Normal-map between normal-compatible manifolds

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Consider two $(n-1)$-dimensional manifolds, S and S' in $\mathbb{R}^n$. We say that they are normal-compatible when the closest projection of each one onto the other is a homeomorphism. We give a tight condition under which S and S' are normal-compatible. It involves the minimum feature size of S and of S' and the Hausdorff distance between them. Furthermore, when S and S' are normal-compatible, their Frechet distance is equal to their Hausdorff distance. Our results hold for arbitrary dimension n.

Keywords: local feature size, minimum feature size, surface homeomorphism, isotopy, normal-projection

1. Introduction

For clarity, let us first consider two nearly similar smooth curves, C and C', in the plane. Then we will generalize the proposed concepts to hypersurfaces in arbitrary dimension. Such pairs of curves appear in several applications. Consider the following examples. C may be an approximation of C' produced for simplification [17] or compression [20]. C and C' may be consecutive frames of a 2D animation [23] or the contours of an organ in consecutive cross-sections [5].

In these applications, it is often important to establish a one-to-one mapping (homeomorphism) between C and C'. For example, one may need to map onto C' the values of attributes (such as color) associated with points along C. Amongst all possible mappings, one is of particular interest: the normal mapping.

To each point q of C', the normal-map p_C associates the closest point $p = p_C(q)$ to q on C. Then each point q of C' may be expressed as the normal offset $q = p + d(p)N(p)$ where $N(p)$ is a unit vector normal to C at p. We say that p is the normal projection of q onto C and can express C' as a deformation of C completely defined by the normal displacement field $d(p)$. Such a deformation (or its inverse which we called the OrthoMap in [9]) may be used to construct a 2D animation that will evolve C into C' or a surface in 3D that will interpolate two consecutive cross-sections, C and C'. Furthermore, to support multi-resolution graphics and compressed progressive transmission, C' may be encoded as a composition of its simplified or faired version, C, and of the details encoded in the normal displacement field d.

The difficulty lies in the fact that for arbitrary curves C and C' the normal map is neither single valued nor one-to-one. For example, the line passing through $p \in C$ and having for direction the normal $N(p)$ to C at p may not intersect C'. Furthermore, two points q_1 and q_2 of C' may map onto the same point on C. In this paper, we formulate a precise condition which guarantees that the normal-map is one-to-one. This condition involves the Hausdorff distance h between C and C' and also the minimum f of the local feature size values (introduced in [2]) for each curve. The precise definitions of these concepts are reviewed further in the paper. In particular, we derive a constant $c = 2-\sqrt{2}$, such that when $h < cf$, the normal-map p_C and the normal-map $p_{C'}$ are both one-to-one. Furthermore, we demonstrate that our condition is tight by producing an example of C and C' where $h = cf$ and for which the normal-map is not one-to-one. Finally, we discuss the extension of this condition to surfaces in 3D and more generally to $(n-1)$-dimensional manifolds in $\mathbb{R}^n$.

A normal-map between surfaces has been used for the compression of triangle meshes [14] and may provide a solution for tracking texture coordinate from one frame to the next in 3D animations [22] . We anticipate that the simple condition derived in this paper will enable some applications to ensure that the curve or surface pairs they generate are homeomorphic under normal-map. The results reported here are formulated for smooth sets. Mapping them to the discrete representations (polygons, triangle meshes) commonly used in geometric modeling and graphic is not straightforward and is not discussed in the paper. We believe that such a mapping requires interpreting these discrete representations not as continuous pointsets, but as an incomplete model representing samples on a smooth curve or surface and their connectivity. This interpretation has strong implications on how the Hausdorff distance, the normal-map, the medial axis, and the local feature size should be computed (see for example [24] for references about such computations).

2. Intuitive overview

This part introduces the concepts and provides intuitive formulations of our results.

Let S and S' be two compact connected codimension one manifolds in $\mathbb{R}^n$. The normal mapping p_S from S' onto S is the (multivalued) map that associates with each point q on S' its normal projection $p_S(q)$ defined as the set of closest points to q on S. We call this map *the normal map of S onto S'* (see section 3). In general p_S is not a bijection (two different points p and q on S' may have the same images $p_S(p) = p_S(q)$) neither single valued (the closest point of a point p on S' may not be uniquely defined) - see figure 2. The set of points p for which $p_S(p)$ is not unique is the cut $C(S)$ of S which is the medial axis of the complement S^c of S [7] (see definition 4). When the normal-map is bijective, we call its inverse the *OrthoMap(S, S')*.

Definition 1. The two manifolds S and S' are normal-compatible if the two normal maps $p_S : S' \to S$ and $p_{S'} : S \to S'$ are homeomorphisms.

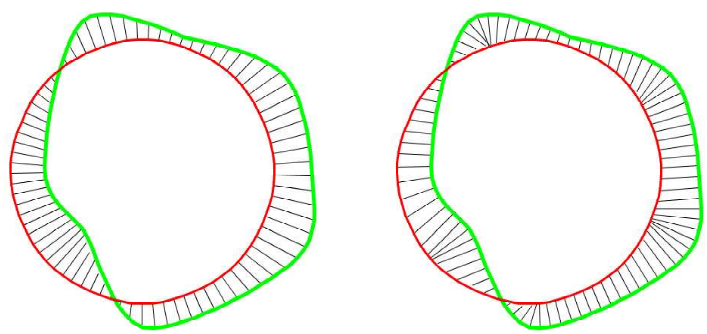

Fig. 1. Two normal-compatible curves

Figure 1 shows two curves that are normal-compatible and illustrates, in 2D, that the normal map between them is bijective.

Our condition insuring that p_S is bijective involves the Hausdorff distance between S and S' and the notion of r-regularity of the surfaces.

The definition of Hausdorff distance $d_H(S, S')$ between S and S' is given in definition 2, section 3.

