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Discrete Critical Values: a General Framework for Silhouettes Computation

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Abstract

Many shapes resulting from important geometric operations in industrial applications such as Minkowski sums or volume swept by a moving object can be seen as the projection of higher dimensional objects. When such a higher dimensional object is a smooth manifold, the boundary of the projected shape can be computed from the critical points of the projection. In this paper, using the notion of polyhedral chains introduced by Whitney, we introduce a new general framework to define an analogous of the set of critical points of piecewise linear maps defined over discrete objects that can be easily computed. We illustrate our results by showing how they can be used to compute Minkowski sums of polyhedra and volumes swept by moving polyhedra.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling —Curve, surface, solid, and object representations

1. Introduction

Many shapes resulting from classical geometric operations (silhouettes, caustics, Minkowski sums, volume swept by a moving object...) can be seen as the projection of higher dimensional objects. This point of view allows to relate the boundary of such shapes to the critical values of the projection maps defining them. In the smooth setting, a critical value of a smooth map ϕ is a value $\phi(x)$ at a point x for which the differential map $D\phi_x$ does not have a full rank. The sets of critical values arising for generic smooth maps have been extensively studied [AV85]. In particular, for a smooth map $\phi : M \longrightarrow \mathbb{R}^d$ defined over a manifold M without boundary of dimension $n \geq d$, the boundary points of $\phi(M)$ are critical values of ϕ. More generally, changes in the topology of the preimages $y \mapsto \phi^{-1}(\{y\})$ arise at critical values (See for example, figure 1). The computation of silhouettes, envelopes or boundaries of swept volumes based upon these properties of the critical values of smooth maps have been widely considered in the last decades - see [EI07,PP00,MS90,RBIM01] for example. Although the smooth setting provides a powerful mathematical framework, it is not well-suited to deal with polyhedral shapes and maps. In practice, the attempts to adapt the smooth approach to the polyhedral setting often leads to rather complicated algorithms facing robustness issues. An alternative approach, which is the one adopted in this paper, consists in defining a counterpart of the notion of critical value in the context of polyhedral shapes and maps. For real valued functions defined on polyhedra or more generally on so-called *convex complexes*, Banchoff [Ban67] has introduced a theory of critical points and a discrete analog of Morse theory. Banchoff's theory has proven to be powerful for generalizing various notions of smooth differential topology (Morse-Smale complexes,...) to the discrete setting - see [EHZ03, EHNP03] for example.

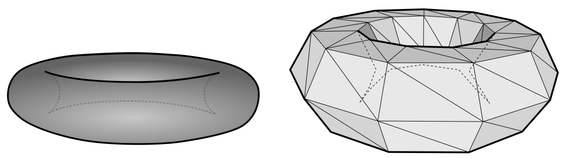

Figure 1: *Two-dimensional linear projections of torus with the silhouettes (bold and dashed curves) corresponding to the critical values of the projection maps. On the left in the classsical setting of smooth functions defined over smooth manifolds. On the right, our counterpart in the context of piecewise linear functions defined over simplicial complexes.*

In this paper, we consider the case of multivalued continuous piecewise linear maps defined on polyhedral objects. Inspired by the theory of critical points introduced in [Ban67] and closely related to the notion of singularity chains for piecewise linear maps [BAN75], our work introduces a general framework that allows to define and compute an analogous of critical values of smooth functions in the world of piecewise linear functions defined over discrete objects. Using the framework of polyhedral chains [Whi57], we formally define this set of critical values as polyhedral cycles and call them *silhouette cycles* (see definition in Section 2 below). We establish key properties on silhouette cycles which allow us to provide a simple and general algorithm for their computation. Note that Edelsbrunner and Harer [EH02] propose, in a similar framework, an alternate notion of singularities using Betti numbers. In contrast, the core of our work consists in proving (see theorems 3.6 and 4.1 below) that our simpler notion based on Euler characteristics allows to compute boundaries of projections and therefore is sufficient for the targeted applications. We illustrate its practical relevance by showing how it can be used to design robust algorithms for the computation of Minkowski sums and swept volumes that are of prime interest in many applications such as geometric modeling & animation, design & manufacturing. However, the analysis of the complexity and the description of data structures for optimal algorithms are beyond the scope of the paper.

The paper is organized as follows. Section 2 quickly presents the necessary mathematical notions. Section 3 and 4 introduce our framework and establish the main results. Practical illustrations are given in section 5.

2. Mathematical Preliminaries

The results of this paper rely upon two mathematical notions that we introduce quickly (for a complete and detailed presentation the reader is referred to the references below). The first one is the classical concept of combinatorial manifold that extends the notion of triangulation in 2D and 3D. The second one, less classical, has been introduced by H. Whitney in the setting of Geometric Integration Theory [Whi57]. It extends the notion of polyhedron and provides a well-suited framework to describe silhouettes and images of maps considered in the sequel.

Simplicial complexes and combinatorial manifolds [Mun93, BB04] A *(finite) simplicial complex* K in $\mathbb{R}^n$ is a (finite) collection of simplices in $\mathbb{R}^n$ such that every face of a simplex of K is also in K and the intersection of any two simplices of K is either empty or a common face of each of them. In this paper, all the complexes are finite. By *face* (resp. *proper face*) of a simplex τ we mean a simplex σ spanned by a subset (resp. a proper subset) of the vertices of τ. Similarly, one says that a simplex τ of the simplicial complex is a *coface* (resp. a *proper coface*) of a simplex σ if σ is a face (resp. a proper face) of τ. The *support* of K, denoted by $|K|$, is the union of the simplices of K. The dimension of K is defined as the maximal dimension of the simplices of K and, given $k \leq \dim(K)$, the *k-skeleton* of K, denoted by $Sk_k(K)$ is the subcomplex of K consisting of the simplices of dimension at most k.

Given a simplex σ in K, the *(open) star* of σ, denoted $\mathrm{St}(\sigma)$, is the union of the relative interiors of all the simplices having σ as a face. The *closed star* of σ, denoted $\overline{\mathrm{St}}(\sigma)$, is the closure of the open star of σ, i.e. the union of all the simplices having σ as a face. The *link* of σ, denoted $\mathrm{Lk}(\sigma)$ is the union of all the simplices lying in $\overline{St}(\sigma)$ that are disjoint from σ, as illustrated on figure 2. A simplicial complex K is a *combinatorial n-manifold* if the link of any k-simplex of K is homeomorphic to $\mathbb{S}^{n-k-1}$, where $\mathbb{S}^{n-k-1}$ denotes the $n-k-1$ dimensional sphere. A *combinatorial n-manifold with boundary* is a simplicial complex for which the set of simplices is partitioned into two categories: the *interior k-simplices*, for which the link is homeomorphic to $\mathbb{S}^{n-k-1}$ and the *boundary k-simplices* for which the link is homeomorphic to the unit ball $\mathbb{D}^{n-k-1}$ of dimension $(n-k-1)$.

