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Computational Geometry 40 (2008) 156–170

Computational Geometry
Theory and Applications

www.elsevier.com/locate/comgeo

Smooth manifold reconstruction from noisy and non-uniform approximation with guarantees

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Received 21 November 2006; received in revised form 29 May 2007; accepted 3 July 2007

Available online 14 July 2007

Communicated by R. Fleischer

Abstract

Given a smooth compact codimension one submanifold S of $\mathbb{R}^k$ and a compact approximation K of S, we prove that it is possible to reconstruct S and to approximate the medial axis of S with topological guarantees using unions of balls centered on K. We consider two notions of noisy-approximation that generalize sampling conditions introduced by Amenta et al. and Dey et al. The first one generalizes uniform sampling based on the minimum value of the local feature size. The second one generalizes non-uniform sampling based on the local feature size function of S. The density and noise of the approximation are bounded by a constant times the local feature size function. This constant does not depend on the surface S. Our results are based upon critical point theory for distance functions. For the two approximation conditions, we prove that the connected components of the boundary of unions of balls centered on K are isotopic to S. We consider using both balls of uniform radius and also balls whose radii vary with the local level of detail of the manifold. For the first approximation condition, we prove that a subset (known as the λ-medial axis) of the medial axis of $\mathbb{R}^k \setminus K$ is homotopy equivalent to the medial axis of S. Our results generalize to smooth compact submanifolds S of $\mathbb{R}^k$ of any codimension.

Keywords: Surface reconstruction; Shape approximation; Distance function; Medial axis

1. Introduction and related works

1.1. Motivation and previous work

Algorithms for surface reconstruction from point samples are required in many application areas such as reverse engineering, medical imaging or, more generally, each time a geometric model of an object must be built from (finite) measures. In last years, many such algorithms have been designed that, starting from a set of 3D point samples, build a polyhedral approximation of the sampled object.

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0925-7721/$ – see front matter
doi:10.1016/j.comgeo.2007.07.001

In the following of a paper of Amenta et al. [1], a family of reconstruction algorithms that provide topological guarantees have been designed, the most recent of them allowing to deal with noisy samples [13,21]. Here topological guarantee means that, under some assumptions on the sampled surface S and the sampling, the algorithm builds a geometric model that is homeomorphic or even isotopic to S. However, in these works, the proofs of the topological correctness are deeply intricated with the details of the algorithms or with some specificities of 3D Voronoi diagrams. The *poles* introduced in [1] play a central role: they allow to approximate the medial axis and the normals of the surface S from the Voronoi diagram of the sample. The relations between Voronoi diagrams, poles and medial axes show that capturing the topology of the surface or capturing the topology of its medial axis are strongly related problems. This suggests that these topological correctness proofs could be better understood in a more general mathematical framework. The expected outcomes of this framework are conditions and associated algorithms able to produce a topologically correct approximation of an object given partial and inaccurate geometrical approximations, not necessarily by finite sample points, in any dimension and for non-smooth objects. Based upon the critical point theory for distance functions to compact sets, this point of view has already brought some results on the medial axis topology and approximation [5,20] and the computation of homotopy and homology groups of compact sets [6]. More recently, this approach has allowed to propose sampling conditions guaranteeing a topologically correct reconstruction of non-smooth objects in any dimension [4]. Beside this result for non-smooth objects, the smooth case deserves a specific study because it allows simpler sampling conditions with better constants.

A recent work of S. Smale et al. [22] considers the question for smooth submanifolds of any dimension in Euclidean spaces. They introduce a uniform sampling condition related to the *reach* that is the minimum distance between the manifold and its medial axis. Based on this sampling condition, they show that an offset of the sampling bears the homotopy type of the sampled manifold.

1.2. Contribution

This paper presents some results obtained as a continuation of [22] in the light of our mathematical framework. First, under similar uniform sampling conditions, we extend the result of [22] to get isotopic approximations of hypersurfaces as well as a reconstruction of the medial axis of the manifold with the right homotopy type. Our uniform sampling condition is merely a ratio between the reach of the sampled manifold S and the Hausdorff distance between the sample and S. This sampling model allows a noise level whose amplitude is of the order of magnitude of the sampling density. It requires no sparsity condition, and allows approximation by any compact sets, such as finite sets of geometric primitives like triangles (polygon soup). Secondly, we extend the results to non-uniform sampling conditions. Our non-uniform sampling condition (see Section 6) is the same as the notion of noisy r-sample without sparsity of [21] generalized to any compact sampling and not restricted to finite set of points. Theorem 6.2 below can be seen as an extension of the class of algorithms initiated by [1], considering a sampling density related to the local feature size, for any dimension of the manifold and the ambient space. The associated proposed algorithm is extremely simple: it consists of taking a union of balls centered on the sampling set. The radii of the balls may freely vary in prescribed intervals depending upon the local feature size of the manifold. From a more practical point of view it merely means computing an alpha-shape [15].

However, the algorithm suggested by Theorem 6.2 requires an oracle: for each sample point we would need a lower bound of the local feature size of the projection of the point on the surface. In fact, usual local feature size based algorithms implicitly assume that one is able to adapt the density and accuracy of the sampling to the local feature size in order to produce a good sampling. So, in practice, ensuring that a sampling is a “good” sampling may require our oracle. Still, if the sampling is assumed good, algorithms described in [2,12] does not require any oracle. We believe that, for exact sampling condition, the oracle information is contained in the poles. Indeed, in presence of noise, a sparsity condition (see [13]) is required in order to still extract some information about “filtered poles”. On another hand, the authors of [21], write that this sparsity condition “does not seems strictly necessary” and they drop it in their “noisy r-sample” condition (similar to ours). In fact the price they have to pay in order to relax the sparsity condition is the need of an oracle to filter the poles. The proposed filtering of the poles requires the knowledge of the reach (denoted $\mathrm{lfs}(S)$ in [21]). They just need a “global oracle” because they use a “noisy r-sample” condition related to the ratio between the minimum and maximum of the local feature size function over the surface (similar to the one in Theorem 6.1). In this case, the non-uniformity of the sampling cannot be fully exploited. In contrast, Theorem 6.2 allows a local sampling density independent on this ratio. Of course, there is no hope to get rid of the oracle if we consider a noisy sampling without any sparsity condition. For example, given points sampled on a surface, if one replaces each point by a dense sampling of a tiny sphere, the relevant topology depends on the scale at which one observes the resulting points cloud. Our oracle plays the role of a local scale parameter. We believe that, for the sake of clarity and deeper understanding, the problem of reconstruction under non-uniform, local feature size related sampling conditions, should be split into two simpler independent problems. First, assuming minimal noise and/or sparsity conditions, how can one derive a lower bound on the local feature size exploiting only the point set. Secondly, starting from the sample point and the oracle, how can one produce a topologically correct reconstruction. Theorem 6.2 answers the second problem in a general setting.

This paper is an extended version of [8]. It is organized as follow. Section 2 gives some definitions and recall results on distance functions and medial axis. In Section 3 one defines uniform noisy sampling and studies distance function to such sampling. Section 4 presents topology guaranteeing algorithms for surface reconstruction with uniform sampling conditions. Section 5 gives results about topology guaranteeing algorithms for Medial axis approximation. Section 6 states result for surface reconstruction with non-uniform sampling conditions.

2. Mathematical preliminaries and distance functions

Throughout the paper, we use the following notations. For any set $X \subset \mathbb{R}^k$, $\overline{X}$, X^c and ∂X denote respectively the closure, the complement and the boundary of X. For any $x \in \mathbb{R}^k$ and any $r > 0$, $\mathbb{B}(x, r)$ is the open ball of center x and radius r. Given two spaces X and Y, two maps $f : X \to Y$ and $g : X \to Y$ are said *homotopic* if there is a continuous map H, $H : [0, 1] \times X \to Y$, such that $\forall x \in X, H(0, x) = f(x)$ and $H(1, x) = g(x)$. X and Y are said *homotopy equivalent* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y (see [19], chap. 0, for more details and examples). Homotopy equivalence between topological sets enforces a one-to-one correspondence between connected components, cycles, holes, tunnels, cavities, or higher dimensional topological features of the two sets, as well as the way these features are related. More precisely, if X and Y have same homotopy type, then their homotopy and homology groups are isomorphic. When $Y \subset X$, one says that Y is a deformation retract of X if one can continuously deform X onto Y i.e. there exists a continuous map $H : [0, 1] \times X \to X$ such that for any $x \in X$, $H(0, x) = x$ and $H(1, x) \in Y$ and for any $y \in Y, t \in [0, 1]$, $H(t, y) = y$. In this case, X and Y are homotopy equivalent.