A manifold S is r-regular if every point of it may be approached from both sides by an open ball of radius r that is disjoint from S. More precisely, the r-thinning $Tr(A)$ of a set A is the difference between A and the union of open balls with center out of A and the r-filleting $Fr(A)$ of A is defined as $Tr(A^r)$ where A^r is the union of balls of radius r and center in A. The manifold S is said to be r-regular [3] if $Fr(S) = S$. Note that $Fr(S)$ contains all points that cannot be reached by a ball of radius r whose interior does not interfere with S. The values r for which S is r-regular are related to the minimum feature size mfs(S) [2,1] which is defined as the

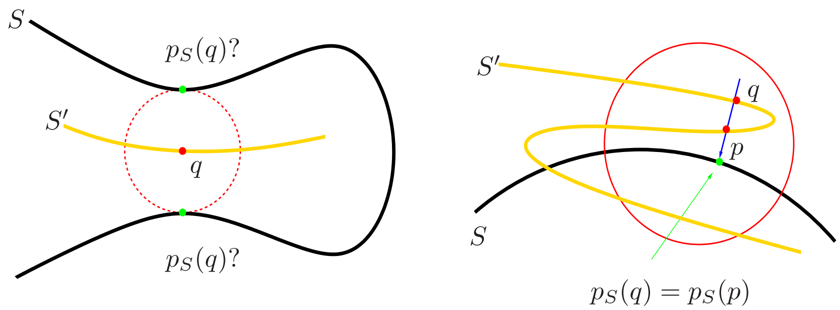

Fig. 2. When normal map is not single valued or not one-to-one

minimum distance between S and its cut $C(S)$. The surface S is r-regular if and only if $r \leq \mathrm{mfs}(S)$ (see lemma 2).

The following theorem is the main result of this paper (see section 4 for a detailled statement):

Theorem 1 *If S and S' are both r-regular for*

$$r = d_H(S, S')/(2 - \sqrt{2})$$

then S and S' are normal-compatible.

In terms of minimum feature size, the condition of theorem is equivalent to

$$d_H(S, S') < (2 - \sqrt{2}) \min(\mathrm{mfs}(S), \mathrm{mfs}(S')).$$

Moreover when the condition is satisfied, p_S allows to define an explicit isotopy between S and S' (see formal definition 5), i.e. a continuous deformation of S' into S (see corollary 1 and [8,21]). The proof of the theorem is cast in a precise mathematical formalism which is necessary for giving the proof in arbitrary dimensions. This is the object of the three next sections. We will also prove that our condition is tight in the following sense: if one replaces $2 - \sqrt{2}$ by any constant $c > 2 - \sqrt{2}$ in our condition, then there exists two surfaces S and S' satisfaying the condition for which the normal-map is not a bijection.

The theorem has a global nature and will disqualify sets for which the mapping is homeomorphic even though the conditions of the theorem do not hold. Consider for instance two shapes, S_1, and S'_1, for which the conditions of the theorem hold. Assume for example that $d_H(S_1, S'_1) = h$ and that $\mathrm{mfs}(S_1) = 2h$ and $\mathrm{mfs}(S'_1) = 3h$. Now consider S_2 and S'_2 to be versions of S_1, and S'_1 scaled by 2 and translated by a sufficiently large distance (see figure 3). Clearly, the conditions of the theorem hold for building normal-maps between S_1 and S'_1, and between S_2 and S'_2. Now, let S be the union of S_1 with S_2. Let S' be the union of S'_1 and S'_2. Clearly, S and S' are normal-compatible and thus, the normal maps between them are homeomorphisms. Yet, S and S' do not satisfy the global conditions of the theorem, because $d_H(S_2, S'_2)$ exceeds $(2 - \sqrt{2})\mathrm{mfs}(S_1)$.

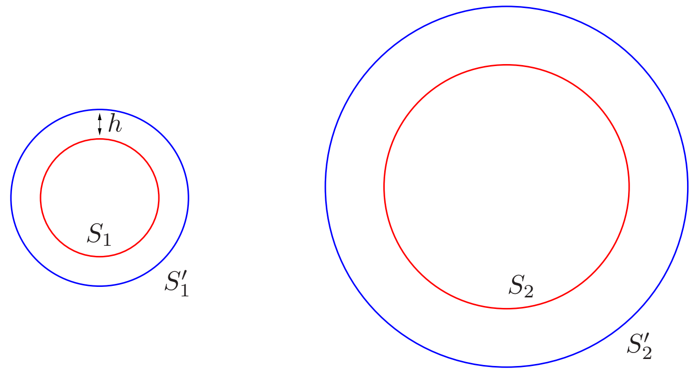

Fig. 3. Normal-compatible curves that does not satisfy our condition

Hence, it is desirable to seek a more local version of our theorem. It appears that it is not possible to give a completely local version of our theorem (see fig. 4) but it follows from the proof that one can give a semi-local version of it. The precise formulation of this result is given at the end of section 4

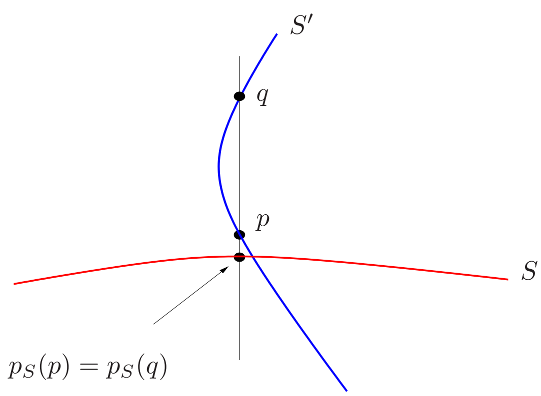

Fig. 4. An example showing two locally near pieces of curves with big mfs and no one-to-one normal-map

3. Mathematical preliminaries

In the following, all the considered manifolds are compact connected codimension one and $\mathcal{C}^k$-smooth submanifolds of $\mathbb{R}^n$ for some integer $k \geq 1$. Codimension one submanifolds of $\mathbb{R}^n$ are called hypersurfaces. Let S be such an hypersurface. Its complement $S^c = \mathbb{R}^n \setminus S$ is the disjoint union of two connected open sets. The bounded component S_i is called the interior component and the unbounded one S_e is called the exterior component of the complement S^c. One thus has $\mathbb{R}^n = S_e \cup S \cup S_i$.

In the following, to measure the distance between two shapes one uses the Hausdorff distance and the Frechet distance.