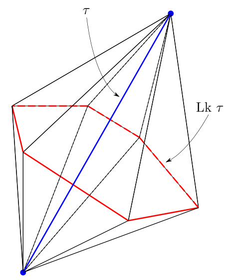

Figure 2: *The edge τ (in blue), its star (in black) and its link (in red).*

Polyhedral chains [Whi57] are extensions of the classical notions of polyhedra and polyhedral regions. They can be seen as formal sums of weighted polyhedra and are defined in the following way. Let $P_1, \cdots, P_m$ be bounded non-overlapping oriented polyhedral regions in $\mathbb{R}^r$ and let $a_1, \cdots, a_m$ be real coefficients. The formal sum $A = \sum_{i=1}^m a_i P_i$ is a *polyhedral r-chain* in $\mathbb{R}^r$. Choosing an orientation of $\mathbb{R}^r$, A is also determined by the piecewise constant, real valued function $A(.)$ defined over $\mathbb{R}^r$, which equals a_i or $-a_i$ in the interior of P_i according as P_i is oriented like or opposite to $\mathbb{R}^r$ for $i = 1, \cdots m$ and equals 0 elsewhere. Since A is entirely defined by $A(.)$ and vice-versa, we use in the sequel the same notation A for an r-chain and its corresponding function. The notion of equality between r-chains is defined using their corresponding functions: two r-chains A and B are equal, and we write $A = B$, if the corresponding functions are equal except in a finite set of polyhedral cells of dimension less than r. As we will see in the following, the ability of discarding a finite set of cells of dimension less than r to prove the equality of two r-chains is extensively used in the proofs. The set of polyhedral r-chains in $\mathbb{R}^r$ naturally inherits a structure of linear space: the product of a chain A with a real a being represented by the function $aA(.)$

and the sum of two chains A and B being represented by the sum of functions $A(.)+B(.)$. Now, a *polyhedral r-chain A in* $\mathbb{R}^d$ ($d \geq r$) consists of a finite set of distinct oriented r-dimensional affine subspaces together with a polyhedral r-chain in each. Again, A can be defined as a function where, once an orientation of each r-plane has been chosen, $A(.)$ is defined in each r-planes. The sum of two such r-chains A and B is defined by summing $A(.)$ and $B(.)$ in each r-planes occurring in A and B. The *support* of a r-chain A, denoted by $|A|$ is the closure of the union of the polyhedral cells on which $A(.)$ is not equal to 0.

A *polyhedral convex cell* in $\mathbb{R}^r$ (or *cell* in the following for short) is a non empty bounded intersection of a finite number of closed half-spaces bounded by hyperplanes. Any r-chain A can be expressed as a finite sum $A = \sum a_i \sigma_i$ where each σ_i is a polyhedral convex cell of dimension r and a_i is a real number. Such a decomposition is obviously non unique. Given a polyhedral convex r-cell σ, its *boundary* $\partial\sigma$ is the $(r-1)$-chain which is equal to the sum of the oriented $(r-1)$-dimensional faces of σ. The *boundary* of the r-chain $A = \sum a_i \sigma_i$ is defined as the $(r-1)$-chain $\partial A = \sum a_i \partial P_i$. An important property of the boundary is that it does not depend on the decomposition of A into convex cells and thus defines a linear operator on chains. Moreover the boundary operator satisfies the following property: for any chain A, we have $\partial\partial A = 0$. A *(polyhedral) cycle* is a chain A such that $\partial A = 0$. This boundary operator is the counterpart, in the world of polyhedral chains, of the boundary operator on chains used in simplicial homology theory. It is also related to the notion of *winding number* since for example if $B = \partial A$ is a 1-chain in $\mathbb{R}^2$, the winding number of B around a point x gives the value $A(x)$.

It is important to notice that in general, the support of the boundary of an r-chain A in $\mathbb{R}^r$ is different from the topological boundary $\partial_t |A| = closure(|A|) \setminus interior(|A|)$ of the support of A (see figure 3 left).

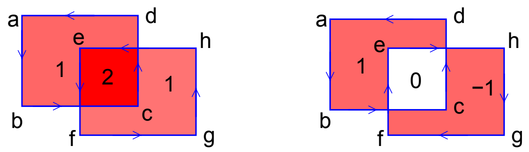

Figure 3: *The rectangles $P_1 = (abcd)$ and $P_2 = (efgh)$ are endowed with the same orientation in $\mathbb{R}^2$. Left: the 2-chain $A = P_1 + P_2$ is defined by the sum of the characteristic functions of P_1 and P_2; its support is the union of the two rectangles. Right: the 2-chain $B = P_1 - P_2$; its support is the symmetric difference of P_1 and P_2. ∂A and ∂B are the sum of the oriented blue segments.*

Euler Characteristic Given a simplicial complex K, we recall that the Euler characteristic of K is the alternate sum $\chi(K) = \sum_{\sigma\in K} (-1)^{\dim(\sigma)}$. The Euler characteristic satisfies the following fundamental *additivity property*: if $K = K_1 \cup K_2$ where K_1 and K_2 are two sub-complexes of K then $\chi(K) = \chi(K_1) + \chi(K_2) - \chi(K_1 \cap K_2)$. Moreover, the Euler characteristic is a topological invariant: if K and K' are two simplicial complexes with homeomorphic supports $|K|$ and $|K'|$ then $\chi(K) = \chi(K')$ and if X is a topological space homeomorphic to $|K|$ then $\chi(X) = \chi(K)$ by definition.

3. χ-projection chains and silhouette cycles of maps defined over simplicial complexes

The class of maps we consider are the continuous maps $\pi : K \longrightarrow \mathbb{R}^d$ where K a simplicial complex with values in $\mathbb{R}^d$ such that the restriction of π to any simplex of K is linear. Given such a map $\pi : K \longrightarrow \mathbb{R}^d$, its image $\pi(K)$ is a polyhedron in $\mathbb{R}^d$. Notice that its (topological) boundary $\partial_t \pi(K) = closure(\pi(K)) \setminus interior(\pi(K))$ cannot be determined by local computations in K. More precisely, deciding whether a point $\pi(p) \in \mathbb{R}^d$, $p \in K$, is on $\partial_t \pi(K)$ cannot be done by only considering the map π in a neighborhood of p (see figure 4). This generally makes the computation of $\partial_t \pi(K)$ prohibitively tricky and expensive. Taking advantage of the framework of polyhedral chains, instead of considering the image $\pi(K)$ as a simple polyhedron, we introduce a chain, called the χ-projection chain, that carries topological information about the fibers $\pi^{-1}(x)$, $x \in \mathbb{R}^d$, of π. Its boundary (as a chain), called the silhouette cycle, appears as a weak version of the actual topological boundary of $\pi(K)$. The two main results of this paper, presented in the two next sections, state that the silhouette cycle can be easily obtained by local computations in K and that in many cases of practical importance the support of the silhouette cycle contains the topological boundary of $\pi(K)$ (theorems 3.6 and 4.1).

