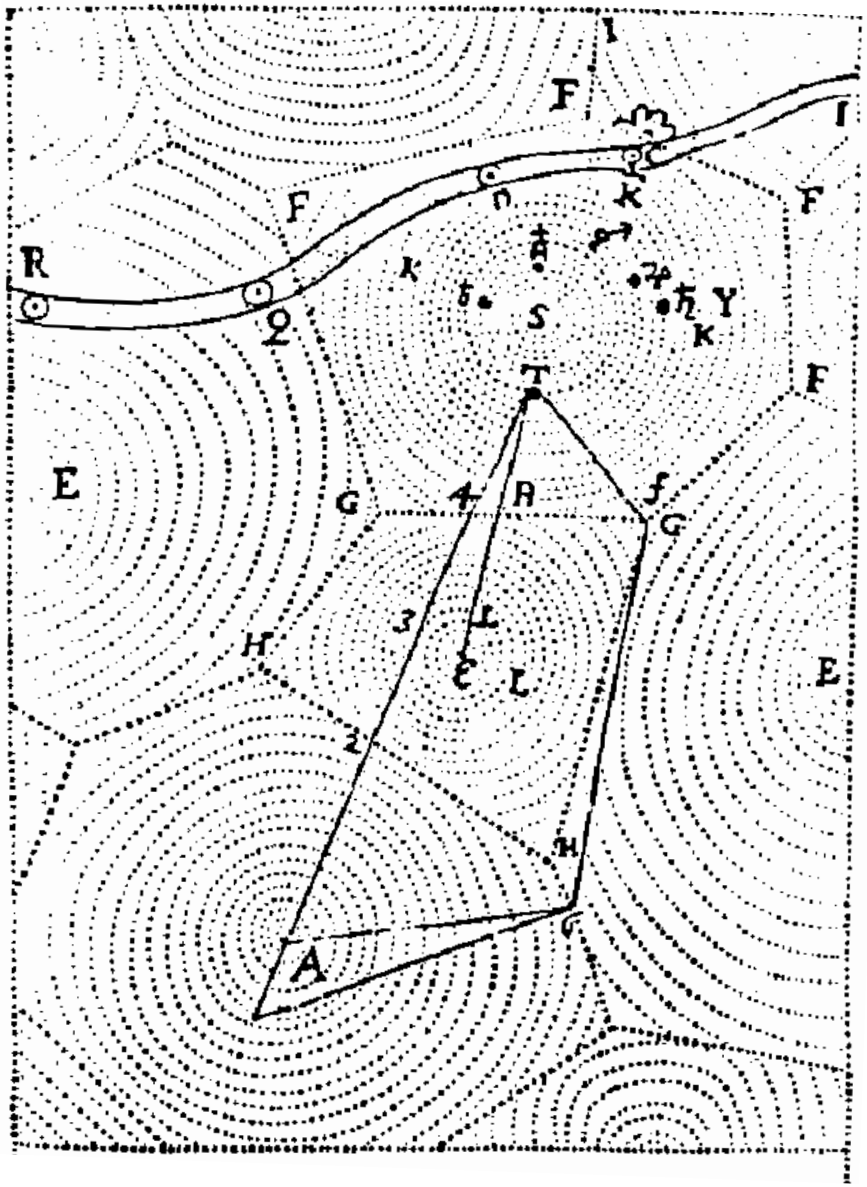
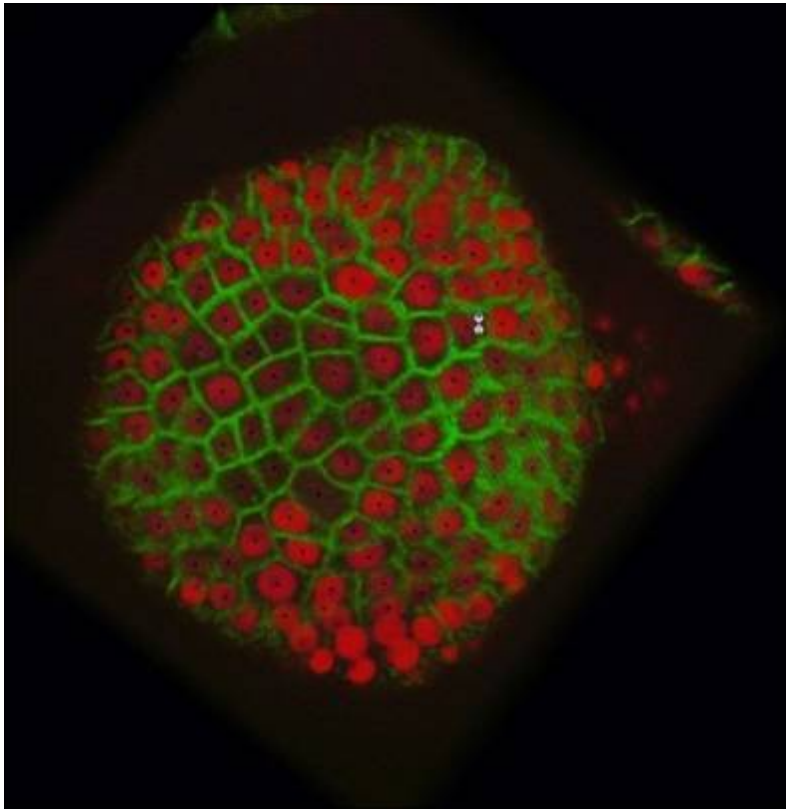
