sqrt{\frac{\varepsilon}{\mu t}}$. Thus

$$\sup_{spt(N(K'_{r,t}))} |\psi - id| \leq \sqrt{\left(\frac{\varepsilon}{\mu}\right)^2+900\frac{\varepsilon}{\mu t}}.$$

Assuming t does not exceed the reach of $\partial K_{r,t}$, we have that the jacobian of the space component of ψ (*i.e.* the projection) is bounded by $\left(\frac{t}{t-\frac{\varepsilon}{\mu}}\right)^{d-1}$ (see [Fed59]). Under the same assumptions, the jacobian of the Gauss map of $K_{r,t}$ is upper bounded by $\left(\frac{1}{t}\right)^{d-1}$. Thus, the jacobian of the normal component of ψ is upper bounded by the product $\left(\frac{1}{t}\,\frac{t}{t-\frac{\varepsilon}{\mu}}\right)^{d-1}$. We then have

$$\sup_{spt(N(K'_{r,t}))} |1+J_{d-1}(\psi)| \leq 1+\left(\frac{1+t^2}{\left(t-\frac{\varepsilon}{\mu}\right)^2}\right)^{\frac{d-1}{2}}.$$

Using similar arguments, it is easily shown that $M(N(K'_{r,t}))$ is bounded by $\mathcal{H}^{d-1}(\partial K'_{r,t})\left(1+\frac{1}{t^2}\right)^{\frac{d-1}{2}}$, which concludes the proof of the lemma. □

Step 3
In the following, we only consider the mean curvature in $\mathbb{R}^3$, the proof for the other curvature measures being similar. By combining previous equations, one has:

$$\begin{aligned} \phi^H_{K_r{}^c}(f) - \phi^H_{K'^c_r}(f) &= N(\overline{K_r{}^c})(\bar f\omega^H) - N(\overline{K'^c_r})(\bar f\omega^H) \\ &= F_{-t\sharp}\partial E(\bar f\omega^H). \end{aligned}$$

It is obvious that F_{-t} is $\sqrt{1+t^2}$-Lipschitz. Then, one has $|F_{-t\sharp}\partial E(\bar f\omega^H)| \leq \left(1+t^2\right)^{\frac{d-1}{2}} |\partial E(\bar f\omega^H)|$. Therefore, since $\|\omega^H\|_\infty \leq 2$, $\|d\omega^H\|_\infty \leq 4$, one has $\|d(\bar f\omega^H)\|_\infty = \|d\bar f\wedge\omega^H + \bar f\wedge d\omega^H\|_\infty \leq 6\ \sup(Lip(\bar f),1)$, and then by Stokes theorem:

$$\begin{aligned} & |F_{-t\sharp}\partial E(\bar f\omega^H)| \\ \leq\ & \left(1+t^2\right)^{\frac{d-1}{2}} E(d(\bar f\omega^H)) \\ \leq\ & 6\left(1+t^2\right)^{\frac{d-1}{2}} M(E)\sup(Lip(\bar f),1). \end{aligned}$$

Since $Lip(\bar f) = Lip(f)$, one gets by taking $t = \mu r/2$:

$$\begin{aligned} & |\phi^H_{K_r{}^c}(f) - \phi^H_{K'^c_r}(f)| \\ \leq\ & k(r,\mu,d)\ \sup(Lip(f),1)\ \mathcal{H}^{d-1}(\partial K'_{r,t})\ \sqrt{\varepsilon}, \end{aligned}$$

where $k(r,\mu,d)$ only depends on r, μ and d. Now, since $\phi^H_{K_r}(f) = -\phi^H_{K_r^c}(f)$ and $\phi^H_{K'_r}(f) = -\phi^H_{K'^c_r}(f)$, the previous inequality still hold for K_r and K'_r. Using the bound on $\mathcal{H}^{d-1}(\partial K'_{r,t})$ in terms of covering number of K_r [CCM07] yields almost the desired bound. The only difference is that the obtained bound does not take advantage of the fact that f might have a small support. In [CCLT08], the proof is done locally in a Borel set that contains the support of the function f: this gives a better bound with the covering number of an offset of the support of f; however, the proof is more complicated since a second current (related to the boundary of $B\cap\partial K'_{r,t}$, where B is the chosen Borel set) appears in Equation (5).