Definition 2. Let A and B be two compact subsets of $\mathbb{R}^n$. The Hausdorff distance between A and B is defined by

$$d_H(A,B) = \max\left(\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A) \right)$$

The Hausdorff distance $d_H(A,B)$ between two compact sets A and B may also be defined in terms of r-thickening. The r-thickening A^r of A is the union of all open balls of radius r and center in A. Note that A^r is the Minkowski sum of A with an open ball of radius r and center at the origin. The r-thickening operator was used as a tool for offsetting, rounding and filleting operations [19] and for shape simplification [26]. The Hausdorff distance, $d_H(A,B)$, between two sets A and B is the infimum of the radii r such that $A \subset B^r$ and $B \subset A^r$. The Hausdorff distance defines a distance on the space of compact subsets of $\mathbb{R}^n$: $d_H(A,B) = 0 \Rightarrow A = B$, $d_H(A,B) = d_H(B,A)$ and $d_H(A,C) \leq d_H(A,B) + d_H(B,C)$ (see [6]). It thus allows to measure the discrepancy between any two compact sets, even when they are not homeomorphic. When one wants to measure the discrepancy between two homeomorphic compact sets A and B, Hausdorff distance does not provide any information about how far one has to move points of A to B in order to realize an homeomorphism. In other words, two compact sets A and B may have very small Hausdorff distance but any homeomorphism between them move some points of A far away (see figure 5). So instead of considering Hausdorff distance, it may be more relevant for homeomorphic shapes to consider Frechet distance as a discrepancy measure.

Definition 3. Let S and S' be two compact homeomorphic submanifolds of $\mathbb{R}^n$. Let $\mathcal{F} = \{f : S \to S' : f \textit{ is an homeomorphism}\}$ be the set of all homeomorphisms between S and S'. Given such an homeomorphism f, $\sup_{x\in S} d(x, f(x))$ is the maximum displacement of the points of S by f. The Frechet distance between S ans S' is the infimum of this maximum displacement among all the homeomorphisms. It is defined by

$$d_F(S,S') = \inf_{f\in\mathcal{F}} \sup_{x\in S} d(x, f(x)).$$

It is an easy and classical exercise to check that Frechet distance satisfies the properties defining a distance and that one always has

$$d_H(S,S') \leq d_F(S,S').$$

In general Frechet distance is much more difficult to compute than Hausdorff distance since one has to find an infimum among all the homeomorphisms between S and S'. Nevertheless, we will see that under the hypothesis of our theorem 1, Hausdorff and Frechet distances are equal: in this case the normal map minimizes the maximum displacement of points of S. More precisely, one has the following result.

Lemma 1. *Let S and S' be two compact hypersurfaces in $\mathbb{R}^n$. If S and S' are normal-compatible, then*

$$d_H(S, S') = d_F(S, S').$$

Proof. Without loss of generality, on can suppose that $d_H(S, S') = \sup_{y \in S'} d(y, S)$. Thus, S and S' being compact, there exists two points $p \in S$ and $q \in S'$ such that $d(p, q) = \sup_{y \in S'} d(y, S) = d_H(S, S')$. It follows that the maximum displacement of points of S' by p_S is equal to $d(p, q)$. It also follows that for any homeomorphism $f : S' \to S$, $d(q, f(q)) \geq d(p, q)$. This proves that p_S minimize the maximum displacement of points of S' to S and $d_F(S, S') = d_H(S, S')$.

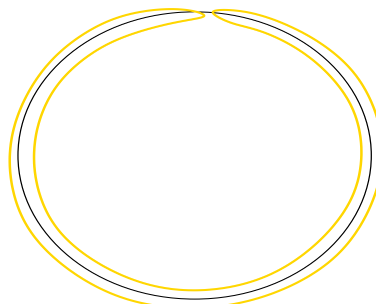

Fig. 5. Two homeomorphic curves with small Hausdorff distance and big Frechet distance.

Definition 4. Let S be a smooth compact connected $\mathcal{C}^2$ codimension one manifold in $\mathbb{R}^n$. The cut $C(S)$ of S is defined as the set of points of $\mathbb{R}^n$ that have more than one nearest neighbor on S:

$$C(S) = \{x \in \mathbb{R}^3 : \exists y, z \in S, y \neq z, d(x, y) = d(x, z) = d(x, S)\}.$$

$C(S)$ decompose into the interior cut $C_i(S)$ and the exterior cut $C_e(S)$, each one contained in S_i and S_e respectively. The minimum feature size of S, denoted mfs(S) is defined as

$$\text{mfs}(S) = \inf_{x \in S} d(x, C(S)).$$

The cut of S is the medial axis of the complement S^c of S that has been introduced by Blum in [7]. Representation of shapes by their medial axis has been extensively studied by various authors (see [27], [4], [10], [11] for example). Interior and exterior cuts are the medial axes of the open sets S_i and S_e respectively. Notice that, since S is a smooth $\mathcal{C}^2$ compact manifold, mfs(S) is a positive real number. Minimum feature size has been first introduced by Federer [13] (who named it *reach* in the setting of geometric measure theory. The minimum feature size relates to the notion of r-regularity in the following way.

Lemma 2. *Let S be a smooth compact $\mathcal{C}^2$ hypersurface embedded in $\mathbb{R}^n$ and let $r > 0$. The manifold S is r-regular if and only if $r \leq \mathrm{mfs}(S)$.*

Proof. It follows from lemma 3 *iii)* below that if $r \leq \mathrm{mfs}(S)$ then S is r-regular. Suppose now that $r > \mathrm{mfs}(S)$. The r-thickening S^r contains at least one point p_0 of $\mathcal{M}(S)$. The maximal open ball centered at p_0 that does not intersect S has a radius $r_0 < r$ and its boundary meets S in two points p and q. This implies that any open ball of radius $r > r_0$ that is tangent to S at p and on the same side of S as p must intersect S. So, p cannot be approached from both sides by an open ball of radius r that is disjoint from S.

Recall that since S is a codimension one submanifold in $\mathbb{R}^n$ it is orientable and one can continuously choose at any point $x \in S$ a unit vector $N(x)$ which is normal to S. The following lemma summarizes classical results from differential geometry (see [25], vol. I, chap. 9 for instance or [15] for more precise results on tubular neighborhoods) that we will use in the following of the paper.