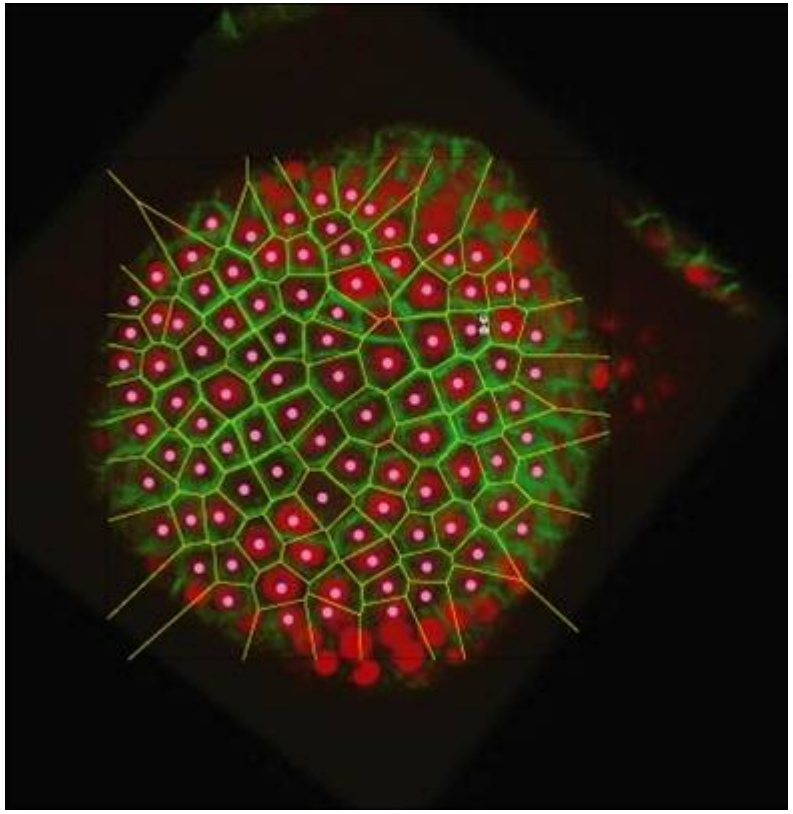
Voronoi diagrams in nature