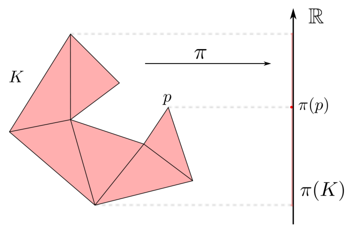

Figure 4: *A map π from a 2-dimensional complex K to $\mathbb{R}$. Considering the restriction of π to a neighborhood of p suggests that $\pi(p)$ is a boundary point of $\pi(K)$ which is obviously not the case.*

3.1. χ-projection chain and silhouette cycle

Definition 3.1 Let K be a simplicial complex and $\pi : K \longrightarrow \mathbb{R}^d$ a continuous map which is linear on each simplex of K. The χ-projection chain $p_\pi(K)$, associated to the pair (K, π), is the polyhedral d-chain in $\mathbb{R}^d$ corresponding to the function $\chi \circ \pi^{-1}$ which associate to each point $x \in \mathbb{R}^d$ the integer $\chi(\pi^{-1}(x))$.

Note that, by the definition of equality between d-chains, the function $x \mapsto \chi(\pi^{-1}(x))$ defines $p_\pi(K)$ up to a finite set of polyhedral cells of dimension less than d. In particular, if K is a simplicial complex of dimension less than d or if $\pi(|K|)$ is contained in a finite union of affine subspaces of dimension $d-1$, then $\chi(\pi^{-1}(x))$ is equal to 0 except on $\pi(|K|)$ which is of dimension less than d, so the d-chain $p_\pi(K)$ is equal to 0.

The main advantage of considering the Euler characteristic of the fibers of π to define the χ-projection chain is that it immediately inherits the important additivity property from the Euler characteristic.

Lemma 3.2 (Additivity of the χ-projection chain) Let K_1 and K_2 be two subcomplexes of K such that $K = K_1 \cup K_2$. If π_1, π_2 and $\pi_{1,2}$ denote respectively the restrictions of π to K_1, K_2 and $K_1 \cap K_2$, one has $p_\pi(K) = p_{\pi_1}(K_1) + p_{\pi_2}(K_2) - p_{\pi_{1,2}}(K_1 \cap K_2)$.

Proof Since for every $p \in \mathbb{R}^d$, $\pi^{-1}(p) = \pi_1^{-1}(p) \cup \pi_2^{-1}(p)$, the additivity of the Euler characteristic gives
$\chi(\pi^{-1}(p)) = \chi(\pi_1^{-1}(p)) + \chi(\pi_2^{-1}(p)) - \chi(\pi_1^{-1}(p) \cap \pi_2^{-1}(p)) = \chi(\pi_1^{-1}(p)) + \chi(\pi_2^{-1}(p)) - \chi(\pi_{1,2}^{-1}(p))$ □

Definition 3.3 Let K be a simplicial complex and $\pi : K \to \mathbb{R}^d$ a continuous map which is linear on each simplex of K. The silhouette cycle $s_\pi(K)$ of (K, π) is the boundary of the χ-projection chain $p_\pi(K)$: $s_\pi(K) = \partial p_\pi(K)$. It is a $(d-1)$-chain in $\mathbb{R}^d$.

The definition of the χ-projection chain and the linearity of the boundary operator on the space of d-chains, easily gives the following properties of the silhouette cycle.

Lemma 3.4 (Additivity of the silhouette cycle) If K_1 and K_2 are two subcomplexes of K such that $K = K_1 \cup K_2$ then $s_\pi(K) = s_\pi(K_1) + s_\pi(K_2) - s_\pi(K_1 \cap K_2)$

Lemma 3.5 The map $y \mapsto \chi\left[\pi^{-1}(\{y\})\right]$ is constant on each connected component of the complement of the support of the silhouette cycle.

Example: Let P be a polygonal region in $\mathbb{R}^2$ and let Γ be a piecewise linear curve in $\mathbb{R}^2$ homeomorphic to a segment (see figure 5). Triangulating the product $P \times \Gamma$ gives a 3-dimensional simplicial complex K in $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. The image of the map $\pi : K \to \mathbb{R}^2$ defined by $\pi(x, \gamma) = x + \gamma$ is the union of the translated polygonal regions $T_\gamma(P)$, $\gamma \in \Gamma$. In other words, $\pi(K)$ is the area swept by P moving along Γ. The points of $\pi(K)$ are exactly the points $x \in \mathbb{R}^2$ such that $\pi^{-1}(x)$ is not empty. Since $\pi^{-1}(x)$ is homeomorphic to a finite union of polygonal curves in Γ, $\chi(\pi^{-1}(x))$ is equal to the number of connected components of $\pi^{-1}(x)$ and $\pi(K)$ is equal to the support of $p_\pi(K)$. The figure 5 shows the χ-projection chain and the silhouette cycle when P is a square and Γ a very simple polygonal curve. The pink (resp. red) part represent the points where the χ-projection chain is equal to 1 (resp. 2). The bold black self-intersecting curve represents the silhouette cycle. Figure 6 depicts a similar situation where the polygon P is not convex, while the trajectory is a line segment.

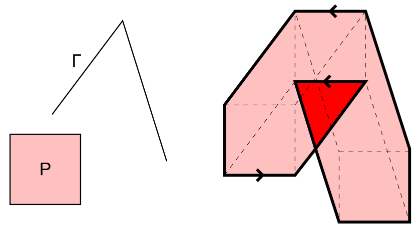

Figure 5: *The χ-projection chain and the silhouette cycle of a 2D polygon P swept along a piecewise linear curve Γ. The pink (resp. red) part represent the points where the χ-projection chain is equal to 1 (resp. 2). The bold black self-intersecting curve represents the silhouette cycle.*