Lemma 3. *Let S be a compact smooth $\mathcal{C}^k$ hypersurface without boundary embedded in $\mathbb{R}^n$.*
i) The map $\varphi : S \times] - \mathrm{mfs}(S), \mathrm{mfs}(S)[\to \mathbb{R}^n$ defined by $\varphi(x,t) = x + tN(x)$ is a $\mathcal{C}^{k-1}$-diffeomorphism onto its image $T = \{x \in \mathbb{R}^n : d(x,S) < \mathrm{mfs}(S)\}$.
ii) For any $t \in] - \mathrm{mfs}(S), \mathrm{mfs}(S)[$, the offset hypersurface $S_t = \{x + tN(x) : x \in S\}$ is a smooth $\mathcal{C}^k$ hypersurface. For any $x \in S$, $N(x)$ is the normal vector to S_t at $x + tN(x)$.
iii) Let $x \in S$ and $-\mathrm{mfs}(S) < t < \mathrm{mfs}(S)$. The open ball $\mathbb{B}(x + tN(x), |t|)$ does not contain any point of S. Moreover, the sphere $\mathbb{S}(x + tN(x), |t|)$ intersects S only at point x.

Using notations of the lemma, recall that φ is a $\mathcal{C}^{k-1}$-diffeomorphism when φ is an homeomorphism and when both itself and its inverse are differentiable $(k-1)$ times with continuous $(k-1)th$ derivatives. The projection $p_{T\to S}$ along the normals of S is defined on T by: for any $y \in T$, $p_{T\to S}(y)$ is the first coordinate of $\varphi^{-1}(y)$. This projection is a $\mathcal{C}^{k-1}$-map. Notice that $p_{T\to S}(y)$ is the nearest neighbor of y on S: $d(y, p_S(y)) = d(y,S) = \inf_{x\in S} d(y,x)$.
It follows that if S' is another smooth surface contained in T, then the restriction of $p_{T\to S}$ to S' is the normal map $p_S : S' \to S$. In this case, the normal map is single-valued and $\mathcal{C}^{k-1}$.

One can extend the normal map p_S to $\mathbb{R}^n$ as a multivalued map in the following way: for any $y \in \mathbb{R}^n$, $p_S(y)$ is the set of points x of S such that $d(x,y) = d(y,S)$. The cut $C(S)$ of S is the set of points where the (extended) normal-map is multivalued.

We use in the proof the notion of topological covering. A definition is given below. See [18] for more formal mathematical definition and properties of coverings.

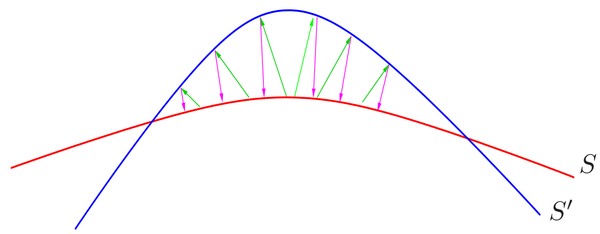

Fig. 6. Example of normal-map: pink arrows represent normal-map p_S and green arrows represent normal-map $p_{S'}$

A map $p : S' \to S$ is a *topological covering* of S if there exists a non empty discrete set F (finite or infinite denumerable) satisfying the following property: for any point $x \in S$, there exists a neighbourhood V of x and an homeomorpism Φ between $p^{-1}(V)$ and $V \times F$ such that $p_1 \circ \Phi = p$ where $p_1 : V \times F \to V$ is the canonical projection. If F is finite, the cardinality of F is known as the number of sheets of the covering. The simplest examples of topological coverings are canonical projections $p_1 : V \times F \to V$ where F is a discrete set; such coverings are said to be *trivial*.

Finally, we prove that normal-compatibility not only implies that the considered manifolds S and S' are homeomorphic but also that they are isotopic. Recall the definition of isotopy.

Definition 5. (Isotopy and ambient isotopy)
An *isotopy* between S and S' is a continuous map $F : S \times [0,1] \to \mathbb{R}^n$ such that $F(.,0)$ is the identity of S, $F(S,1) = S'$, and for each $t \in [0,1]$, $F(.,t)$ is a homeomorphism onto its image.
An *ambient isotopy* between S and S' is a continuous map $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that $F(.,0)$ is the identity of $\mathbb{R}^n$, $F(S,1) = S'$, and for each $t \in [0,1]$, $F(.,t)$ is a homeomorphism of $\mathbb{R}^n$.

Restricting an ambient isotopy between S and S' to $S \times [0,1]$ thus yields an isotopy between them. It is actually true that if there exists an isotopy between S and S', then there is an ambient isotopy between them [16], so that both notions are equivalent in our case.

4. Normal-compatibility and isotopy

Theorem 1. *Let S and S' be two compact $\mathcal{C}^k$, $k \geq 1$, surfaces embedded in $\mathbb{R}^n$ such that*

$$d_H(S,S') < (2-\sqrt{2})\min(\mathrm{mfs}(S), \mathrm{mfs}(S')).$$

Then S and S' are normal-compatible, i.e. the normal projections $p_S : S' \to S$ and $p_{S'} : S \to S'$ are $\mathcal{C}^{k-1}$-diffeomorphisms.

Moreover Hausdorff and Frechet distance between S and S' are equal: $d_H(S, S') = d_F(S, S')$.

Notice that if $k = 1$, the hypothesis of theorem does not ensure that $\mathrm{mfs}(S) > 0$ and $\mathrm{mfs}(S') > 0$. It follows from geometric measure theory (see [12]) that the normals of S and S' need to satisfy some Lipschitz condition to ensure positiveness of mfs. Such a condition is always satisfied if the hypersurfaces are $\mathcal{C}^2$. Remark that in previous theorem the equality between Hausdorff and Frechet distance is an immediate consequence of lemma 1.

The condition of theorem implies that each surface separates the interior and the exterior parts of the cut of the other one. Notice the converse is not true: separation of interior and exterior parts of the cuts does not imply normal-compatibility as it is shown with two pieces of curves on figure 7. Another example of two closed curves is given in section 6 where we prove the tightness of our condition.