The solar system (Descartes)

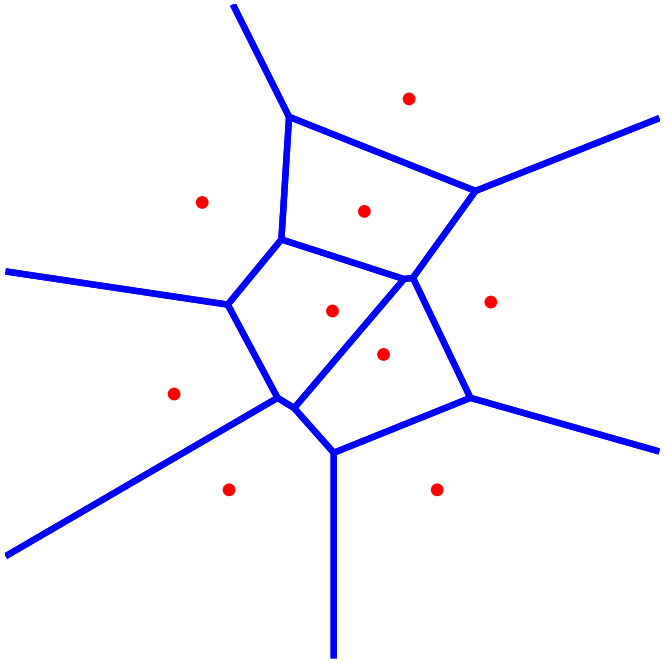


Growth of merystem





Euclidean Voronoi diagrams



Voronoi cell $\quad V(p_i) = \{x : \|x - p_i\| \leq \|x - p_j\|, \ \forall j\}$

Voronoi diagram (P) $\quad = \{$ collection of all cells $V(p_i)$, $p_i \in P$ $\}$

Voronoi diagrams and polytopes

Polytope

The intersection of a finite collection of half-spaces : $\mathcal{V} = \bigcap_{i \in I} h_i^+$

- Each Voronoi cell is a polytope
- The Voronoi diagram has the structure of a cell complex
- The Voronoi diagram of P is the projection of a polytope of $\mathbb{R}^{d+1}$

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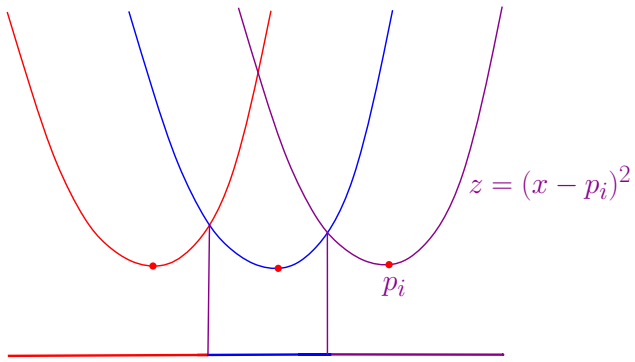
Voronoi diagrams and polyhedra

- Vor$(p_1, \ldots, p_n)$ is the minimization diagram of the n functions $\delta_i(x) = (x - p_i)^2$



- $\arg\min(\delta_i) = \arg\max(h_i)$ where $h_{p_i}(x) = 2\, p_i \cdot x - p_i^2$

- The minimization diagram of the δ_i is also the maximization diagram of the affine functions $h_{p_i}(x)$

- The faces of Vor(P) are the projections of the faces of $\mathcal{V}(P) = \bigcap_i h_{p_i}^+$

$h_{p_i}^+ = \{x : x_{d+1} > 2p_i \cdot x - p_i^2\}$

Note !

the graph of $h_{p_i}(x)$ is the hyperplane tangent to $\mathcal{Q}: \quad x_{d+1} = x^2 \quad$ at $\quad (x, x^2)$

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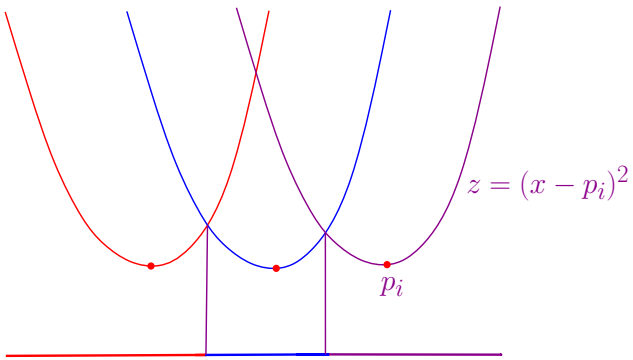