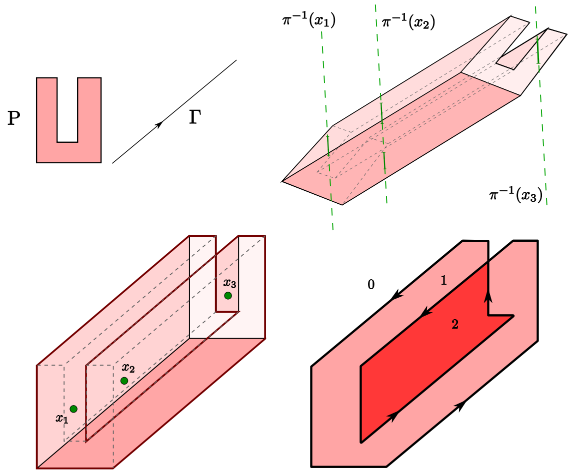

Figure 6: *The χ-projection chain and the silhouette cycle (lower right) of a non convex 2D polygon P swept along a linear segment Γ (upper left). Upper right: the product complex and the inverse images by the projection. Lower left the product seen from the projection direction.*

3.2. A local formula for the silhouette cycle

Let K be a simplicial complex and let $\pi : K \to \mathbb{R}^d$ a continuous map which is linear on each simplex of K. From now on we assume that an orientation of $\mathbb{R}^d$ has been chosen and that the images of the simplices of K by π satisfy the following generic property:

(G) the image $\pi(\sigma)$ of any simplex σ of K of dimension at most d is a non degenerate simplex in $\mathbb{R}^d$, i.e. the smallest affine subspace containing $\pi(\sigma)$ has same dimension as σ.

Assuming Property (G) allows us to give a local formula to compute the silhouette cycle. The constraint it imposes can be easily overcome in practice as discussed in section 5.1. We also assume in the sequel that any simplex of K of dimension less than d has at least one coface of dimension d.

Note that such an assumption is always satisfied when K is a combinatorial d-manifold.

Assume that an arbitrary orientation on each $(d-1)$-face of K is given. For any $(d-1)$-face t of K the simplex $T = \pi(t)$ inherits an orientation from t. The *normal vector* n_T *to* T is the unit length vector normal to T such that the orientation of T together with n_T define the chosen orientation of $\mathbb{R}^d$. The hyperplane H_T passing through T is the boundary in $\mathbb{R}^d$ of two closed half-spaces H_T^+ and H_T^-, H_T^+ being the one in which n_T is pointing inside.

Theorem 3.6 The silhouette cycle $s_\pi(K)$ of (K,π) is equal to a $(d-1)$-chain in $\mathbb{R}^d$ obtained as a weighted sum of the oriented images $T = \pi(t)$ of the $(d-1)$-faces of K. The weight w_T of the $(d-1)$-simplex $T = \pi(t)$ is given by

$$w_T = \sum_{\substack{\partial\sigma \ni t \\ \pi(\sigma)\subseteq H_T^+}} (-1)^{\dim(\sigma)-d} - \sum_{\substack{\partial\sigma \ni t \\ \pi(\sigma)\subseteq H_T^-}} (-1)^{\dim(\sigma)-d} \qquad (1)$$

where, by a small abuse of notation $\partial\sigma$ denotes here the set of proper faces of σ.

By providing an explicit formula for the silhouette cycle, this theorem shows that its support is contained in the image of the $(d-1)$-skeleton of K and the computation of the weight of the image $T = \pi(t)$ of any $(d-1)$-simplex t boils down to computing on which side of the supporting hyperplane of T the images of the vertices of the cofaces of t are.

Notice that the silhouette cycle does not depend on the arbitrary choice of the orientation of the $(d-1)$-skeleton of K. Indeed, if the orientation of a $(d-1)$-face t is reversed, then the sign of w_T in the previous formula and the orientation of T are reversed, the silhouette cycle remaining unchanged.

The proof of theorem 3.6 is a consequence of the following fundamental lemma relating the silhouette cycle to the Euler characteristic of the fibers of π when K is reduced to a simplex.

Lemma 3.7 Let Σ be a simplex and π a linear map from Σ to $\mathbb{R}^d$ satisfying the generic property (G). Let t be a $(d-1)$-face of Σ and let $p^+ \in H_T^+ \setminus H_T$ and $p^- \in H_T^- \setminus H_T$ be two points that are arbitrarily close to the (relative) interior of $T = \pi(t)$. One has:

$$\chi(\pi^{-1}(p^+)) - \chi(\pi^{-1}(p^-)) = \sum_{\substack{\partial\sigma \ni t \\ \pi(\sigma)\subseteq H_T^+}} (-1)^{\dim(\sigma)-d} - \sum_{\substack{\partial\sigma \ni t \\ \pi(\sigma)\subseteq H_T^-}} (-1)^{\dim(\sigma)-d} \qquad (2)$$

As a consequence $s_\pi(\Sigma)$ satisfies the equation (1) of theorem 3.6.

Proof Let n be the dimension of Σ. First observe that if $n \leq d-1$ then there is a $(d-1)$ hyperplane of $\mathbb{R}^d$ that contains $\pi(\Sigma)$. As a consequence the two sides of equation (2) are equal to 0 for any $p \in \mathbb{R}^d$. The lemma thus follows immediately in this case. So we can now assume that $n \geq d$. The map π being linear, the image $\pi(\Sigma)$ is a convex polyhedron in $\mathbb{R}^d$ and for any $p \in \mathbb{R}^d$, $\pi^{-1}(p)$ is either a convex set or empty so that $\chi(\pi^{-1}(p))$ is equal to 0 or 1. This implies that $s_\pi(\Sigma) = \partial p_\pi(\Sigma)$ is equal to the sum of the faces of the convex polyhedron $\pi(\Sigma)$, i.e. a weighted sum of the images of the $(d-1)$-simplices of Σ. The weight of each of these images is exactly given by the left-hand side of equation (2). It is thus sufficient to prove Equation (2) to get the conclusion of the lemma. We distinguish two cases.

Case 1: t is a $(d-1)$-simplex of Σ such that T is in the (topological) boundary of $\pi(\Sigma)$. $\pi(\Sigma)$ being convex, it is contained in one of the two half-spaces H_T^+ or H_T^- defined by the plane containing T. The left hand side of equation (2) is equal to $+1$ or -1. If it is for example equal to -1, then the right hand side of equation (2) is equal to

$$-\sum_{\substack{\partial\sigma \ni t \\ \pi(\sigma)\subseteq H_T^-}} (-1)^{\dim(\sigma)-d} = -\sum_{k=d}^{n} (-1)^{k-d} \binom{n-d+1}{k-d+1}$$

$$= \sum_{j=1}^{n-d+1} (-1)^j \binom{n-d+1}{j} = (1-1)^{n-d+1} - 1 = -1$$

the first equality following from the fact that the number of k-dimensional cofaces of a $(d-1)$-simplex in a n-simplex is equal to $\binom{n-d+1}{k-d+1}$.