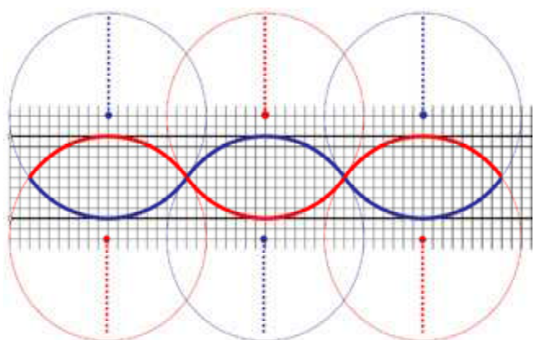

Fig. 7. Separation of interior and exterior cuts does not imply normal-compatibility

If the Hausdorff distance between two compact surfaces S and S' in $\mathbb{R}^3$ is small with respect to their minimum feature sizes, it turns out that they are isotopic [8]. More precisely, it is proven in [8] that if two surfaces S and S' embedded in $\mathbb{R}^3$ are such that $d_H(S, S') < \min(\mathrm{mfs}(S), \mathrm{mfs}(S'))$ then there exists an isotopy between S and S'. But in [8], the proof of the existence of the isotopy is not constructive and only works for surfaces in $\mathbb{R}^3$. We extend this result in the case of smooth hypersurfaces by giving explicit homeomorphism and isotopy between S and S'. Moreover, our result remains true in any dimension. As an immediate consequence of previous theorem one deduces that S' and S are isotopic.

Corollary 1. *The map* $F : S' \times [0, 1] \to \mathbb{R}^n$ *defined by*

$$F(x, t) = p_S(x) + (1 - t)\tilde{d}(x, p_S(x))N(p_S(x)),$$

where $\tilde{d}(x, p_S(x)) =< xp_S(x), N(p_S(x)) >$ *denotes the signed distance, is an isotopy between* S' *and* S*. Moreover* F *is an* $d_H(S, S')$*-isotopy: for any* $x \in S', t \in [0, 1]$, $d(x, F(t, x)) < d_H(S, S')$.

The proof of theorem 1 adapts easily to give a semi-local result involving the local feature size of S and S' introduced in [2]. For any $x \in S'$, denote by $\mathrm{lfs}(x)$ the distance from x to the cut $C(S')$ of S':

$$\mathrm{lfs}(x) = d(x, C(S')) = \inf_{y \in C(S')} d(x, y).$$

For any point $y \in S$, one also denotes $\mathrm{lfs}(y)$ the distance from y to the cut $C(S)$ of S. Introduce the two functions ρ and α defined on S' by

$$\rho(x) = \min(\mathrm{lfs}(x), \mathrm{lfs}(p_S(x))),$$

$$\alpha(x) = d_H(S \cap \mathbb{B}(x, \sqrt{2}\rho(x)), S' \cap \mathbb{B}(x, \sqrt{2}\rho(x)))$$

where $\mathbb{B}(x, \sqrt{2}\rho(x))$ denotes the ball of center x and radius $\sqrt{2}\rho(x)$.

Theorem 2. *Let S and S' be two compact $\mathcal{C}^k$, $k \geq 1$, hypersurfaces embedded in $\mathbb{R}^n$ such that*

$$\alpha(x) < (2 - \sqrt{2})\rho(x)$$

for any $x \in S'$. The normal maps $p_S : S' \to S$ and $p_{S'} : S \to S'$ are $\mathcal{C}^{k-1}$-diffeomorphisms. In particular, S and S' are normal-compatible.

5. Proof of theorem

We must prove that the projection of S' onto S along the normals of S is one-to-one. The proof proceeds in two steps. First, one proves that S' cannot be tangent to one of the normals to S. To do that, one supposes that there exists a point x where S' is tangent to a normal of S. This means that the normals of S at $p_S(x)$ and S' at x are orthogonal. Such a condition implies that the Hausdorff distance between S and S' cannot remain small (relatively to the mfs of S and S') in a neighborhood of x which leads to a contradiction. Second, using a topological argument one deduces that S' intersects each normal to S restricted to the mfs(S)-thickening of S in exactly one point. More precisely, one shows that S' intersects each normal segment of length mfs on each side of S at exactly one point. In the following, the tangent hyperplane to a hypersurface S at a point $x \in S$ is denoted T_xS.

First step: *For any $x \in S'$, T_xS' is transverse to $N(p_S(x))$.*

Suppose this is not the case, that is there exists a point $x \in S'$ such that $N(p_S(x))$ is colinear to the tangent plane T_xS'. Let $\rho = \min(\mathrm{mfs}(S), \mathrm{mfs}(S'))$ and $\alpha = d_H(S, S') < (2 - \sqrt{2})\rho$ be the Hausdorff distance between S and S'. Without loss of generality, one can suppose that there exists $0 \leq t < \alpha$ such that $x = p_S(x) + tN(p_S(x))$. Since $S \subset T'_\alpha = \{x \in \mathbb{R}^3 : d(x, S') < \alpha\}$ and $S' \subset T_\alpha = \{x \in \mathbb{R}^3 : d(x, S) < \alpha\}$ one has that $x \in T_\alpha$. Consider the two open balls of radius ρ which are tangent to T_xS' at x and let a' and b' be their centers. They do not intersect the surface S' because $\mathrm{mfs}(S') \geq \rho$. In the same way, consider the offset surface $\tilde{S}$ of S that passes through x and consider the two open balls of radius

$\rho - t$ which are tangent to $T_x\tilde{S}$ at x. Their centers a and b are on the normal to S issued from $p_S(x)$ and one can suppose that a is on the same side as x of S. Since $\tilde{S} \subset S_t = \{x \in \mathbb{R}^3 : d(x, S) = t\}$ is an offset surface of S, the local feature size of $\tilde{S}$ is greater than $\rho - t$. It follows that $\mathbb{B}(a, \rho - t)$ and $\mathbb{B}(b, \rho - t)$ do not intersect $\tilde{S}$. Moreover, their tangent planes at x are orthogonal to the tangent planes of $\mathbb{B}(a', \rho)$ and $\mathbb{B}(b', \rho)$ (see figure 8).