Two subsets X and Y of $\mathbb{R}^k$ are isotopic if there is a continuous map $F : X \times [0, 1] \to \mathbb{R}^k$ such that $F(., 0)$ is the identity of X, $F(X, 1) = Y$, and for each $t \in [0, 1]$, $F(., t)$ is a homeomorphism onto its image. Notice that isotopy is a stronger condition than homeomorphy.

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^k$ with compact boundary $K = \overline{\mathcal{O}} \cap \mathcal{O}^c$ and let $\mathcal{R}$ be the function defined on $\mathcal{O}$ by $\mathcal{R}(x) = d(x, K)$ for all $x \in \mathcal{O}$. For any point $x \in \mathcal{O}$, we denotes by $\Gamma(x)$ the set of closest boundary points:

$$\Gamma(x) = \{y \in K:\ d(x, y) = d(x, K)\}.$$

Definition 2.1. The medial axis $\mathcal{M}$ of $\mathcal{O}$ is the set of points $x \in \mathcal{O}$ that have at least two closest boundary points:

$$\mathcal{M} = \{x \in \mathcal{O}:\ |\Gamma(x)| \geqslant 2\}.$$

For a compact subset K of $\mathbb{R}^k$, the medial axis $\mathcal{M}(K)$ of K is the medial axis of its complement $\mathbb{R}^k \setminus K$.

In the setting of classical differential calculus, $\mathcal{M}$ is the set of critical points of $\mathcal{R}$: the function $\mathcal{R}$ is differentiable and non-critical on $\mathcal{O} \setminus \overline{\mathcal{M}}$ (see [16]). In a more general setting, one can consider intuitively that a point x is regular for $\mathcal{R}$ if one can find a direction issued from x such that the derivative of $\mathcal{R}$ along this direction is positive. Otherwise, x is said to be critical. Such an intuition coincides with the notion of critical point classically used in non-smooth analysis [11] and in Riemannian geometry (see [10,18]). A formal and rigorous definition is given below.

Definition 2.2. (See [17, p. 360].) A point $x \in \mathcal{O}$ is critical for the distance function $\mathcal{R}$ if and only if it is contained in the convex hull of $\Gamma(x)$.

Remark 1. Notice that such a definition implies that the critical points of $\mathcal{R}$ are contained in the convex hull of K (because the convex hull of $\Gamma(x)$ is contained in the convex hull of K).

Some of the properties of the distance function to a compact set are quite similar to the smooth functions ones. In particular, they satisfy an Isotopy Lemma [17], that we reproduce below. For any $\rho \in \mathbb{R}_+$ one denotes by $\overline{O}_\rho$ the open offset $\mathcal{O}_\rho = \{x \in \mathcal{O}\colon\ \mathcal{R}(x) > \rho\}$ and by $\mathcal{R}^{-1}(\rho) = \{x \in \mathcal{O}\colon\ \mathcal{R}(x) = \rho\}$ the ρ-level set of $\mathcal{R}$.

Proposition 2.3. *If* $0 < \rho_1 < \rho_2$ *are such that* $(\overline{\mathcal{O}}_{\rho_1} \setminus \mathcal{O}_{\rho_2})$ *does not contain any critical point of* $\mathcal{R}$*, then all the levels* $\mathcal{R}^{-1}(\rho)$*,* $\rho \in [\rho_1, \rho_2]$*, are homeomorphic topological manifolds and*

$$\overline{\mathcal{O}}_{\rho_1} \setminus \mathcal{O}_{\rho_2} = \{x \in \mathcal{O}\colon\ \rho_1 \leqslant \mathcal{R}(x) \leqslant \rho_2\}$$

is homeomorphic to $\mathcal{R}^{-1}(\rho_1) \times [\rho_1, \rho_2]$*. As a consequence,* $\mathcal{O}\rho_1$ *and* $\mathcal{O}\rho_2$ *are homeomorphic.*

In the following, we also consider some subset of the medial axis known as λ-medial axis [5]. For any point $x \in \mathcal{O}$ one denotes by $\mathcal{F}(x)$ the radius of the smallest ball containing $\Gamma(x)$. We thus define a function $\mathcal{F} : \mathcal{O} \to \mathbb{R}_+$ which is upper semi-continuous (see [5]) and satisfies $\mathcal{F}(x) \neq 0$ if and only if $x \in \mathcal{M}$. Given a positive real $\lambda > 0$ one defines the λ-medial axis of $\mathcal{O}$ as the closed subset $\mathcal{M}_\lambda$ of $\mathcal{M}$ defined by

$$\mathcal{M}_\lambda = \{x \in \mathcal{O}\colon\ \mathcal{F}(x) \geqslant \lambda\}.$$

Topological properties of the medial axis and its subsets have been studied in [5,9,20] for bounded open sets. In the following we consider unbounded open sets that are the complement of compact subsets of $\mathbb{R}^k$. To avoid problems with non-bounded open sets, we consider the complement of these compacts restricted to a sufficiently big ball.

Definition 2.4. Let $K \subset \mathbb{R}^k$ be a compact subset of $\mathbb{R}^k$ and let $D > 0$ be the distance between the origin O of $\mathbb{R}^k$ and the farthest point of K from O. The *bounded medial axis* of K, denoted $\mathcal{BM}(K)$, is the medial axis of the complement of K intersected with the open ball $\mathbb{B}(O, 10D)$ of center O and radius $10D$:

$$\mathcal{BM}(K) = \mathcal{M}(\mathbb{B}(0, 10D) \setminus K).$$

In the same way one defines the *bounded λ-medial axis* of K as

$$\mathcal{BM}_\lambda(K) = \mathcal{M}_\lambda(\mathbb{B}(0, 10D) \setminus K).$$

Such a definition will be used in Section 5. It should be considered as a technical trick to suitably state the results of Section 5. Notice that $\mathbb{B}(0, 10D) \setminus K$ and $\mathbb{R}^k \setminus K$ are homeomorphic and thus homotopy equivalent. It is proven in [20] that $\mathcal{BM}(K)$ and $\mathbb{B}(0, 10D) \setminus K$ are homotopy equivalent. It is also proven in [5] that if $\lambda <$ *reach*(K) (see next section for a definition of reach), then $\mathcal{BM}_\lambda$ and $\mathbb{B}(0, 10D) \setminus K$ are homotopy equivalent.

3. Distance function to a noisy approximation

Results of this section are closely related to results in [22]. Mostly, we restate and extend some propositions of [22] in terms of critical points of distance functions.

Let $S \subset \mathbb{R}^k$ be a compact smooth manifold and let $\mathcal{M}$ be its medial axis. The *local feature size* of S is the function $\mathrm{lfs} : S \to \mathbb{R}_+$ defined by

$$\mathrm{lfs}(x) = d(x, \mathcal{M}) = \inf\{d(x, y)\colon\ y \in \mathcal{M}\}.$$

The infimum τ of lfs is known as the *reach* of S [16]. Remind that the distance of a point $x \in \mathbb{R}^k$ to S is denoted by $\mathcal{R}(x) = \inf\{d(x, y)\colon\ y \in S\}$. For any point $x \in \mathbb{R}^k \setminus \mathcal{M}$, the projection $\Pi(x)$ of x on S is the unique point on S such that $d(x, \Pi(x)) = \mathcal{R}(x)$.

We also denote by $f : \mathbb{R}^k \setminus (S \cup \mathcal{M}) \to \mathbb{R}_+ \cup \{+\infty\}$ the function defined by $f(x)$ is the distance between $\Pi(x)$ and the first intersection point of the half-line $[\Pi(x), x)$ (which is normal to S) with the closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ (see Fig. 1). Notice that lfs and f are related by

$$\mathit{reach}(S) \leqslant \mathrm{lfs}(\Pi(x)) \leqslant f(x) \quad \text{for any } x \in \mathbb{R}^k \setminus (S \cup \mathcal{M}). \tag{1}$$

Notice that for $x \in \mathbb{R}^k \setminus (S \cup M)$, the half-line $L_{\Pi(x)}$ issued from $\Pi(x)$ and passing through x is normal to S at $\Pi(x)$.