Case 2: t is a $(d-1)$-simplex of Σ such that the (relative) interior of T is contained in $\pi(\Sigma)$. Then $\pi^{-1}(p^+)$ and $\pi^{-1}(p^-)$ are both non empty, so the left hand side of equation (2) is equal to 0. To evaluate the right hand side of the equation let n_1 and n_2 be the numbers of vertices of Σ contained respectively in H_T^+ and H_T^- (one thus has $n_1 + n_2 = n - d + 1$). Consider the restriction of π to the simplex generated by the union of the vertices of t and the n_1 vertices of Σ projected in H_T^+. T is contained in the boundary of the image of this simplex and we can apply the case 1 which gives

$$1 = \sum_{\substack{\partial\sigma \ni t \\ \pi(\sigma)\subseteq H_T^+}} (-1)^{\dim(\sigma)-d}$$

Similarly, applying case 1 to the simplex generated by the union of the vertices of t and the n_2 vertices of Σ projected in H_T^- gives

$$-1 = -\sum_{\substack{\partial\sigma \ni t \\ \pi(\sigma)\subseteq H_T^-}} (-1)^{\dim(\sigma)-d}$$

As a consequence the right hand side of equation 2 is equal to 0. □

Proof [of the Theorem 3.6]
Note that if K is a simplex, then the result is given by the lemma 3.7. Now notice that the righ-hand side of Equation (1) satisfies the same additivity property as the silhouette cycle $s_\pi(K)$ (Lemma 3.4). More precisely, let K_1 and K_2 be two subcomplexes of K such that $K = K_1 \cup K_2$ and let t be a $(d-1)$-simplex of K. If w_{T_1}, w_{T_2} and $w_{T_{12}}$ are the weights

given by the equation 1 when one restricts the sum to the cofaces of t that are contained in K_1, K_2 and $K_1 \cap K_2$ respectively then $w_{T_1} + w_{T_2} = w_T + w_{T_{12}}$. The proof of the theorem follows by a simple induction on the number of simplices of K. $\square$

Remark 1 The previous results can be generalized to convex cell complexes introduced in [Ban67]. Roughly speaking, in a convex cell complex, the required property on cells of simplicial complexes to be simplices is weakened to the property to be polyhedral convex cells and the restriction of the map π to each convex cell of the complex is required to be affine (thus the value of π on the vertices is constrained by this condition). This generalization does not introduce new ideas but can be useful in some practical applications where the complexity of a convex cell complex may be simpler than the complexity of a simplicial complex having the same support.

4. Topological boundary of images of combinatorial manifolds

Our goal is to compute the topological boundary $\partial_t \pi(K)$ of the image $\pi(K)$ of a simplicial complex by a piecewise linear map π. However, in general, the value of the function associated to the χ-projection chain, which is the Euler characteristic of a fiber $\pi^{-1}(p)$, may be equal to zero even if $\pi^{-1}(p)$ is non empty. In this case, $\partial_t \pi(K)$ is not necessarily a subset of the support of the silhouette cycle $s_\pi(K)$. Fortunately, this problem can be overcome in many practical situations thanks to Theorem 4.1 stated below. In section 5 we illustrate on two cases (computation of Minkowski sums and swept volumes) how Theorems 3.6 and 4.1 allow us to design simple and reliable algorithms.

We call the *outer skin* of a bounded set $E \subset \mathbb{R}^d$, the topological boundary of the unbounded connected component of the complement of E in $\mathbb{R}^d$.

Theorem 4.1 Let $n \geq d \geq 1$ be two integers such that $n-d$ is even (or zero). Let K be a combinatorial n-manifold, with or without boundary, and $\pi : K \to \mathbb{R}^d$ be a piecewise linear map satisfying the generic condition (G). Then, $\partial_t \pi(K)$ is a subset of the support of the silhouette cycle $s_\pi(K)$. In particular, the outer skin of $\pi(K)$ coincides with the outer skin of the support of the silhouette cycle $s_\pi(K)$.

The proof of Theorem 4.1 relies on the key technical Lemma 4.2 below. For $x \in \mathbb{R}^d$, let us denote by K_x^i the set of simplices of K of dimension i whose image have x in their relative interior:

$$K_x^i = \{\tau \in K, \dim \tau = i, x \in \operatorname{Int} \pi(\tau)\}$$

where $\operatorname{Int} \pi(\tau)$ denotes the relative interior of $\pi(\tau)$. Note that generically the cardinal of K_x^{d-1} is 0 for $x \in \mathbb{R}^d$ and 1 for $x \in \partial_t \pi(K)$. For an integer i, we denote by $Sk_i(K)$ the i-skeleton of K defined in Section 2. Observe that $\partial_t \pi(K) \subset \pi(Sk_{d-1}(K))$ is $(d-1)$-dimensional. For technical reasons, we consider points in $\partial_t \pi(K)$ which are neither in the image of $(d-2)$-simplices from K nor in the intersection of two non overlapping images of $(d-1)$-simplices. We therefore exclude from $\partial_t \pi(K)$ a set of points included in a finite union of $(d-2)$-dimensional affine spaces and has therefore a zero $(d-1)$-dimensional measure. Removing this set of points, we get the following subset B of $\partial_t \pi(K)$ of $\pi(K)$ which contains "almost all" the points in $\partial_t \pi(K)$:

$$\begin{aligned} \mathrm{B} &= (\partial_t \pi(K) \setminus \pi(Sk_{d-2}(K))) \\ &\cap \left\{ x \in \mathbb{R}^d, \dim \operatorname{Aff} \left(\bigcup_{\tau \in K_x^{d-1}} \pi(\tau) \right) = d-1 \right\} \end{aligned}$$

where, for a set X, $\operatorname{Aff} X$ denotes the affine space spanned by X.

Lemma 4.2 For any $x \in \mathrm{B}$ there is $\varepsilon > 0$ such that, for any $y \in B(x, \varepsilon) \cap \operatorname{Int} \pi(K)$:

$$\chi\left[\pi^{-1}(y)\right] = \sum_{\tau \in K_x^{d-1}} \chi\left[\operatorname{Lk} \tau\right]$$

where $\operatorname{Lk} \tau$ denotes the link of τ in K.

The proof of this lemma is technical and postponed to Appendix A. Figure 7 illustrate the lemma in a simple case.

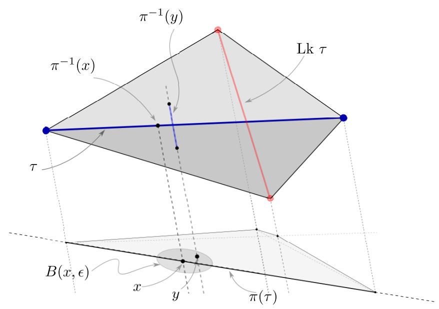

Figure 7: *The inverse image of a point y in a neighborhood of x is homeomorphic to* $\operatorname{Lk} \tau$.