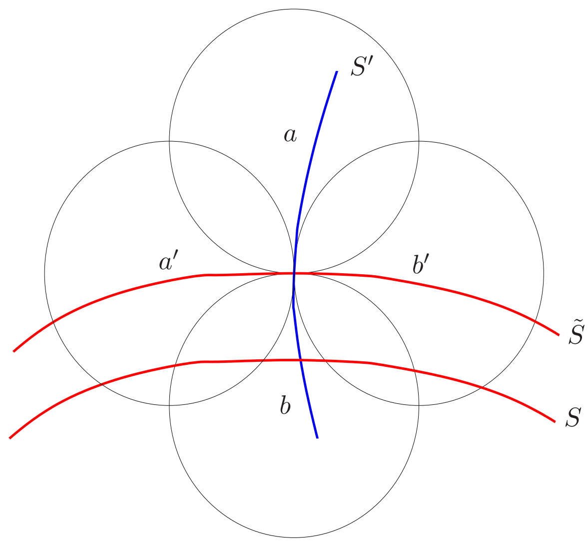

Fig. 8. Intersection of tangent balls

Since the two lines (ab) and $(a'b')$ intersect at x, the points a, b, a', b' and x are coplanar. Let P be the plane that contains them. It follows from lemma 3 *ii)* that P is transverse to the tangent planes of S' and $\tilde{S}$ at x (it contains the normals to S' and $\tilde{S}$ at x). So the intersection of S' and $\tilde{S}$ with P in a neighbourhood of x are smooth plane curves. The four balls $\mathbb{B}(a, \rho - t)$, $\mathbb{B}(b, \rho - t)$, $\mathbb{B}(a', \rho)$ and $\mathbb{B}(b', \rho)$ intersect P along four discs of radius $\rho - t$ and ρ and centers a, b, a' and b' respectively.

The ball of center a and radius $\rho - \alpha$ is contained in the complement of T_α. Since $S' \subset T_\alpha$, $S' \cap P$ is contained in the complement of the disc of center a and radius $\rho - \alpha$. It is also contained in the complement of the discs of radius ρ and centers a' and b' (see figure 9).

Now consider the segment that joins a to the first point of intersection of the line (aa') with the disc of radius ρ and center a'. The square length of (aa') is equal to $\rho^2 + (\rho - t)^2$ (see figure 10). Since $\alpha < (2 - \sqrt{2})\rho$, one has $(2\rho - \alpha)^2 > \rho^2 + (\rho - t)^2$, so the disc of center a' and radius ρ and the disc of center a and radius $\rho - \alpha$ intersect. It follows that the part of $S' \cap P$ which is on the same side of $\tilde{S}$ as a remains in the "triangular" area Δ_a delimited by the circles $C(a', \rho)$, $C(b', \rho)$, $C(a, \rho - \alpha)$ (see

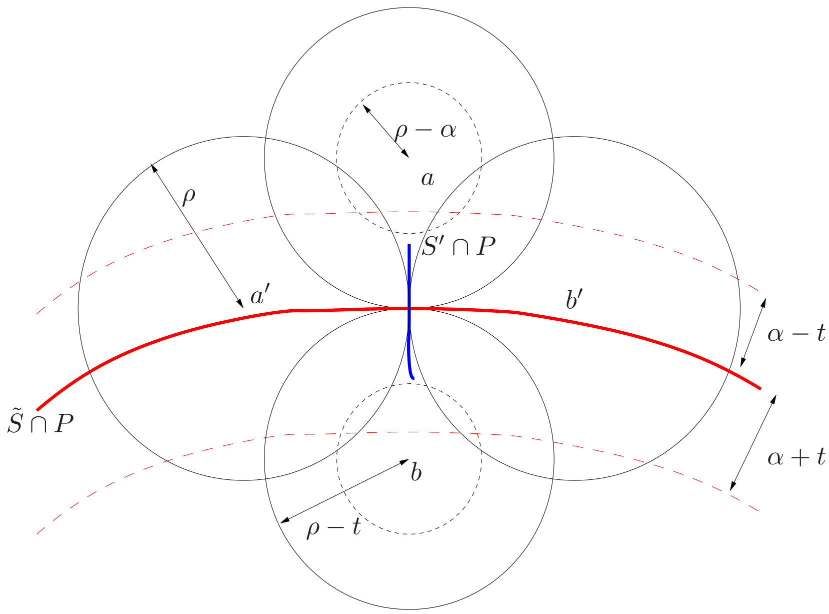

Fig. 9. Intersection with the plane P near from x

figure 10). Since $S' \cap P$ is a smooth curve it has to intersect the boundary of Δ_a in at leas two points: a contradiction. This concludes the proof of the first step.

Second step: *S' intersects each normal of S in M in exactely one point.*

This step is a classical fact from algebraic topology. We recall it here. First consider the case where S and S' are connected. Since S' is transverse to each fiber of the normal bundle M of S, the normal-map p_S defines a topological covering of S (see section 3 the definition of covering). Thus S' being compact, there exists a positive integer l such that for any $x \in S$, $N(x) \cap S' \cap M$ is a set of l points. Recall that there exists a diffeomorphism φ between $T_\rho = \{x \in \mathbb{R}^n : d(x, S) < \rho\}$ and $S \times] - \rho; \rho[$, so that T_ρ has two sides $\varphi(S \times \{-\rho\})$ and $\varphi(S \times \{\rho\})$. One knows from [8] that S' separates the two sides of T_ρ: any continuous path from one side of T_ρ to the other has to meet S'. Recall now that S and S' are connected. For any $x \in S'$, the vector $N(p_S(x))$ defines a transverse orientation of S' and points inside the same connected component of $\mathbb{R}^n \setminus S'$ that we can suppose to be S'_e without loss of generality. Suppose that $l \geq 2$ and let $x \in S$. The normal line to S at x intersects S' in l points in M. Among these points, denote by y the farthest from x (see figure 11). Locally, the connected component of $\mathbb{R}^n \setminus S'$ that belongs "over" y is S'_e. But the one that belongs below y belongs over another point of intersection. So this connected component is also S'_e. This contradicts the fact that S' separates $\mathbb{R}^n$ in two connected components.