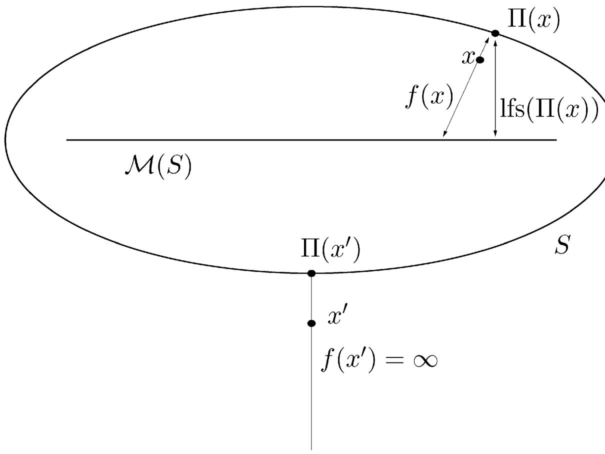

Fig. 1. Definition of function f.

Definition 3.1. The normal segment $N_{\Pi(x)}$ passing through x is the connected component of $L_{\Pi(x)} \setminus \overline{\mathcal{M}}$ that contains $\Pi(x)$.

Notice that $N_{\Pi(x)}$ is unbounded if and only if $f(x) = +\infty$.

Definition 3.2. A compact set $\mathcal{K} \subset \mathbb{R}^k$ is a uniform noisy ε-approximation of S if it satisfies

$$d_H(\mathcal{K}, S) < \varepsilon\tau$$

where τ is the reach of S and d_H denotes the Hausdorff distance.

Notice that we do not make any assumption neither on the finiteness nor on the geometric structure of $\mathcal{K}$. The case when $\mathcal{K}$ is a finite set of points is of particular interest for applications, but as mentioned in the introduction, considering some other sets may be relevant from a practical point of view.

In the following $\mathcal{K} \subset \mathbb{R}^k$ denotes a uniform noisy ε-approximation of S. Let $\tilde{\mathcal{R}}$ be the distance function to the compact set $\mathcal{K}$. The functions $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are related by the following inequality

$$\left|\mathcal{R}(x) - \tilde{\mathcal{R}}(x)\right| < \varepsilon\tau \quad \text{for any } x \in \mathbb{R}^k. \tag{2}$$

A key argument in the paper is the following lemma.

Lemma 3.3. *(See also [22, Proposition 7.1].) Let $\varepsilon < 1/6 \simeq 0.1667$ and let $\mathcal{K}$ be a uniform noisy ε-approximation of S. Let $x \in \mathbb{R}^n \setminus S$ satisfying one of the two following conditions*:

condition 1: *$f(x)$ is finite and*

$$\frac{5}{2}\varepsilon\tau < \mathcal{R}(x) < \left(1 - \frac{7}{2}\varepsilon\right) f(x).$$

condition 2: *$f(x) = +\infty$ and $\mathcal{R}(x) > \varepsilon\tau$.*

Then x is not a critical point of $\tilde{\mathcal{R}}$.

Moreover, if one denotes by $N_{\Pi(x)}$ the normal segment passing through x, then $\tilde{\mathcal{R}}$ is strictly increasing along $N_{\Pi(x)} \cap \mathcal{R}^{-1}([\frac{5}{2}\varepsilon\tau, (1 - \frac{7}{2}\varepsilon) f(x)]) \cap \mathbb{B}(\Pi(x), f(x))$ in case 1.

In case 2, *$\tilde{\mathcal{R}}$ is strictly increasing along $N_{\Pi(x)} \cap \mathcal{R}^{-1}(]\varepsilon\tau, +\infty[)$.*

Before proving this lemma, one has to notice that its first part may be viewed as a variant of a result in [22] (Proposition 7.1). A result of the same kind is also obtained in [14] under more restrictive hypothesis (only finite sets of point sampled exactly on a surface $S \subset \mathbb{R}^3$ are considered).

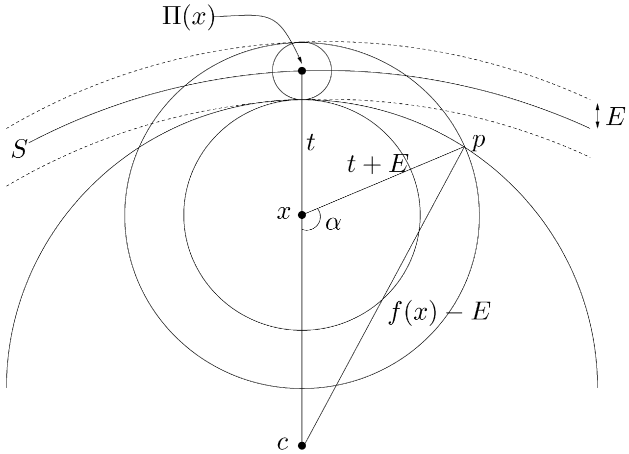

Fig. 2.

Proof. To simplify notations, one introduces $E = \varepsilon\tau$. Let S_E be the offset manifold $S_E = \{x \in \mathcal{R}^k\colon\ d(x, S) = E\}$. All the points of $\mathcal{K}$ are contained in the tubular neighborhood $Tub_E(S) = \{x \in \mathbb{R}^k\colon\ d(x, S) < E\}$.

First, suppose that x satisfies condition 1. Let $c \in \overline{\mathcal{M}}$ be such that $d(c, \Pi(x)) = f(x)$, x is contained in the segment $[c, \Pi(x)]$ and the ball $\mathbb{B}(c, f(x))$ of center c and radius $f(x)$ is tangent to S at $\Pi(x)$ (see Fig. 2). Notice that, since the open ball $\mathbb{B}(c, f(x))$ is contained in $\mathbb{R}^k \setminus S$, the ball $\mathbb{B}(c, f(x) - E)$ is contained in the complement of $Tub_E(S)$.

Denoting by $t = d(x, \Pi(x)) = \mathcal{R}(x)$, it follows from hypothesis that the ball $\mathbb{B}(x, t - E)$ does not contain any point of $\mathcal{K}$. Since $\mathbb{B}(\Pi(x), E)$ intersects $\mathcal{K}$ and is contained in $\mathbb{B}(x, t + E)$, the ball $\mathbb{B}(x, t + E)$ contains at least one point of $\mathcal{K}$. The radius of the maximal ball $\mathbb{B}_{\max}(x)$ contained in $\mathbb{R}^k \setminus \mathcal{K}$ with center x is thus contained in $[t - E; t + E]$. Moreover, the points of $\mathcal{K}$ which are on the boundary of $\mathbb{B}_{\max}(x)$ are contained in $\mathbb{B}(x, t + E) \setminus \mathbb{B}(c, f(x) - E)$. It follows from the definition of critical point that whenever the part of the sphere $\mathbb{S}(x, t + E)$ which is not contained in $\mathbb{B}(c, f(x) - E)$ is less than an hemisphere, x is a regular point of $\tilde{\mathcal{R}}$. This condition is equivalent to $\cos\alpha < 0$ where α is the angle between $[xc]$ and any segment joining x to a point p of the $(k-2)$-sphere $\mathbb{S}(x, t + E) \cap \mathbb{S}(c, f(x) - E)$ (see Fig. 2).