6. Proof of Proposition 1

We first recall that the curvature measures can be computed directly by using the normal cycle. In dimension 3, the mean curvature measure and the Gaussian curvature measure are given by $\phi^H_V(f) = N(V)(\bar f\omega^H)$ and $\phi^G_V(f) = N(V)(\bar f\omega^G)$ where $\bar f(p,n) = f(p)$, ω^G and ω^H are two particular 2-differential forms on $\mathbb{R}^3\times\mathbb{R}^3$, that do not depend on V (see [Coh04, CM03] for more details).

$i)$. Since $\partial\mathbb{B}$ is a smooth surface, one has

$$\Phi^H_{\mathbb{B}}(B) = \int_{B\cap\partial\mathbb{B}} H(p)dp = \frac{2}{r}\,Area(B\cap\partial\mathbb{B})$$

and

$$\Phi^G_{\mathbb{B}}(B) = \int_{B\cap\partial\mathbb{B}} G(p)dp = \frac{1}{r^2}\, Area(B\cap\partial\mathbb{B}),$$

where $G(p)$ is the Gaussian curvature and $H(p)$ is the mean curvature of $\mathbb{B}$ at p.

ii) Since the curvature measures are additive [Coh04, CM03] and since C is one-dimensional, one has:

$$\begin{aligned} & \Phi^H_{\mathbb{B}_1\cup\mathbb{B}_2}(B\cap C) \\ = & \Phi^H_{\mathbb{B}_1}(B\cap C) + \Phi^H_{\mathbb{B}_2}(B\cap C) - \Phi^H_{\mathbb{B}_1\cap\mathbb{B}_2}(B\cap C) \\ = & -\Phi^H_{\mathbb{B}_1\cap\mathbb{B}_2}(B\cap C). \end{aligned}$$

We now need to describe the normal cycle of $\mathbb{B}_1\cap\mathbb{B}_2$ “above" $C\cap B$. Since $\mathbb{B}_1\cap\mathbb{B}_2$ is convex, its normal cycle “above" $C\cap B$ is just the 2-current defined by integration over the set $S_C(\mathbb{B}_1\cap\mathbb{B}_2)$ equal to:

$$\left\{(m,\xi),\ m\in C\cap B,\ \|\xi\| = 1 \text{ and } \forall q\in\mathbb{B}_1\cap\mathbb{B}_2\ \overrightarrow{mq}.\xi\le 0\right\}.$$

Let $\alpha = arcsin\left(\frac{A_1A_2}{2r}\right)$. In an suitable frame, the set $S_C(\mathbb{B}_1\cap\mathbb{B}_2)$ can be parametrized by $f: [0,\beta]\times[-\alpha,\alpha]\to\mathbb{R}^3\times\mathbb{S}^2$ defined by:

$$f(u,v) = \left(\begin{pmatrix} r\cos\alpha\cos u \\ r\cos\alpha\sin u \\ 0\end{pmatrix}, \sin(v)\begin{pmatrix}0\\0\\1\end{pmatrix} + \cos(v)\begin{pmatrix}\cos u\\ \sin u\\ 0\end{pmatrix}\right)$$

Let $(m,\xi) = f(u,v)\in S(\mathbb{B}_1\cap\mathbb{B}_2)$. We put

$$e_1 = \begin{pmatrix}-\sin u\\ \cos u\\ 0\end{pmatrix},\ e_2 = \cos(v)\begin{pmatrix}0\\0\\1\end{pmatrix} - \sin(v)\begin{pmatrix}\cos u\\ \sin u\\ 0\end{pmatrix}.$$

Then (e_1,e_2,ξ) is a direct orthonormal basis of $\mathbb{R}^3$. We put $\varepsilon_1 = (e_1,0)$, $\varepsilon_2 = (e_2,0)$, $\tilde{\varepsilon_1} = (0,e_1)$ and $\tilde{\varepsilon_2} = (0,e_2)$. One has [Coh04, CM03]:

$$\omega^H = \varepsilon_1\wedge\tilde{\varepsilon_2} + \tilde{\varepsilon_1}\wedge\varepsilon_2 \quad\text{and}\quad \omega^G = \tilde{\varepsilon_1}\wedge\tilde{\varepsilon_2}.$$

Furthermore, one has:

$$\frac{\partial f}{\partial u}(u,v) = (r\cos\alpha\, e_1, \cos v\, e_1) \quad\text{and}\quad \frac{\partial f}{\partial v}(u,v) = (0,e_2).$$