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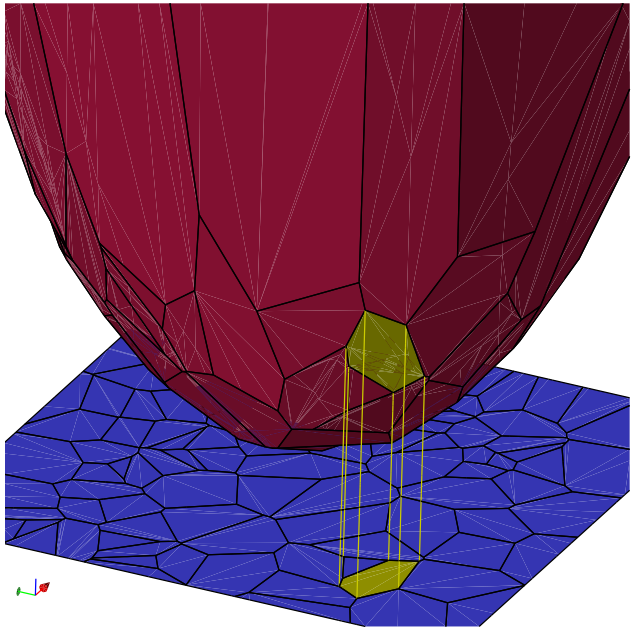

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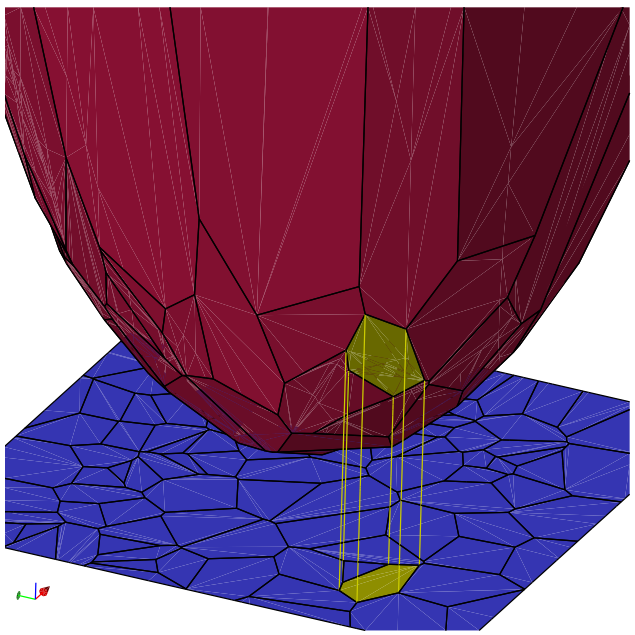

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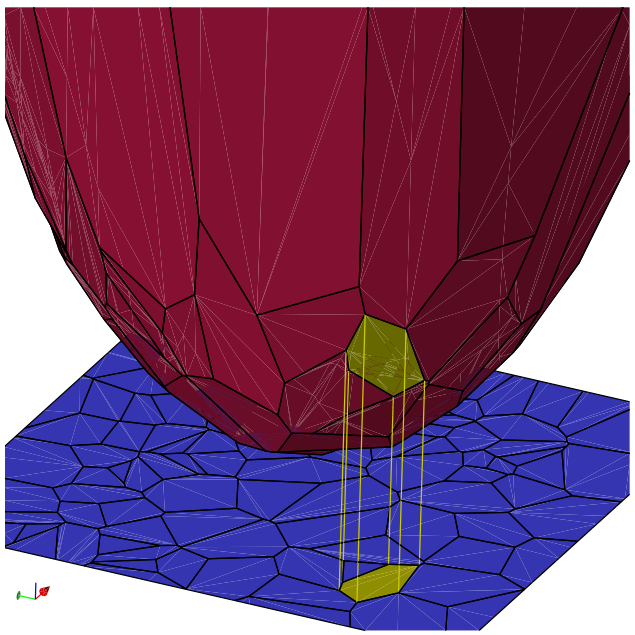