Using the relations between the lengths of the edges of the triangle (cxp), the cosine of the angle α satisfies the following relation

$$\big(f(x) - E\big)^2 = \big(f(x) - t\big)^2 + (t + E)^2 - 2\big(f(x) - t\big)(t + E)\cos\alpha.$$

Since $(f(x) - t)(t + E) > 0$, $\cos\alpha < 0$ if and only if

$$\big(f(x) - t\big)^2 + (t + E)^2 - \big(f(x) - E\big)^2 = 2\big(t^2 + t\big(E - f(x)\big) + f(x)E\big) < 0.$$

The discriminant of this equation is equal to

$$E^2 - 6f(x)E + f(x)^2 = f(x)^2\big(\big(\tau\varepsilon/f(x)\big)^2 - 6\big(\tau\varepsilon/f(x)\big) + 1\big).$$

It is positive whenever $\varepsilon < 3 - 2\sqrt{2} \simeq 0.1715$ because $\tau \leqslant f(x)$. In this case, the roots of equation $t^2 + t(E - f(x)) + 2f(x)E = 0$ are given by

$$t_- = \frac{f(x) - E - f(x)\sqrt{1 - 6E/f(x) + E^2/f(x)^2}}{2}$$

and

$$t_+ = \frac{f(x) - E + f(x)\sqrt{1 - 6E/f(x) + E^2/f(x)^2}}{2}.$$

Using that $\sqrt{1-u} > 1 - u$ for any $u \in]0, 1]$ and that $\tau \leqslant f(x)$, one immediately deduces that

$$t_+ > f(x) - \frac{7}{2}E + \frac{E^2}{2f(x)} = f(x)\left(1 - \frac{7}{2}\frac{\tau}{f(x)}\varepsilon + \left(\frac{\tau}{f(x)}\right)^2\frac{\varepsilon^2}{2}\right) > \left(1 - \frac{7}{2}\varepsilon\right)f(x)$$

and

$$t_- < \frac{5}{2}E - \frac{E^2}{2f(x)} = f(x)\left(\frac{5}{2}\frac{\tau}{f(x)}\varepsilon - \left(\frac{\tau}{f(x)}\right)^2\frac{\varepsilon^2}{2}\right) < \frac{5}{2}\varepsilon\tau.$$

The first statement of the lemma follows from that if $t = \mathcal{R}(x) \in]t_-, t_+[$, then x is a regular point of $\tilde{\mathcal{R}}$.

Suppose now that x satisfies condition 2. From $f(x) = +\infty$, one deduces that x is not contained in the convex hull of S. Since moreover $\mathcal{R}(x) > \varepsilon\tau$, one has that x is not contained in the convex hull of $\mathcal{K}$. Since we know (Remark 1) that the critical points are contained in the convex hull of S, it follows that x is not a critical point.

To prove the second part of the lemma, we just have to remark that, under both conditions, the angle between the vector collinear to $N_{\Pi(x)}$ and pointing away from S and any segment joining x to a point of $\mathcal{K} \cap \mathbb{S}(x, \tilde{\mathcal{R}}(x))$ is greater than $\pi/2$. $\mathcal{K} \cap \mathbb{S}(x, \tilde{\mathcal{R}}(x))$ being compact, the infimum of these angles is greater than $\pi/2$. It follows from [17, Lemma 1.5] that the function $\tilde{\mathcal{R}}$ restricted to $N_{\Pi(x)}$ is strictly increasing around x. □

Remark 2. The previous proof gives slightly better bounds than in the statement of the lemma. In the proof, we only use that $\varepsilon < 3 - 2\sqrt{2}$. The condition $\varepsilon < 1/6$ in the statement of the lemma ensures that $\frac{5}{2}\varepsilon < (1 - \frac{7}{2}\varepsilon)$. Some refinements on the bounds may be found in [22].

Remark 3. Notice that previous lemma defines an open set depending only upon S and ε in which $\tilde{\mathcal{R}}$ does not contain any critical point independently of the choice of a uniform noisy ε-approximation of S.

Using the inequality (1) one immediately deduces the following more global statement.

Lemma 3.4. *Let $\varepsilon < 1/6 \simeq 0.1667$ and let $\mathcal{K}$ be a uniform noisy ε-approximation of S. If $x \in \mathbb{R}^n$ is such that*

$$\frac{5}{2}\varepsilon\tau < \mathcal{R}(x) < \left(1 - \frac{7}{2}\varepsilon\right)\tau$$

then x is not a critical point of $\tilde{\mathcal{R}}$.

Moreover, if one denotes by $N_{\Pi(x)}$ the normal segment passing through x, then the function $\tilde{\mathcal{R}}$ is strictly increasing along the segment $\mathbb{B}(\Pi(x), f(x)) \cap N_{\Pi(x)} \cap \mathcal{R}^{-1}([\frac{5}{2}\varepsilon\tau, (1 - \frac{7}{2}\varepsilon)\tau])$.

An immediate consequence of the lemma is that the critical points of $\tilde{\mathcal{R}}$ are located in a neighborhood of $S \cup \mathcal{M}$.

Corollary 3.5. *Under hypothesis of the previous lemma, if $x \in \mathbb{R}^k \setminus (S \cup \mathcal{M})$ is a critical point of $\tilde{\mathcal{R}}$, then*

$$d(x, S) < \frac{5}{2}\tau\varepsilon \quad \text{or} \quad d(x, \mathcal{M}) < \frac{7}{2}f(x)\varepsilon.$$

In particular, if $d = \sup\{d(x, y)\colon x \in S, y \in \mathcal{M}\}$, then

$$d(x, S) < \frac{5}{2}\tau\varepsilon \quad \text{or} \quad d(x, \mathcal{M}) < \frac{7}{2}d\varepsilon.$$

Lemma 3.4 also implies the following result which is a particular case of [22], Proposition 7.1.

Corollary 3.6. *(See [22, Proposition 7.1].) Let $\varepsilon < 1/8 \simeq 0.125$ and let $\mathcal{K}$ be a uniform noisy ε-approximation of S. If $\alpha \in [\frac{7}{2}\varepsilon\tau, (1 - \frac{9}{2}\varepsilon)\tau]$, then S is a deformation retract of the union of balls*

$$U_\alpha = \bigcup_{e \in \mathcal{K}} \mathbb{B}(e, \alpha) = \tilde{\mathcal{R}}^{-1}\big([0, \alpha[\big).$$

Proof. For any $p \in S$, denote by $T_pS^\perp$ the affine subspace passing through p and orthogonal to S at p. One just has to prove that for any point $p \in S$, $T_pS^\perp \cap U_\alpha$ is star-shaped (in this case $H(x, t) = x + t\overrightarrow{x\Pi(x)}$ is the required deformation retraction). Let $x \in T_pS^\perp \cap U_\alpha$ and let $y \in [p, x[$. If $\mathcal{R}(y) \geqslant \frac{5}{2}\varepsilon\tau$, it follows from second part of above lemma that $\tilde{\mathcal{R}}(y) < \tilde{\mathcal{R}}(x) = \alpha$. If $\mathcal{R}(y) < \frac{5}{2}\varepsilon\tau$ then $\tilde{\mathcal{R}}(y) < \frac{5}{2}\varepsilon\tau + \varepsilon\tau \leqslant \alpha$ (because $|\mathcal{R}(z) - \tilde{\mathcal{R}}(z)| < \varepsilon\tau$ for any $z \in \mathbb{R}^k$). In both cases, $y \in U_\alpha \cap T_pS^\perp$. □

4. Applications to hypersurface reconstruction

Lemma 3.4 implies the following result for hypersurfaces embedded in $\mathbb{R}^k$ (i.e. $(k-1)$-dimensional submanifolds of $\mathbb{R}^k$).

Theorem 4.1. *Let S be a smooth compact connected hypersurface embedded in $\mathbb{R}^k$ with positive reach $\tau > 0$. Let $0 < \varepsilon < 1/10$ and let $\mathcal{K}$ be a uniform noisy ε-approximation of S. For any value $\alpha \in [\frac{7}{2}\varepsilon\tau, (1-\frac{9}{2}\varepsilon)\tau]$ the boundary of the union U_α of balls of radii α and centers the point of $\mathcal{K}$,*

$$U_\alpha = \bigcup_{e \in \mathcal{K}} \mathbb{B}(e, \alpha),$$

contains two connected components, each of one isotopic to S.

Proof. We use the second part of Lemma 3.3 of the previous section to prove that the restriction of Π to any connected component $\tilde{S}_\alpha$ of the boundary of U_α is an homeomorphism. The isotopy between S and $\tilde{S}_\alpha$ is realized by "pushing" $\tilde{S}_\alpha$ onto S along the normals of S. Since S is a connected hypersurface, $\mathbb{R}^k \setminus S$ contains two connected components denoted by $\mathcal{O}_i$ and $\overline{O}_e$. Let us consider, for example, the component $\tilde{S}_\alpha$ of the boundary of U_α which is contained in $\mathcal{O}_i$. Since α is a regular value of $\tilde{\mathcal{R}}$, $\tilde{S}_\alpha$ is a compact $\mathcal{C}^0$ hypersurface in $\mathcal{O}_i$ (Proposition 2.3).