Proof [of Theorem 4.1] We apply Lemma 4.2 in the special situation of the theorem. In our case, the simplices τ of the Lemma 4.2 are $d-1$ simplices and, according to the definition of combinatorial manifolds given in section 2, the link $\operatorname{Lk} \tau$ in K is either homeomorphic to a $(n-d)$-sphere $\mathbb{S}^{n-d}$ or to a $(n-d)$-disk $\mathbb{D}^{n-d}$. Recall that the Euler characteristic of a k-sphere is 2 when k is even and the Euler characteristic of a disk is always 1. In the former case one has $\chi\left[\operatorname{Lk} \tau\right] = \chi\left[\mathbb{S}^{n-d}\right] = 2$ since $n-d$ is even and in the latter case $\chi\left[\operatorname{Lk} \tau\right] = \left[\mathbb{D}^{n-d}\right] = 1$. Therefore, in any neighborhood of a point $x \in B \subset \partial_t \pi(K)$, the function $y \mapsto \chi\left[\pi^{-1}(y)\right]$ defining the χ-projection chain takes a strictly positive value inside $\pi(K)$ and obviously the value 0 outside. Using Lemma

3.5 we get that x must belong to the support of the silhouette cycle. Since $\partial_t \pi(K)$ is the closure of B and since the support of the silhouette cycle is closed we get that $\partial_t \pi(K)$ is a subset of the support of $s_\pi(K)$ denoted $|s_\pi(K)|$. Since this support is obviously a subset of $\pi(K)$ we get $\partial_t \pi(K) \subset |s_\pi(K)| \subset \pi(K)$ and it follows that the unbounded component of the complement of $\pi(K)$ coincides with the unbounded component of the complement of $|s_\pi(K)|$ which proves our claim on outer skins. □

5. Application to the computation of generalized sweeps

In this section we illustrate the applicability of Theorems 3.6 and 4.1 on two important practical cases.

5.1. Minkowski sums in 2D and 3D

The Minkowski sums $A \oplus B$ of two solids A and B, defined as $A \oplus B = \{a+b, a \in A, b \in B\}$ can be expressed as the image, by the projection $(a,b) \mapsto a+b$ of the product $A \times B$. Let us first consider the Minkowski sum of two simple polygons in $\mathbb{R}^2$. Let P_1 and P_2 be the respective curves bounding the polygons. Both P_1 and P_2 may be seen as combinatorial 1-manifolds and the product $P_1 \times P_2$ is a combinatorial 2-manifold after the triangulation of each product of an edge of P_1 by and edge of P_2 into two triangles. The Minkowski sum $P_1 \oplus P_2$ is the image of the projection $\pi : P_1 \times P_2 \rightarrow \mathbb{R}^2$ defined by $\pi(x_1, x_2) = x_1 + x_2$. Theorem 4.1 applies since $n - d = 2 - 2$ is even. The simplest algorithm to compute $s_\pi(P_1 \times P_2)$ consists in iterating on the 1-simplices (the edges) of the product $P_1 \times P_2$ and, for each of these simplices compute the weight by the formula of Theorem 3.6. Each edge for which the corresponding weight is non-zero can be added to the output (connecting the corresponding images of the vertices of the product). If the edges are given the right weight/orientation, the output is the silhouette cycle which support contains the outer skin of $P_1 \oplus P_2$. Notice that in this case the silhouette cycle precisely coincides with the cycle obtained in the context of the kinetic framework [GRS83, BGRR96].

For the three-dimensional case, the product of the two boundaries P_1 and P_2 of two polyhedra, each being a combinatorial 2-manifold, is a combinatorial 4-manifold. Again, $P_1 \oplus P_2$ is the image of the projection $\pi : P_1 \times P_2 \rightarrow \mathbb{R}^3$ defined by $\pi(x_1, x_2) = x_1 + x_2$ but unfortunately one cannot apply directly Theorem 4.1 because $4 - 3 = 1$ is odd. Nevertheless, notice that the outer skin of $P_1 \oplus P_2$ is the same as the one of the image by π of the product of the combinatorial 2-manifold corresponding to the boundary of one of the polyhedron by the combinatorial 3-manifold corresponding to a triangulation (by tetrahedra) of the second polyhedron (or, alternatively, a thickening of the boundary of it). Now this product is a combinatorial manifold of dimension 5 and one can apply Theorem 4.1 ($5 - 3 = 2$ is even!). Again, the simplest algorithm consists in computing the silhouette cycle by iterating on the 2-simplices (triangles) of the product. For each of these triangles, if the associated weight given by theorem 3.6 is non zero, one adds its image by the projection π to the output. The resulting silhouette cycle coincides again with the output given by the 3-dimensional kinetic framework [GRS83, BGRR96] while our algorithm is simpler and more general.

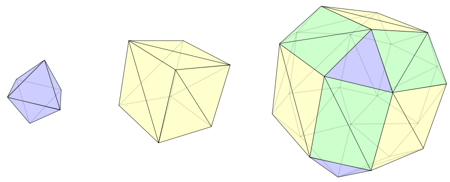

Figure 8: *Minkowski sums between an octahedron and a cube*

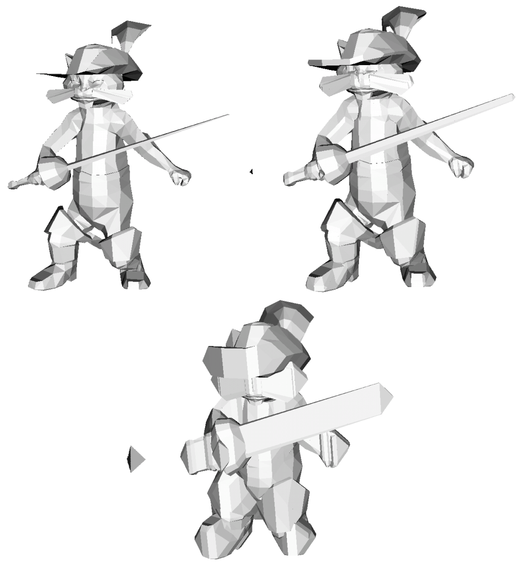

Figure 9: *Minkowski sums between Puss in Boots (upper left) and a small tetrahedron (upper right) and a big one (bottom).*

The discussion on the optimal complexity of this approach is beyond the scope of this paper but an adequate data structure should allow to accelerate the previous algorithm with fast rejection tests avoiding to consider individually all the 2-simplices in the product. A simple but robust implementation of the algorithm for 3-dimensional Minkowski sums has been realized and figures 8 and 9 show some results. The robustness of this implementation relies on three points. The use of exact predicates for the evaluation of the weight given by theorem 3.6, the use of symbolic perturbation [Ede87, EM90] to manage the situations where the generic condition (G) is not satisfied and, last but not least, the fact that the output of the algorithm is expressed as a polyhedral cycle. This polyhedral cycle is represented by a set of oriented triangles sharing a set of vertices and the zero boundary condition allows to define what is called a winding number in other settings. As explained for example in [MS07], such a representation is non redundant and any small change (such as the effect of rounding errors in the coordinates of projected points) does not break the validity of the output and results only in a local and controlled geometric perturbation. In fact, in the particular case of Minkowski sums, it was more convenient in the implementation to consider the product $P_1 \times P_2$ as a convex cell complex whose cells are the products of the simplices of P_1 by the simplices of P_2 since the restriction of the projection $\pi(x_1, x_2) = x_1 + x_2$ to each such convex product cell is affine. This is not possible in the situation considered in the next section if the trajectory contains not only translations but also rotations.