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Voronoi diagrams and polytopes

Lifting map

The faces of $\mathrm{Vor}(P)$ are the projection of the faces of the polytope

$$\mathcal{V}(P) = \bigcap_i \; h_{p_i}^+$$

where h_{p_i} is the hyperplane tangent to paraboloid $\mathcal{Q}$ at the lifted point (p_i, p_i^2)

Corollaries

- The size of $\mathrm{Vor}(\mathcal{P})$ is the same as the size of $\mathcal{V}(P)$
- Computing $\mathrm{Vor}(\mathcal{P})$ reduces to computing $\mathcal{V}(P)$

Voronoi diagrams and polytopes

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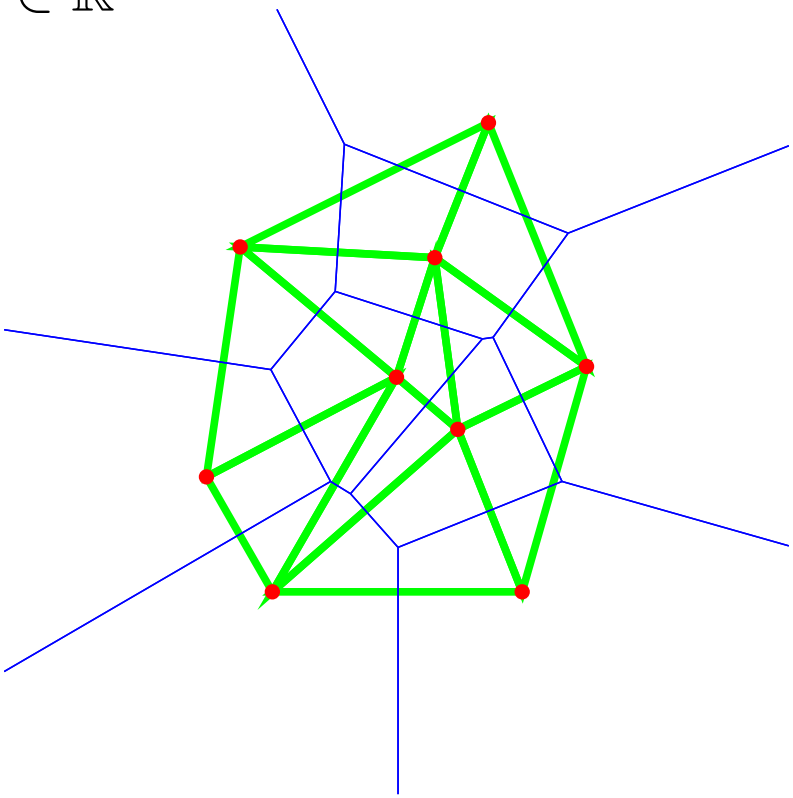
Voronoi diagram and Delaunay triangulation

Finite set of points $P \in \mathbb{R}^d$



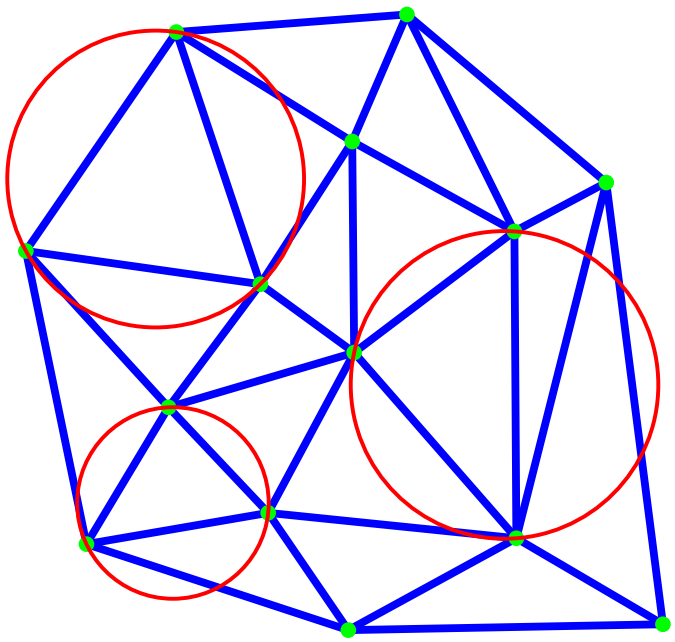
- The Delaunay complex is the nerve of the Voronoi diagram
- It is not always embedded in $\mathbb{R}^d$