Claim. *For any $p \in S$, the normal segment N_p issued from p, normal to S and pointing into $\mathcal{O}_i$ meets $\tilde{S}_\alpha$ in exactly one point in $\mathcal{R}^{-1}([\frac{5}{2}\varepsilon\tau, (1-\frac{7}{2}\varepsilon)\tau])$.*

First notice that $\tilde{\mathcal{R}}(p) < \varepsilon\tau < \alpha$. Since $\tilde{\mathcal{R}}$ is continuous and unbounded on N_p there exists some point $x \in N_p$ such that $\tilde{\mathcal{R}}(x) = \mathrm{lfs}(p) > \alpha$ and N_p intersects $\tilde{S}_\alpha$. Now, let $y \in N_p$ be such that $\tilde{\mathcal{R}}(y) = \alpha$. Inequality (2) implies $\frac{5}{2}\varepsilon\tau < \mathcal{R}(y) < (1-\frac{7}{2}\varepsilon)\tau$. It follows from Lemma 3.4 of the previous section that $\tilde{\mathcal{R}}$ is strictly increasing along the segment $N_p \cap \mathcal{R}^{-1}([\frac{5}{2}\varepsilon\tau, (1-\frac{7}{2}\varepsilon)\tau])$, so y is the unique point of N_p satisfying $\tilde{\mathcal{R}}(y) = \alpha$. This proves the claim.

The end of the proof of theorem now follows easily from the claim: the restriction of Π to $\tilde{S}_\alpha$ is thus a continuous bijective map. The hypersurfaces S and $\tilde{S}_\alpha$ being compact, it is thus an homeomorphism. □

Remark 4. Assuming connectedness of S in previous theorem is not necessary. By taking care of the definition of S_α, one can easily give a similar statement when S contains several components.

Remark 5. The previous proof can be adapted to manifolds S of any codimension. One thus obtains that $\tilde{S}_\alpha$ is a $\mathcal{C}^0$ hypersurface isotopic to the boundary of the tubular neighborhood of S of sufficiently small radius.

5. Application to topology guaranteeing approximations of the medial axis

In this section, one uses classical results from distance function theory and the results of [6,7] to relate the topology of the bounded λ-medial axes of $\mathcal{K}$ to the topology of $\mathbb{R}^k \setminus S$.

Let S be a smooth compact submanifold of $\mathbb{R}^k$ of any codimension and let $\mathcal{BM}_\lambda$ be the bounded λ-medial axis of S. If $\mathcal{K}$ is a uniform noisy ε-approximation of S and $\lambda > 0$ is a positive real number, one denotes by $\mathcal{BM}_\lambda(\mathcal{K})$ the bounded λ-medial axis of $\mathcal{K}$ and by $U_\lambda = \tilde{\mathcal{R}}^{-1}([0, \lambda[)$ the union of balls of radii λ and centers the points of $\mathcal{K}$.

Lemma 5.1. *Let $\varepsilon < 1/8 \simeq 0.125$ and let $\mathcal{K}$ be a uniform noisy ε-approximation of S. If $\lambda \in [\frac{7}{2}\varepsilon\tau, (1-\frac{9}{2}\varepsilon)\tau]$, then $\mathbb{R}^k \setminus U_\lambda$ is a deformation retract of $\mathbb{R}^k \setminus S$.*

Proof. Since $\lambda \in [\frac{7}{2}\varepsilon\tau, (1-\frac{9}{2}\varepsilon)\tau]$, it follows from (2) and Proposition 3.3 that λ is a regular value of $\tilde{\mathcal{R}}$ and the boundary $\tilde{S}_\lambda = \tilde{\mathcal{R}}^{-1}(\lambda)$ of U_λ is a $\mathcal{C}^0$ hypersurface in $\mathbb{R}^k$. The same argument as in the proof of the claim in previous section implies that for any $x \in U_\lambda \setminus S$, the normal segment issued from $\Pi(x)$, normal to S and passing through x meets $\tilde{S}_\lambda$ in exactly one point $\varphi(x)$ in $\mathcal{R}^{-1}([0, (1-\frac{7}{2}\varepsilon)\tau])$. Since $\tilde{S}_\lambda$ is a $\mathcal{C}^0$ hypersurface and the field of half-lines issued from S and normal to S is smooth, we thus define a continuous application $\varphi : \overline{U_\lambda} \setminus S \to \tilde{S}_\lambda$ which is the identity on $\tilde{S}_\lambda$.

Now,we proved in corollary 3.6 that for any $x \in U_\lambda \setminus S$, the segment $[\Pi(x), \varphi(x)]$ is contained in $\overline{U_\lambda}$. The map $\Phi : (\mathbb{R}^k \setminus S) \times [0,1] \to \mathbb{R}^k \setminus U_\alpha$ defined by $\Phi(x,t) = x + t\overrightarrow{x\varphi(x)}$ if $x \in U_\alpha \setminus S$ and $\Phi(x,t) = x$ otherwise, is the desired deformation retraction. □

It follows from previous lemma that $\mathbb{R}^k \setminus S$ and $\mathbb{R}^k \setminus U_\lambda$ are homotopy equivalent. Using results from [6,7], we can relate the homotopy type of $\mathbb{R}^k \setminus U_\lambda$ to the one of $\mathcal{BM}_\lambda(\mathcal{K})$.

Theorem 5.2. *Let S be a smooth compact submanifold of $\mathbb{R}^k$. Let $\varepsilon < 1/8$ and let $\mathcal{K}$ be a uniform noisy ε-approximation of S. For any value $\lambda \in [\frac{7}{2}\varepsilon\tau, (1 - \frac{9}{2}\varepsilon)\tau]$, $\mathcal{BM}_\lambda(\mathcal{K})$ and $\mathbb{R}^k \setminus S$ are homotopy equivalent.*

Proof. It is proven in [5, Theorem 2], that if λ is not a critical value of $\tilde{\mathcal{R}}$, then the open set $\mathbb{R}^k \setminus \overline{U_\lambda}$ and the bounded λ-medial axis $\mathcal{BM}_\lambda(\mathcal{K})$ are homotopy equivalent. Since, in our case, $\lambda \in [\frac{7}{2}\varepsilon\tau, (1 - \frac{9}{2}\varepsilon)\tau]$, it follows from Lemma 3.4 that λ is a regular value of $\tilde{\mathcal{R}}$. The theorem is thus an immediate consequence of Lemma 5.1. □

An important particular case is when $\mathcal{K}$ is a finite sample of points and S is the boundary of a bounded open set $\mathcal{O}$. In this case, the bounded λ-medial axis of $\mathcal{K}$ is a subcomplex $Vor_\lambda(\mathcal{K})$ of the Voronoi diagram $Vor(\mathcal{K})$ of $\mathcal{K}$ (see [5]).

Corollary 5.3. *Let S be a smooth compact hypersurface of $\mathbb{R}^k$ that is the boundary of a bounded open set $\mathcal{O}$. Let $\varepsilon < 1/8$ and let $\mathcal{K}$ be a finite set of points which is a uniform noisy ε-approximation of S. For any value $\lambda \in [\frac{7}{2}\varepsilon\tau, (1 - \frac{9}{2}\varepsilon)\tau]$, $Vor_\lambda(\mathcal{K})$ and $\mathcal{O}$ are homotopy equivalent.*

In [5] we proved that, under hypothesis of the previous lemma, $Vor_\lambda(\mathcal{K})$ is an approximation of $\mathcal{M}_\lambda(\mathcal{O})$ for Hausdorff distance that may be easily computed from the Voronoi diagram of $\mathcal{K}$. So, previous lemma ensures that the algorithm given in [5] to approximate the λ-medial axis of $\mathcal{O}$ provides an output which has the homotopy type of $\mathbb{R}^k \setminus S$. The parameter λ being chosen smaller than τ, it follows that $\mathcal{O}$ has the homotopy type of $\mathcal{M}(\mathcal{O})$ that has itself the homotopy type of $\mathcal{M}_\lambda(\mathcal{O})$ [5, Theorem 2].

Moreover, if one considers [3, p. 13], the λ-medial axes may also be used to provide approximation of the medial axis. This leads to an approximation algorithm of the medial axis of a smooth hypersurface from noisy data sample with topological guarantee: the output is homotopy equivalent to the medial axis of the hypersurface.

6. Non-uniform approximations

In practical applications it may be useful to use non-uniform approximations. For example, one may want to have more precise approximation in the areas where the manifold has a small lfs and less precise approximation in the areas where the manifold has big lfs. In this section, we give a generalization of Lemma 3.3 for non-uniform approximations. Using the same notations as in previous sections, one has the following definition of non-uniform approximation.