The simple but robust implementation mentioned above is usable in practice for inputs polyhedra of reasonable size: its complexity is obviously the size of the product complex, which is $O(nm)$ where n and m are the respective numbers of simplices in the boundaries of the two given polyhedra. As explained above, in 2D and 3D the kinetic framework [GRS83, BGRR96], provides precisely the same output than our algorithm. A naive implementation of the kinetic framework would lead to the same complexity than our simple algorithm. Conversely, in the particular case of Minkowski sums, our algorithm could benefit from the same optimisation proposed for the three-dimensional kinetic framework. As for the kinetic framework, the output of our algorithm is a 2-chain that may self-intersect but whose outer skin coincides exactly with the Minkowski sum outer skin. However, the combinatorial complexity of the polyhedral representation of this outer skin is $O(n^3m^3)$ in the worst case [FH07], in other words, resolving the self-intersectons of a 2-chain with complexity $O(nm)$ may produce a non self-intersecting polyhedron of complexity $O(n^3m^3)$. A natural method for Minkowski sum computation [FH07] consists in three steps. First step decomposes the operands into convex polyhedrons, second step computes all the Minkowski sums between each convex component of the first operand and each convex component of the other operand and third step computes the union of all the resulting convex sums. Since at the end of he second step one gets a collection of polytopes whose outer skin is exactly the outer skin of the actual Minkowski sum, our method can be compared to the two first steps, since the third step consists in resolving all the intersections in order to obtain a classical (non self-intersectiong) polyhedral representation. However, in the worst case, the convex decomposition of a polyhedron can produce up to $O(n^2)$ pieces and therefore the combinatorial complexiy of the result of step two can be $O(n^2m^2)$.

5.2. Sweeping a solid along a trajectory

A second example for which the silhouette cycle allows to determine reliably the outer skin of $\pi(K)$ is the volume swept by a body P along a one dimensional trajectory γ. More formally, let GF(3) be the group of affine transformations acting on $\mathbb{R}^3$. Each element of GF(3) can be represented by an invertible 3×3 matrix together with a translation vector. From this representation GF(3) can be seen as a subset of the vector space $\mathbb{R}^{12}$, which makes it possible to define piecewise linear paths in GF(3). Let P a be combinatorial 3-manifold (with boundary) embedded in $\mathbb{R}^3$ triangulating a 3-dimensional polyhedron and $\gamma : [a,b] \to \mathrm{GF}(3)$ be a continuous, piecewise linear path in GF(3). For example, γ could be the discretization of a trajectory in the space of rigid displacements. In this later case the discretized trajectory is not exactly isometric but the corresponding "chordal error" can be bounded. We require the *local preservation of orientation*, namely that for any parameter t of the discretized trajectory and any tetrahedron (p,q,r,s) of P, $\det\left(\overrightarrow{\gamma(t)p\,\gamma(t)q}, \overrightarrow{\gamma(t)p\,\gamma(t)r}, \overrightarrow{\gamma(t)p\,\gamma(t)s}\right)$ is non zero and has the sign of $\det\left(\overrightarrow{pq}, \overrightarrow{pr}, \overrightarrow{ps}\right)$. A consequence of this property is that if u is an interior vertex of P its image $\gamma(t)u$ belongs to the interior of $\gamma(t)\,\mathrm{St}\,u$ and therefore can not be on the boundary of the set $\gamma(t)|P|$. From this property, one see that the outer skin of the volume swept by the polyhedron P moving along the trajectory γ coincides with the outer skin of the volume swept by the boundary of P moving along the trajectory γ. For example, if the path γ consists in a discretization of a trajectory in the group of rigid body displacements, it is sufficient to avoid a rotation of exactly π betweeen two successive discretization points in order to preserve the orientation along the trajectory. Therefore, in this case, the local preservation of orientation will automatically follow from the choice of a reasonable tolerance on the chordal error of the trajectory discretization. Let us denote by P' the combinatorial 2-manifold corresponding to a triangulation of the boundary of P.

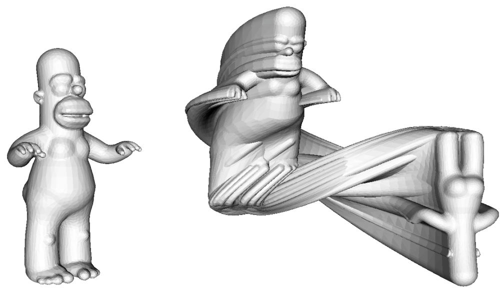

Figure 10: *Homer (left) swept along a rotational trajectory (right)*

Let T, with vertices $t_0 = a, t_1, \ldots, t_n = b$ be the one dimensional simplicial complex associated to a discretization of the parameter space of the trajectory, which means that γ is linear on each interval t_i, t_{i+1}. We denote by K the combinatorial 3-manifold corresponding to the product $T \times P'$, after a consistent triangulation of the convex cells product of simplices of T with simplices of P'. The projection map π is defined by $\pi(t_i, v_j) = \gamma(t_i)v_j$ where $\gamma(t_i)v_j$ represents the image of v_j by the transformation $\gamma(t_i)$, for any vertex $t_i \in T$ and $v_j \in P$ and is linear on any simplex of K. Again, the dimension of the product is 3 and theorem 4.1 applies since $n-d=3-3=0$ is even. Figure 10 shows an example of the support of the silhouette cycle corresponding to a rotational sweep implemented again with a simplest algorithm which iterate on each triangle of the product complex and, for each of these triangles compute the weight asociated to its image by π using the formula of Theorem 3.6.