We then have:

$$f^*\omega^H((1,0),(0,1)) = \omega^H\left(\frac{\partial f}{\partial u}(u,v), \frac{\partial f}{\partial v}(u,v)\right) = r\cos\alpha,$$

$$f^*\omega^G((1,0),(0,1)) = \cos v.$$

Then

$$\Phi^H_{\mathbb{B}_1\cup\mathbb{B}_2}(C) = -\int_0^\beta\int_{-\alpha}^{\alpha} r\cos\alpha\, dudv = -2\beta r\alpha\cos\alpha.$$

Similarly, one has:

$$\Phi^G_{\mathbb{B}_1\cup\mathbb{B}_2}(C) = -\int_0^\beta\int_{-\alpha}^{\alpha}\cos v\, dudv = -2\beta\sin\alpha = -\frac{\beta\, A_1A_2}{r}.$$

iii) Since the 2-form ω^H is mixed and since the support of the normal cycle of $\mathbb{B}_1\cup\mathbb{B}_2\cup\mathbb{B}_3$ “above" p lies in $\{p\}\times\mathbb{R}^3$, one has $\Phi^H_{\mathbb{B}_1\cup\mathbb{B}_2}(\{p\}) = 0$. Since the normal cycle is additive and since $\{p\}$ is 0-dimensional, one has:

$$\Phi^G_{\mathbb{B}_1\cup\mathbb{B}_2\cup\mathbb{B}_3}(\{p\})\Phi^G_{\mathbb{B}_1\cap\mathbb{B}_2\cap\mathbb{B}_3}(\{p\}).$$

$\Phi^G_{\mathbb{B}_1\cap\mathbb{B}_2\cap\mathbb{B}_3}(\{p\})$ is just the area of the set:

$$S(p) = \left\{(p,\xi),\ \|\xi\| = 1 \text{ and } \forall q\in\mathbb{B}_1\cap\mathbb{B}_2\cap\mathbb{B}_3\ \overrightarrow{pq}.\xi\le 0\right\}.$$

The set $S(p)$ is a spherical triangle whose area is given by the following formulae (see [Ber87] page 289):

$$4\arctan\sqrt{\tan\left(\frac{\sigma}{2}\right)\tan\left(\frac{\sigma-\alpha_{1,2}}{2}\right)\tan\left(\frac{\sigma-\alpha_{2,3}}{2}\right)\tan\left(\frac{\sigma-\alpha_{1,3}}{2}\right)}.$$

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8. Conclusion and future work

We have introduced the first notion of anisotropic curvature measure which is Hausdorff stable and applies to a large class of objects, including non manifold and non smooth sets as well as point clouds. Indeed, it is enough to require that some offset has a positive μ-reach, or, equivalently that the critical function of the set is greater than some positive number μ on some interval.

In light of these results, one can introduce a scale dependent variant of the normal cycle. We say that a compact set $K\subset\mathbb{R}^d$ satisfies the (P_α) property if its critical function is greater than some positive μ on an open interval containing $\alpha > 0$. For such K, we define the α-normal cycle as the rectifiable $(d-1)$-current of $\mathbb{R}^d\times\mathbb{S}^{d-1}$, $N_\alpha(K) = F_{-\alpha\sharp}N(K_\alpha)$. The effect of push-forward $F_{-\alpha\sharp}$ is to move the support of the normal cycle closer to K: in simple cases (but not in general) $N_\alpha(K)$ is equal to the normal cycle of the double offset $\overline{K_{\alpha,\alpha}{}^c}$. This current captures in some sense the curvature information at scale α and has two nice properties. First, it coincides with the usual normal cycle for sets with positive reach, more precisely, if a compact set K has a reach greater than α, then $N_\alpha(K) = N(K)$. Second it is Hausdorff stable, more precisely, if K satisfies (P_α), then there are constants C and $\varepsilon_0 > 0$ depending only on K such that if K' is a compact set such that $d_H(K,K') < \varepsilon\le\varepsilon_0$ then $N_\alpha(K)$ and $N_\alpha(K')$ differ by less than $C\sqrt{\varepsilon}$ in the so-called *flat norm* (see for example [Mor87, FX02] for a definition), which implies that the associated curvature measures are also $O\left(\sqrt{\varepsilon}\right)$ close.

We think of several possible future directions. First we think of extending our paradigm to the measure of higher order quantities such as torsion of curves or curvature derivatives. A possible track to define a stable measure for these higher order quantities is to integrate the gradient of a Gaussian function again the normal cycle: this would retrieve informations about the gradient of curvature measure. Associated stability results require more investigations.

Furthermore, although the practical setting is beyond the scope of this paper, we obtain promising results for the estimation of the curvature measures from a noisy point cloud sample and we expect potential applications in the context of point cloud modeling.

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