Definition 6.1. Let $S \subset \mathbb{R}^k$ be a compact manifold with positive reach and let $\varepsilon > 0$. Given $\kappa > 0$, a compact set $\mathcal{K} \subset \mathbb{R}^k$ is a (non-uniform) (κ, ε)-approximation of S if it satisfies the following conditions:

- for any $e \in \mathcal{K}$, $d(e, \Pi(e)) < \kappa\varepsilon\, \mathrm{lfs}(\Pi(e))$,
- for any $p \in S$, there exists a point $e \in \mathcal{K}$ such that $d(p, \Pi(e)) < \varepsilon\, \mathrm{lfs}(p)$.

Notice that in the case where $\mathcal{K}$ is a finite sample of points, the second condition is equivalent to $\Pi(\mathcal{K})$ is an ε-sample of S as defined in [1] and our sampling condition is almost the same as the one introduced in [13] and [21].

6.1. Union of balls of constant radius

Using that the lfs function is 1-Lipschitz, Lemma 3.3 generalizes for non-uniform approximations in the following way.

Lemma 6.2. *Let $\varepsilon < 1/30 \sim 0.034$ and let $\mathcal{K}$ be a (κ, ε)-approximation of S for $\kappa \leqslant 1$. Let $x \in \mathbb{R}^n \setminus S$ satisfying*

$$\frac{25}{2}\varepsilon \operatorname{lfs}\big(\Pi(x)\big) < \mathcal{R}(x) < \left(1 - \frac{35}{2}\varepsilon\right) \operatorname{lfs}\big(\Pi(x)\big).$$

Then x is not a critical point of $\tilde{\mathcal{R}}$.

Moreover, if one denotes by $N_{\Pi(x)}$ the normal segment passing through x, then the function $\tilde{\mathcal{R}}$ is strictly increasing along the segment $N_{\Pi(x)} \cap \mathcal{R}^{-1}([\frac{25}{2}\varepsilon \operatorname{lfs}(\Pi(x)), (1 - \frac{35}{2}\varepsilon) \operatorname{lfs}(\Pi(x))])$.

Notice that the condition $\varepsilon < 1/30$ is necessary to ensure that the hypothesis of lemma is verified for at least a point x (that is $25/2\varepsilon < 1 - 35/2\varepsilon$).

As in the case of uniform approximations, one can deduce from this lemma some result about hypersurface reconstruction.

Theorem 6.1. *Let S be a smooth compact connected hypersurface embedded in $\mathbb{R}^k$ with positive reach $\tau > 0$ and let $M = \sup_{X \in S} \operatorname{lfs}(X)$ the supremum of the function* lfs *restricted to S. Let $0 < \varepsilon < 1/32$ be such that $(27M + 37\tau)\varepsilon < 2\tau$ and let $\mathcal{K}$ be a (κ, ε)-approximation of S for $\kappa \leqslant 1$. For any value $\alpha \in [\frac{27}{2}\varepsilon M, (1 - \frac{37}{2}\varepsilon)\tau]$ the boundary of the union U_α of balls of radii α and centers the point of $\mathcal{K}$,*

$$U_\alpha = \bigcup_{e \in \mathcal{K}} \mathbb{B}(e, \alpha),$$

contains two connected components, each of one isotopic to S.

We skip the proofs of these two results. They are just an adaptation of the proofs given in previous Sections 3 and 4 using that the lfs function is 1-Lipschitz. The main drawback of the previous theorem is twofold. First, it imposes to consider balls of constant radius. Second, the condition on ε involves the ratio between the minimum and the maximum of the lfs function. It thus follows that if ε fulfills condition of Theorem 6.1, then the compact $\mathcal{K}$ is in fact a uniform noisy η-approximation of S for $\eta = 2\min(\frac{1}{32}, \frac{2\tau}{27M+37\tau})M$.

6.2. Union of balls of different radii

We now generalize techniques and results from previous section to improve Theorem 6.1.

In the following, S denotes a smooth compact submanifold of $\mathbb{R}^k$. Let $\varepsilon > 0$, $\kappa > 0$ and $\mathcal{K}$ be a (κ, ε)-approximation of S. For any family $r = (r_e)_{e \in \mathcal{K}}$ of positive real numbers such that $r_e = \alpha_e \operatorname{lfs}(\Pi(e))$, $0 < \alpha_e < 1$, one denotes by $\mathcal{K}(r)$ the union of balls

$$\mathcal{K}(r) = \bigcup_{e \in \mathcal{K}} \mathbb{B}(e, r_e).$$

Theorem 6.2. *Let κ, ε, and $0 < a < b < \frac{1}{3} - \kappa\varepsilon$ be such that*

$$(1 - a')^2 + \left((b' - a') + \frac{b(1 + 2b' - a')}{1 - (b + \kappa\varepsilon)}\right)^2 < \left(1 - \frac{\kappa\varepsilon(1 + 2b' - a')}{1 - (b + \varepsilon)}\right)^2$$

with $a' = (a - \kappa\varepsilon)(1 - \varepsilon) - \varepsilon$ and $b' = \frac{b+\kappa\varepsilon}{1-2(b+\kappa\varepsilon)}$.

Let $\mathcal{K}$ be a (κ, ε)-approximation of S. If $r = (r_e)_{e \in \mathcal{K}}$ is such that $a \leqslant \alpha_e \leqslant b$ then

- *S is a deformation retract of $\mathcal{K}(r)$,*
- *$\mathcal{K}(r)$ is homeomorphic to any tubular neighborhood $\mathcal{R}^{-1}([0, d))$, $d <$ reach(S),*
- *the boundary $\partial\mathcal{K}(r)$ of $\mathcal{K}(r)$ is an hypersurface isotopic to $\mathcal{R}^{-1}(d)$.*

Notice that when S is a codimension one submanifold, $\partial\mathcal{K}(r)$ is isotopic to two copies of S. The inequality involving ε, κ, a and b given in the theorem is a technical condition that will appear in the proof of the theorem. Examples of values for which this condition is satisfied are given at the end of this section.

Proof. The proof follows from a sequence of lemmas.

Let $g:\mathbb{R}^k \setminus \overline{\mathcal{M}(S)} \to \mathbb{R}_+$ defined by $g(x) = \mathcal{R}(x)/\operatorname{lfs}(\Pi(x))$. Since $\mathcal{R}$, lfs and Π are continuous functions, $g = \mathcal{R}/(\operatorname{lfs} \circ \Pi)$ is a continuous function.

Lemma 6.3. *(Isotopy lemma). Let $a, b \in]0, 1[$ be such that $a < b$. The level set $g^{-1}(a)$ is an hypersurface isotopic to $\mathcal{R}^{-1}(d)$ for any $0 < d < reach(S)$ and $g^{-1}([a, b])$ is homeomorphic to $g^{-1}(a) \times [a, b]$. Moreover for any $x \in S$ and any normal segment N_x issued from x, g is strictly increasing along $g^{-1}([a, b]) \cap N_x$.*

Proof. First notice that the second part of the lemma is an obvious consequence of the definition of g and the fact that the restriction of $\operatorname{lfs}' \circ \Pi$ is constant along N_x until it meets $\overline{\mathcal{M}(S)}$. Let $d < reach(S)$ and let $\varphi : \mathcal{R}^{-1}(d) \to g^{-1}(a)$ defined by

$$\varphi(x) = \Pi(x) + a \operatorname{lfs}\bigl(\Pi(x)\bigr)\frac{\overrightarrow{\Pi(x)x}}{d}.$$

This maps is obviously continuous since Π and lfs are. For any point $y \in g^{-1}(a)$ the half-line $[\Pi(y)y)$ is normal to S and has to intersect $\mathcal{R}^{-1}(d)$ in a first point x. One thus has $\varphi(x) = y$ and φ is surjective. If $x, x' \in \mathcal{R}^{-1}(d)$ are such that $x \neq x'$ then $\varphi(x)$ and $\varphi(x')$ are on two distinct half-lines normal to S that cannot intersect in $\mathcal{R}^{-1}([0, reach(S)[)$. So $\varphi(x) \neq \varphi(x')$ and φ is injective. It follows that φ is a continuous bijection between compact sets, so it is an homeomorphism. The proof that $g^{-1}([a, b])$ and $g^{-1}(a) \times [a, b]$ are homeomorphic is done in the same way. □

Next lemma shows that the boundary of $\mathcal{K}(r)$ is enclosed between two sublevel sets of g.