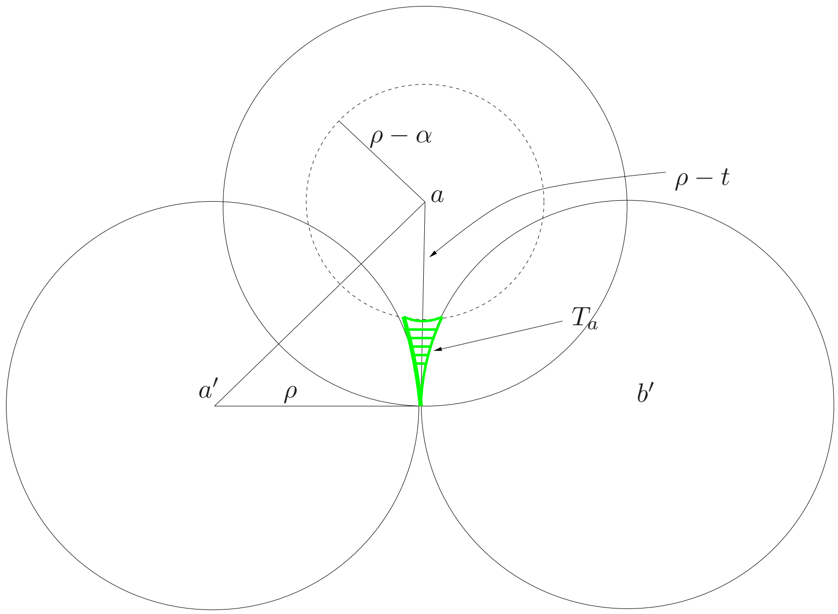

Fig. 10.

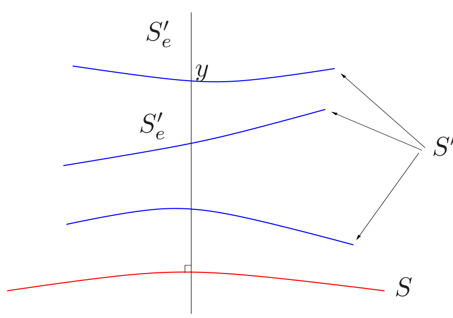

Fig. 11.

To conclude, it remains to consider the case where S or S' is not connected. Using notations of first step, recall that S' is included in the tubular neighborhood $T_\alpha = \{x \in \mathbb{R}^3 : d(x,S) < \alpha\}$ with $\alpha = d_H(S,S') < (2-\sqrt{2})\min(\mathrm{mfs}(S), \mathrm{mfs}(S'))$. Since $\alpha < \mathrm{mfs}(S)$, T_α is the disjoint union of the tubular neighborhoods of size α of each connected component of S. The Hausdorff distance between S and S' being equal to α, each connected component of T_α contains at least one connected component of S'. It follows that the number of connected components of S' is at least as big as the number of connected components of S. In the same way, by symmetry, the number of connected components of S is at least as big as the number of connected components of S'. Thus, S and S' have the same number of connected components. Moreover, each connected component of S' (resp. S) is contained in the tubular neighborhood of size α of a unique connected component of S (resp. S'). So one can apply first part of the second step to each pairs of connected components of S and S' separately.

Remark: To deduce theorem 2 from previous proof it sufficies to remark that step one adapts easily to the semi-local hypothesis of theorem 2: arguments used only depend on the geometry of S and S' in a ball of center x and radius $\sqrt{2}\rho(x)$. Step 2 remains unchanged.

6. Tightness of the bound $2 - \sqrt{2}$

The constant $2 - \sqrt{2}$ involved in theorem 1 is tight in the following sense.

Proposition 1. *Let c be a positive real number such that $c > 2 - \sqrt{2}$. There exists two planar curves C and C' such that $d_H(C, C') < c \min(\mathrm{mfs}(C), \mathrm{mfs}(C'))$ and p_C is not an homeomorphism.*

Notice also that, in the general case, the constant $2 - \sqrt{2}$ is independent of the dimension of the ambient space $\mathbb{R}^n$. The previous proposition is proved by constructing an explicit example of two planar curves C and C'. One can easily derive from this construction, higher dimensional examples showing the optimality of $2 - \sqrt{2}$ in any dimensions (we will not attempt to draw them here.)

Note that we use the word tight instead of necessary or optimal because it is possible that S and S' be normal-compatible without satisfaying our condition. An example of such a situation is given on figure 3.

Let $c > 2 - \sqrt{2}$ be a fixed positive real number. We prove proposition 1 by giving an example of two curves C and C' such that $d_H(C, C') < c \min(\mathrm{mfs}(C), \mathrm{mfs}(C'))$ and p_C is not a a bijection. This example is deduced from the first step of the proof of theorem 1. We first give an example of two curves C and C' being such that $d_H(C, C') = (2 - \sqrt{2}) \min(\mathrm{mfs}(C), \mathrm{mfs}(C'))$ and C' is not transverse to the normals of C. We then obtain the desired curves as a small perturbation of this first example.

The first curves are represented on figure 12. They are made of line segment and pieces of circles of radius 1 that meet orthogonaly on the vertices of a regular orthogonal grid which edges are of length 1. Notice that for clarity, only the half of the grid is represented on figure 12. As an exception, some of the pieces of circles that join the segment line are not centered on the grid vertices (but they remain of radius 1). One easily sees that the minimum feature sizes of C and C' are both equal to 1 and the Hausdorff distance between C and C' is equal to $2 - \sqrt{2}$. It is also clear that C and C' are $\mathcal{C}^1$. At the origin O, the two curves meet orthogonaly, so that C' is tangent to the normal of C at O.