The preceding construction generalizes to the situation where a triangulated solid P is swept along a m-dimensional trajectory γ, for example a robot arm with m degrees of freedom. Formally, P is assumed to be a combinatorial 3-manifold with boundary embedded in $\mathbb{R}^3$, in other words the triangulation of a solid three-dimensional polyhedron with non degenerate tetrahedra. T is a combinatorial m-manifold with or without boundary, which could be for example a discretization of the configuration space of the robot obtained as the product of the discretization of the degrees of freedom. To each vertex v_i of T we assign a space transformation $\gamma(v_i)$, that could be for example the rigid transformation corresponding to the given configuration of the robot. If m is even, the dimension of the product is $n=m+3$ and theorem 4.1 applies since $n-d=m+3-3=m$ is even. Now if m is odd, assuming, as in the case of a one-dimensional trajectory the local preservation of orientation, on can consider the product of T with the combinatorial 2-manifold P' corresponding to the triangulation of the boundary of P. Again, $n-d=m+2-3$ which is even and 4.1 applies.

6. Conclusion

In this paper we have introduced a general framework for the robust computation of Minkowski sums and swept volumes in any dimension. Building on the notion of polyhedral chains, we have introduced a formal notion of critical values for continuous piecewise linear maps defined on simplicial complexes with values in an Euclidean space $\mathbb{R}^d$. In this framework the set of such critical values is described by the silhouette cycle that turns out to be very simply characterized: the computation of the coefficient of each $(d-1)$-simplex only involves the cofaces of the simplex and does not require any geometric construction. We have shown how the topological boundary of the images of combinatorial manifolds can be deduced from the silhouette cycle and as examples of applications we have described simple algorithms for swept volumes and Minkowski sums computations in a polyhedral setting. In the particular case of Minkowski sums, the suggested algorithm, while being simpler to implement in dimension 3 and higher, provides the same combinatorial output as the kinetic framework of Guibas and Al [GRS83, BGRR96]. Practical comparaisons on realistic cases with other methods, including aproximate methods [VM06], is beyond the scope of the paper.

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A. Appendix: proof of Lemma 4.2

Note: This appendix includes a complete proof of the lemma 4.2 which is technical but not necessary for the understanding of the results of the paper.

Lemma 4.2 follows from Lemmas A.1, A.2 and A.3 below. In some non generic situations that are not excluded by condition (G), it may happen that a point $x \in \mathrm{B}$ lies in the interior of two or more overlapping images of $(d-1)$-simplices in K_x^i. The following Lemma says that in this case the open stars of these $(d-1)$-simplices are disjoint.

Lemma A.1 For $x \in \mathrm{B}$ and $\tau_1, \tau_2 \in K_x^{d-1}$

$$\tau_1 \neq \tau_2 \Rightarrow \mathrm{St}\,\tau_1 \cap \mathrm{St}\,\tau_2 = \emptyset,$$

where $\mathrm{St}\,\tau$ denotes the (open) star of τ in K.

Proof Assume that $\mathrm{St}\,\tau_1 \cap \mathrm{St}\,\tau_2 \neq \emptyset$. Because $\tau_1 \neq \tau_2$ this means that τ_1 and τ_2 share a common proper coface σ. Let σ' be the face of σ whose vertices are the union of the set of vertices of τ_1 and the set of vertices of τ_2. Since $\tau_1 \neq \tau_2$, $\dim \sigma' \geq d$. Because $x \in \mathrm{B}$, $\dim \mathrm{Aff}\,[\pi(\tau_1) \cup \pi(\tau_2)] = d-1$. From the linearity of π on σ', $\mathrm{Aff}\,\pi(\sigma') = \mathrm{Aff}\,[\pi(\tau_1) \cup \pi(\tau_2)]$. But if we have simultaneously $\dim \mathrm{Aff}\,\pi(\sigma') = d-1$ and $\dim \sigma' \geq d$, this contradicts the generic condition (G). $\square$

The next lemma states that for a point $x \in \mathrm{B}$, the preimage of points in a neighborhood of x are included in the open stars of simplices of K_x^{d-1}.

Lemma A.2 For any $x \in \mathrm{B}$ there is $\varepsilon > 0$ such that:

$$\pi^{-1}[B(x,\varepsilon)] \subset \bigcup_{\tau \in K_x^{d-1}} \mathrm{St}\,\tau$$

where $B(x,\varepsilon)$ is the open ball with center x and radius ε.

Proof Let K' be the set of simplices of K which are not a coface of a simplex in K_x^{d-1}: $K' = K \setminus \bigcup_{\tau \in K_x^{d-1}} \mathrm{St}\,\tau$. Observe that if $\tau \in K'$ and σ is a face of τ, then $\sigma \in K'$. Therefore K' is a subcomplex of K and thus $\pi(K')$ is a compact subset of $\mathbb{R}^d$ that does not contain x. As a consequence, $d(x, \pi(K')) > 0$. Taking $\varepsilon = \frac{1}{2} d(x, \pi(K'))$, one has $B(x,\varepsilon) \cap \pi(K') = \emptyset$ which gives $\pi^{-1}[B(x,\varepsilon)] \subset K \setminus K' = \bigcup_{\tau \in K_x^{d-1}} \mathrm{St}\,\tau$. $\square$

The next lemma gives a characterization of the topology of $\pi^{-1}(y) \cap \mathrm{St}\,\tau$ for y in a neighborhood of $x \in \mathrm{B}$ and $\tau \in K_x^{d-1}$.

Lemma A.3 For any $x \in \mathrm{B}$ and $\tau \in K_x^{d-1}$, there exists $\varepsilon > 0$ such that for all $y \in B(x,\varepsilon) \cap \mathrm{Int}\,\pi(K)$, $\pi^{-1}(y) \cap \mathrm{St}\,\tau$ is homeomorphic to $\mathrm{Lk}\,\tau$, where $\mathrm{Int}\,\pi(K)$ denotes the interior of $\pi(K)$

Proof The proof of this Lemma is is a little bit technical but the idea is simple. To each proper coface σ of τ corresponds a simplex $\eta_\sigma \subset \mathrm{Lk}\,\tau$ whose vertices are the vertices of σ which are not vertices of τ and the map $\sigma \mapsto \eta_\sigma$ defines a one to one correspondence between the set of proper cofaces of τ and the simplices subsets of $\mathrm{Lk}\,\tau$. Figure 2 illustrates this situation for a 1-simplex τ in a three-dimensional complex, each triangle (resp. tetrahedron) coface of τ corresponds to a vertex (resp. an edge) of the link. Under the hypothesis of the Lemma, $\pi^{-1}(y) \cap \sigma$ is a simplex homeomorphic to η_σ and $\pi^{-1}(y) \cap \sigma_1$ is a face of $\pi^{-1}(y) \cap \sigma_2$ if and only if η_{σ_1} is a face of η_{σ_2}. It results that $\pi^{-1}(y) \cap \mathrm{St}\,\tau$ is homeomorphic to $\mathrm{Lk}\,\tau$ as depicted on Figure 7. The formal proof building explicitly the homeomorphism is rather technical and available in the full version of the paper [CLM09].
$\square$