Lemma 6.4. *Let $a' = (a - \kappa\varepsilon)(1 - \varepsilon) - \varepsilon$ and $b' = \frac{b+\kappa\varepsilon}{1-2(b+\kappa\varepsilon)}$. One has*

$$g^{-1}\bigl([0, a']\bigr) \subset \mathcal{K}(r) \subset g^{-1}\bigl([0, b'[\bigr).$$

Proof. Let x be in $\mathcal{K}(r)^c \cap \overline{\mathcal{M}(S)}^c$ the complement of $\mathcal{K}(r)$ and of $\overline{\mathcal{M}(S)}$ and let $X = \Pi(x)$. $\mathcal{K}$ being a noisy ε-approximation of S, there exists $e \in \mathcal{K}$ such that $d(X, E) < \varepsilon \operatorname{lfs}(X)$ where $E = \Pi(e)$. Since $x \in \mathcal{K}(r)^c$, $d(x, e) \geqslant r_e$. Using that $d(e, E) < \kappa\varepsilon \operatorname{lfs}(E)$ it follows that $d(x, E) > r_e - \kappa\varepsilon \operatorname{lfs}(E) > (a - \kappa\varepsilon)\operatorname{lfs}(E)$. Now, applying triangular inequality one obtains

$$d(x, X) \geqslant d(x, E) - d(X, E) > (a - \kappa\varepsilon)\operatorname{lfs}(E) - \varepsilon \operatorname{lfs}(X). \tag{3}$$

Recall that lfs is a 1-Lipschitz function, so $|\operatorname{lfs}(E) - \operatorname{lfs}(X)| \leqslant d(X, E) < \varepsilon \operatorname{lfs}(X)$ which implies $\operatorname{lfs}(E) > (1 - \varepsilon)\operatorname{lfs}(X)$. From this last inequality and from (3) one deduces

$$d(x, X) > \bigl((a - \kappa\varepsilon)(1 - \varepsilon) - \varepsilon\bigr)\operatorname{lfs}(X). \tag{4}$$

This proves the first inclusion of lemma.

Now, let $x \in \mathcal{K}(r)$ and let $X = \Pi(x)$. There exists $e \in \mathcal{K}$ such that $x \in \mathbb{B}(e, r_e)$. Denoting $E = \Pi(e)$ one has

$$\begin{aligned} d(x, X) \leqslant d(x, E) &\leqslant d(x, e) + d(e, E) &(5)\\ &< r_e + \kappa\varepsilon \operatorname{lfs}(E) &(6)\\ &< (b + \kappa\varepsilon)\operatorname{lfs}(E). &(7)\end{aligned}$$

Since lfs is 1-Lipschitz,

$$\begin{aligned}\bigl|\operatorname{lfs}(E) - \operatorname{lfs}(X)\bigr| &\leqslant d(X, E)\\ &\leqslant d(x, X) + d(x, E)\\ &< d(x, X) + (b + \kappa\varepsilon)\operatorname{lfs}(E)\\ &< 2(b + \kappa\varepsilon)\operatorname{lfs}(E).\end{aligned}$$

It follows that

$$\operatorname{lfs}(E) < \frac{1}{1 - 2(b + \kappa\varepsilon)}\operatorname{lfs}(X). \tag{8}$$

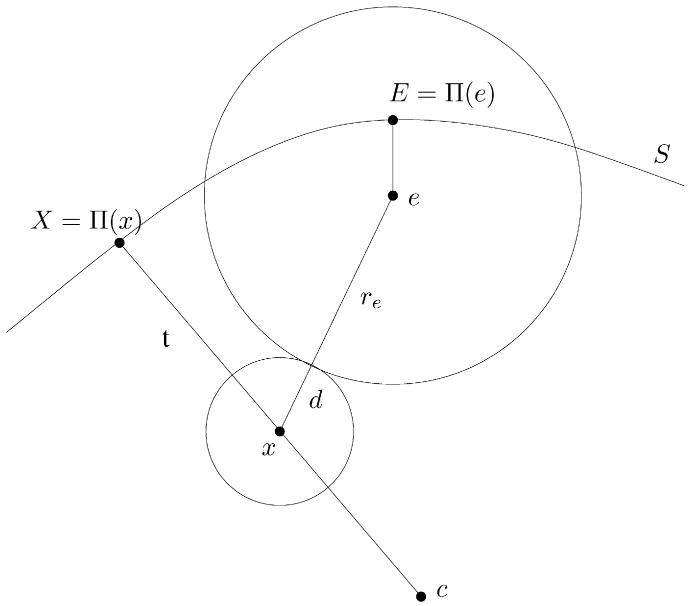

Fig. 3.

Combining inequalities (7) and (8) one obtains that

$$d(x, X) < \frac{b + \kappa\varepsilon}{1 - 2(b + \kappa\varepsilon)} \operatorname{lfs}(X). \tag{9}$$

This proves second inclusion of lemma. □

The next lemma (and its corollary) is the key argument for the proof of Theorem 6.2. It shows that distance function to $\mathcal{K}(r)$ is strictly increasing along the normals of S between $\mathcal{K}(r)$ and $g^{-1}(b')$.

Lemma 6.5. *Let $x \in g^{-1}([a', b']) \setminus \mathcal{K}(r)$, let $X = \Pi(x)$ and let $l_X = [X, x)$ be the normal half-line passing through x. For any $e \in \mathcal{K}$ such that the ball of maximal radius centered on x and contained in $\mathcal{K}(r)^c$ meets $\overline{\mathbb{B}}(e, r_e)$, the distance to e restricted to l_X is strictly increasing in a neighborhood of x.*

Proof. To prove the lemma, one introduces a few notations. Let $E = \Pi(e)$, $d = d(x, \mathbb{B}(e, r_e)) \geqslant 0$, $t = d(x, X)$ and let $c \in l_X$ be the center of the ball of radius $\operatorname{lfs}(X)$ tangent to S at X (see Fig. 3). Notice that $\mathbb{B}(c, \operatorname{lfs}(X)) \cap S = \emptyset$ and that $a' \operatorname{lfs}(X) \leqslant t < b' \operatorname{lfs}(X)$.

To prove that the distance to e restricted to l_X is strictly increasing in a neighborhood of x, it suffices to show that the angle between the vectors $\overrightarrow{xc}$ and $\overrightarrow{xe}$ is greater than $\pi/2$. Such a condition is satisfied as soon as

$$d(x, e)^2 + d(c, x)^2 < d(c, e)^2. \tag{10}$$

Since $\mathbb{B}(c, \operatorname{lfs}(X)) \cap S = \emptyset$, $d(c, E) > \operatorname{lfs}(X)$. Using triangular inequality it follows that $d(c, e) > \operatorname{lfs}(X) - \kappa\varepsilon \operatorname{lfs}(E)$. So inequality (10) is satisfied as soon as

$$d(x, e)^2 + d(c, x)^2 < \big(\operatorname{lfs}(X) - \kappa\varepsilon \operatorname{lfs}(E)\big)^2. \tag{11}$$

Now, using that $d(c, x) = \operatorname{lfs}(X) - t < (1 - a') \operatorname{lfs}(X)$, one obtains that inequality (11) is satisfied as soon as

$$d(x, e)^2 + (1 - a')^2 \operatorname{lfs}(X)^2 < \big(\operatorname{lfs}(X) - \kappa\varepsilon \operatorname{lfs}(E)\big)^2. \tag{12}$$

It now remains to bound $d(x, e)$ and $\operatorname{lfs}(E)$.