Nevertheless, the normal-map of C' onto C is an homeomorphism (notice that it is not differentiable at O). But one can make a small perturbation of our example

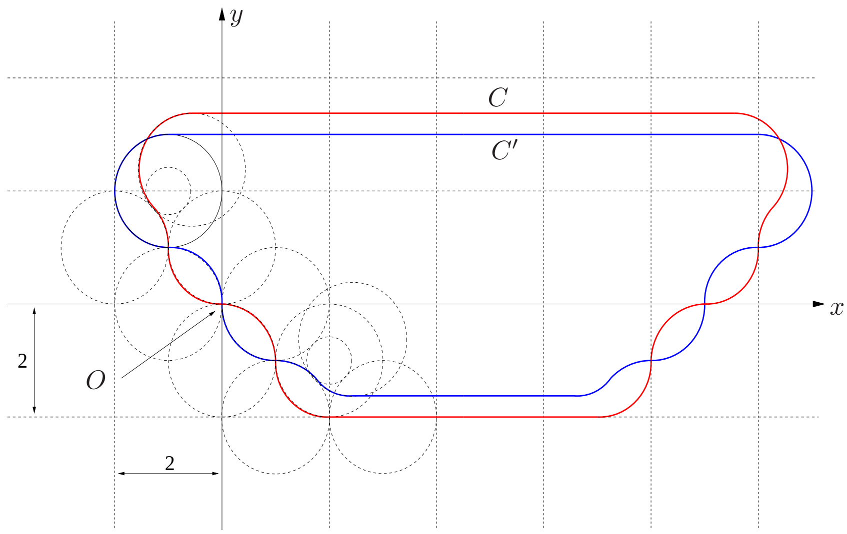

Fig. 12. Two curves being such that $d_H(S, S') = (2 - \sqrt{2}) \min(\mathrm{mfs}(S), \mathrm{mfs}(S'))$ and that meet orthogonaly

in order that the normal-map fails to be one-to-one in a neighborhood of O. This is done in figure 13. Instead of considering the circles centered on the vertices of an orthogonal grid one chooses a non-orthogonal grid. The angle between the two families of parallel lines defining the grid is equal to $\frac{\pi}{2} - \theta$ for a sufficiently small value of $\theta > 0$. The length of the edges of the grid is still equal to 1.
Clearly, the two new curves C and C' are $\mathcal{C}^1$ and their minimum feature sizes are still equal to 1. They may be viewed as continuous deformations of the initial curves. So for $\theta > 0$ sufficiently small, one has $d_H(C, C') < c$. Unlike in first example, the normal projection of C' onto C is not one-to-one: the normal of C at O is the y-axis which is intersected three times in a neighborhood of O (see figure 14).

The previous example may be generalized in higher dimension in the following way. Let $n \geq 3$ be an integer and denote by $x_1, \cdots, x_n$ the coordinates in $\mathbb{R}^n$. Identify the plane that contains the curves C and C' of previous example with the $(x_1 x_2)$-plane in $\mathbb{R}^n$ and identify the subspace generated by $x_3, \cdots, x_n$ with $\mathbb{R}^{n-2}$. The two manifolds $S = C \times \mathbb{R}^{n-2}$ and $S' = C' \times \mathbb{R}^{n-2}$ are $\mathcal{C}^1$ manifolds of codimension one in $\mathbb{R}^n$ that satisfy hypothesis and conclusion of proposition 1.

7. Conclusion

We have proven that one hypersurface can be expressed as the normal offset of another, when the Hausdorff distance between the two hypersurfaces is less than $2 - \sqrt{2}$ times their minimum feature sizes. We have proven that, under this condition, the mapping between one hypersurface and its normal offset is one-to-one.

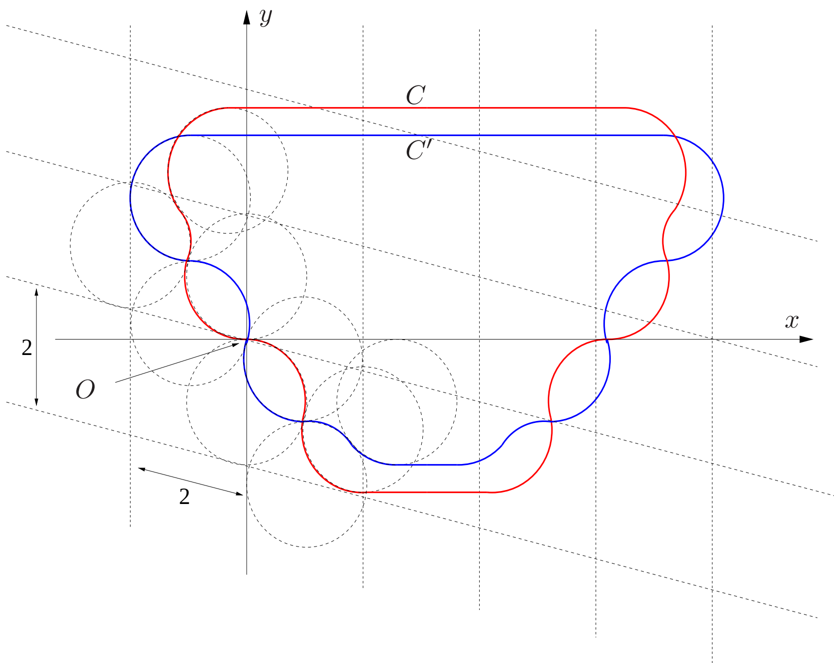

Fig. 13. Example showing that the bound of theorem 1 is tight

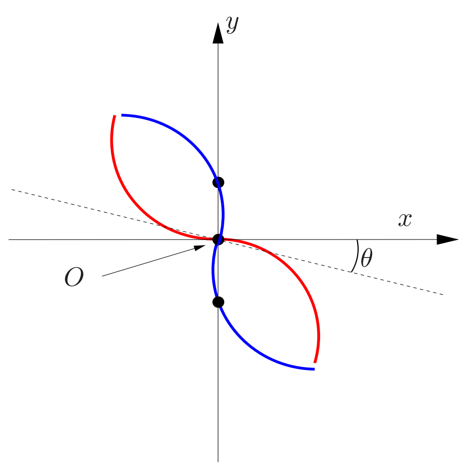

Fig. 14. Zoom of figure 13 in a neighborhood of O: normal projection is not one-to-one.

Furthermore, we have shown that the condition is tight by providing an example where the Hausdorff distance equals the above limit and yet the mapping is not one-to-one.

8. Acknowledgement

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