Empty circumballs

An (open) d-ball B circumscribing a simplex $\sigma \subset \mathcal{P}$ is called empty if

1. $\mathrm{vert}(\sigma) \subset \partial B$
2. $B \cap \mathcal{P} = \emptyset$

$\mathrm{Del}(\mathcal{P})$ is the collection of simplices admitting an empty circumball



Point sets in general position wrt spheres

$P = \{p_1, p_2 \dots p_n\}$ is said to be in general position wrt spheres if $\nexists\, d+2$ points of P lying on a same $(d-1)$-sphere

Theorem [Delaunay 1936]

If P is in general position wrt spheres, the natural mapping

$$f : \mathrm{vert}(\mathrm{Del}P) \to P$$

provides a realization of $\mathrm{Del}(P)$ called the Delaunay triangulation of P.

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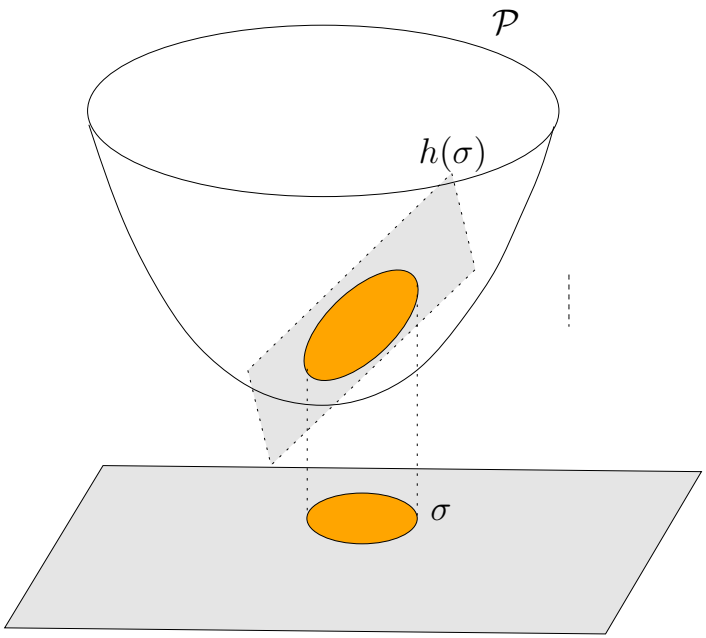
Proof of Delaunay's theorem 1



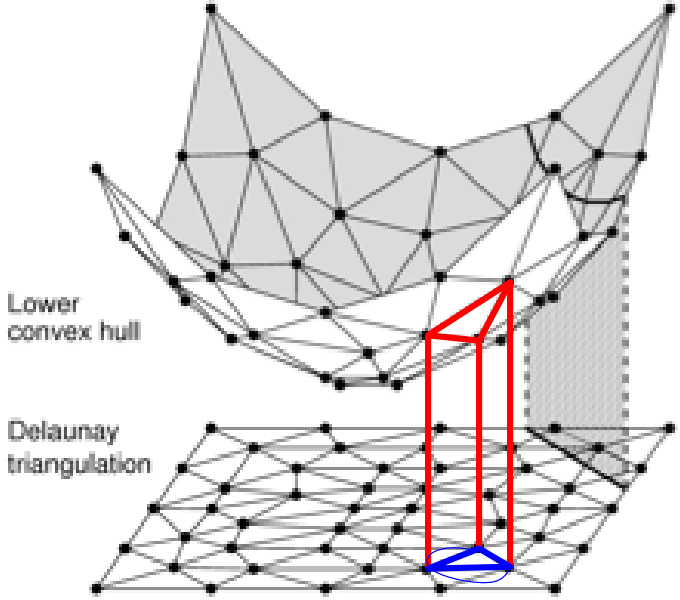
Linearization

$$S(x) = x^2 - 2c \cdot x + s, \quad s = c^2 - r^2$$

$$S(x) < 0 \Leftrightarrow \begin{cases} z < 2c \cdot x - s & (h_S^-) \\ z = x^2 & (\mathcal{P}) \end{cases}$$