One has $d(x, e) = d + r_e < d + b \operatorname{lfs}(E)$. Using Lemma 6.4, one deduces that $d < t - a' \operatorname{lfs}(X) < (b' - a') \operatorname{lfs}(X)$. It follows that

$$d(x,e) < (b' - a')\,\mathrm{lfs}(X) + b\,\mathrm{lfs}(E). \tag{13}$$

Recall that lfs is 1-Lipschitz so $|\mathrm{lfs}(X) - \mathrm{lfs}(E)| \leqslant d(X,E) < t + d + r_e + \kappa\varepsilon\,\mathrm{lfs}(E)$. It follows that

$$\bigl|\mathrm{lfs}(X) - \mathrm{lfs}(E)\bigr| < (2b' - a')\,\mathrm{lfs}(X) + (b + \kappa\varepsilon)\,\mathrm{lfs}(E)$$

which implies

$$\mathrm{lfs}(E) < \frac{1 + 2b' - a'}{1 - (b + \kappa\varepsilon)}\,\mathrm{lfs}(X). \tag{14}$$

Combining this last inequality with (13), one obtains

$$d(x,e) < \left((b' - a') + \frac{b(1 + 2b' - a')}{1 - (b + \kappa\varepsilon)}\right)\mathrm{lfs}(X). \tag{15}$$

One also deduces from (14) that

$$\mathrm{lfs}(X) - \kappa\varepsilon\,\mathrm{lfs}(E) > \left(1 - \frac{\kappa\varepsilon(1 + 2b' - a')}{1 - (b + \kappa\varepsilon)}\right)\mathrm{lfs}(X). \tag{16}$$

From (15) and (16) and dividing by $\mathrm{lfs}(X)^2$ one finally obtains that inequality (12) is satisfied as soon as

$$(1 - a')^2 + \left((b' - a') + \frac{b(1 + 2b' - a')}{1 - (b + \varepsilon)}\right)^2 < \left(1 - \frac{\varepsilon(1 + 2b' - a')}{1 - (b + \varepsilon)}\right)^2. \qquad \square \tag{17}$$

From Lemma 6.5 one deduces the following corollary.

Corollary 6.6. *Let x be as in Lemma* 6.5. *Then the distance function to $\mathcal{K}(r)$ restricted to l_X is strictly increasing in a neighborhood of x.*

Proof. The proof is based upon the compactness of $\mathcal{K}$. We use the same notations as in previous lemma. In the proof of Lemma 6.5, we showed that the angle α_e between the vectors $\vec{xc}$ and $\vec{xe}$ is greater than $\pi/2$ for any point $e \in \mathcal{K}$ satisfying hypothesis of the lemma. To prove corollary it is sufficient to show that the infimum of these angles among all the points $e \in \mathcal{K}$ satisfying hypothesis of Lemma 6.5 is greater than $\pi/2$. Suppose this is not the case. Then there exists a sequence of points e_n such that $\alpha_{e_n} \to \pi/2$ as $n \to \infty$. $\mathcal{K}$ being compact, one can assume without loss of generality that the sequence (e_n) converges to some point $e \in \mathcal{K}$. Distance function to e restricted to l_X is then critical at point x. If e satisfies hypothesis of Lemma 6.5 then $\alpha_e > \pi/2$: a contradiction. Otherwise, since $x \in \mathcal{K}(r)^c$, one has $r_e + d < d(x,e)$. If one replaces in $\mathcal{K}(r)$ the ball $\mathbb{B}(e, r_e)$ by the ball $\mathbb{B}(e, d(x,e) - d)$ one obtains a new set $\mathcal{K}(r')$ which still satisfies hypothesis of Theorem 6.2: since $r_{e_n} = d(x, e_n) - d$ for any n, $r_{e_n} \to d(x,e) - d$; but, using that $a\,\mathrm{lfs}(E_n) < r_{e_n} < b\,\mathrm{lfs}(E_n)$ for all n and that lfs is continuous, it follows that $a\,\mathrm{lfs}(E) < d(x,e) - d < b\,\mathrm{lfs}(E)$. So, one can apply Lemma 6.5 to $\mathcal{K}(r')$ and to points x and e to deduce that distance function to e restricted to l_X is strictly increasing in a neighborhood of x: a contradiction. $\square$

Lemma 6.7. *Restricted to $g^{-1}([0, b'])$, any half-line l_X normal to S and issued from a point $X \in S$ intersects $\partial\mathcal{K}(r)$ in a unique point.*

Proof. Denote by N_X the connected component of $l_X \cap g^{-1}([0, b'])$ that contains X and denote by $t \to x(t) = X + tb'\,\mathrm{lfs}(X)\vec{n_X}$ a parametrization of N_X where n_X is the unitary vector normal to S at X which spans l_X. It follows from Lemma 6.4 that $x(0) \in \mathcal{K}(r)$ and $x(1) \in \mathcal{K}(r)^c$. So N_X intersects $\partial\mathcal{K}(r)$. It follows from Corollary 6.6 that once $x(t) \in \mathcal{K}(r)^c$ distance function to $\mathcal{K}(r)$ restricted to N_X is strictly increasing in a neighborhood of x. So $x(t)$ cannot re-enter into $\mathcal{K}(r)$. This proves the lemma. $\square$

We are now able to prove Theorem 6.2. This is done by "pushing" $\partial\mathcal{K}(r)$ onto $g^{-1}(a')$ along the normals to S. For any $x \in g^{-1}(a')$ denote by $X = \Pi(x)$ and by $\varphi(x)$ the first intersection point of the half-line $l_X = [X, x)$ with $\partial\mathcal{K}(r)$. One thus defines a map $\varphi : g^{-1}(a') \to \partial\mathcal{K}(r)$. Using continuity of Π and of the field of half-lines normal to S, one easily check that φ is continuous. Notice that restricted to $g^{-1}([0, b'])$, l_X intersects $g^{-1}(a')$ in a unique

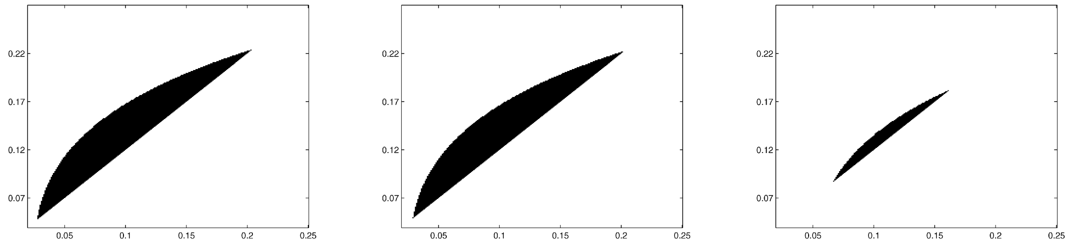

Fig. 4. Values of a and b that fulfill the inequality of Theorem 6.2 are represented in black on the diagrams for $\varepsilon = 0.025$ and different values of κ. Horizontal (resp. vertical) coordinates correspond to a (resp. b). From left to right: $\kappa = 0$, $\kappa = 0.025$, and $\kappa = 0.5$.

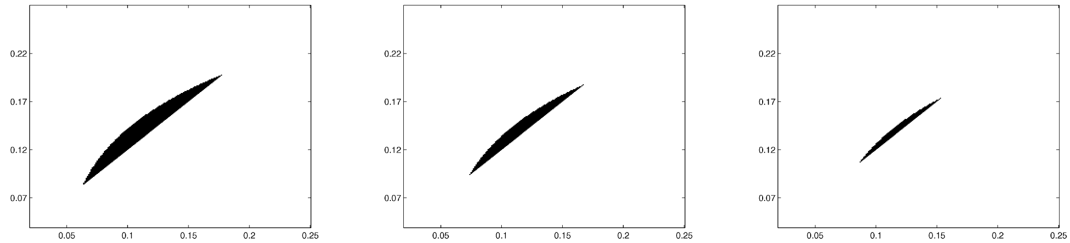

Fig. 5. Values of a and b that fulfill the inequality of Theorem 6.2 are represented in black on the diagrams for $\varepsilon = 0.05$ and different values of κ. Horizontal (resp. vertical) coordinates correspond to a (resp. b). From left to right: $\kappa = 0$, $\kappa = 0.05$, and $\kappa = 0.1$.

point. It follows that φ is a bijection. The subsets $g^{-1}(a')$ and $\partial\mathcal{K}(r)$ being compact, φ is thus an homeomorphism. The map $\Phi : \partial\mathcal{K}(r) \times [0,1] \to \mathbb{R}^k$ defined by $\Phi(x,t) = \varphi(x) - t\overrightarrow{x\varphi(x)}$ is an isotopy between $\partial\mathcal{K}(r)$ and $g^{-1}(a')$. The same map can be easily extended and used to define a deformation retraction of $\mathcal{K}(r)$ onto $g^{-1}([0,a'])$. Proof of Theorem 6.2 now follows from Lemma 6.3. $\square$

It is not difficult to find values ε, κ, a, b that fulfill the condition of Theorem 6.2. For example $\varepsilon < 0.01$, $k \leqslant 1$, $a = 1/15$ and $b = 1/10$ are good choices. Other examples are given in Figs. 4 and 5 where, given ε and κ, black areas of the diagrams represent the values a and b such that ε, κ, a, b fulfill the conditions of the theorem.

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