$$\Leftrightarrow \hat{x} = (x, x^2) \in h_S^-$$

Proof of Delaunay's theorem 2



Proof of Delaunay's th.

P general position wrt spheres
$\Leftrightarrow \hat{P}$ in general position

σ a simplex, S_σ its circumscribing sphere

$\sigma \in \mathrm{Del}(P) \Leftrightarrow S_\sigma$ empty

$\Leftrightarrow \forall i,\ \hat{p}_i \in h^+_{S_\sigma}$

$\Leftrightarrow \hat{\sigma}$ is a face of $\mathrm{conv}^-(\hat{P})$

$$\mathrm{Del}(P) = \mathrm{proj}(\mathrm{conv}^-(\hat{P}))$$

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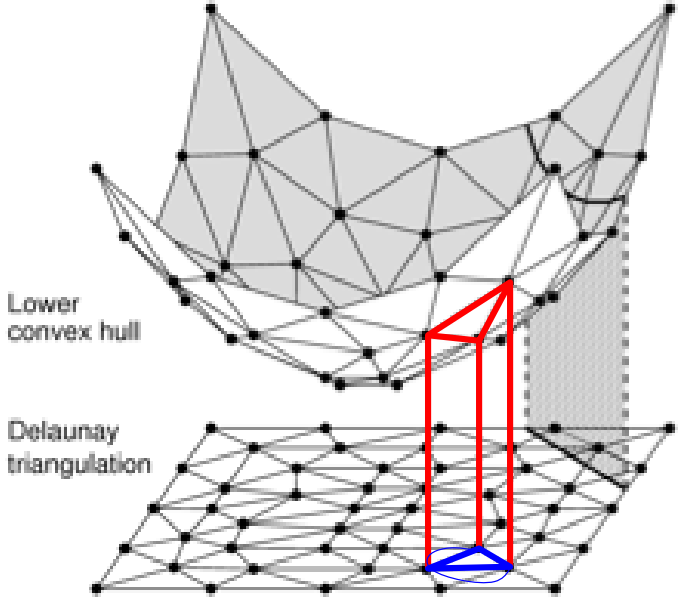

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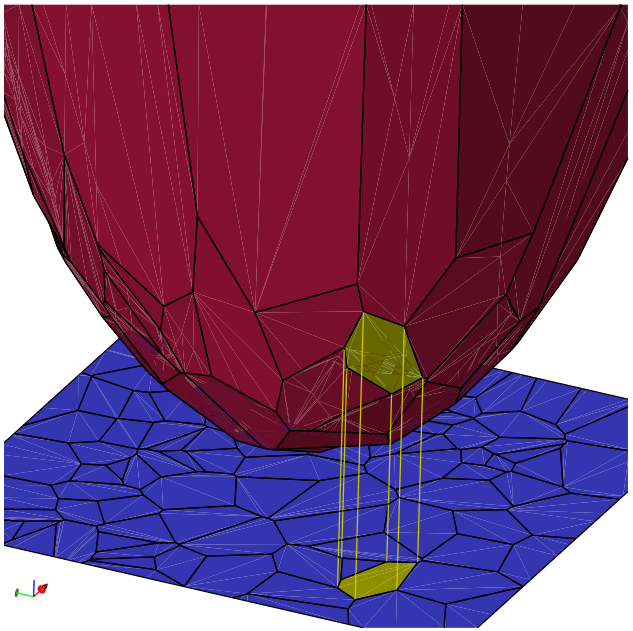
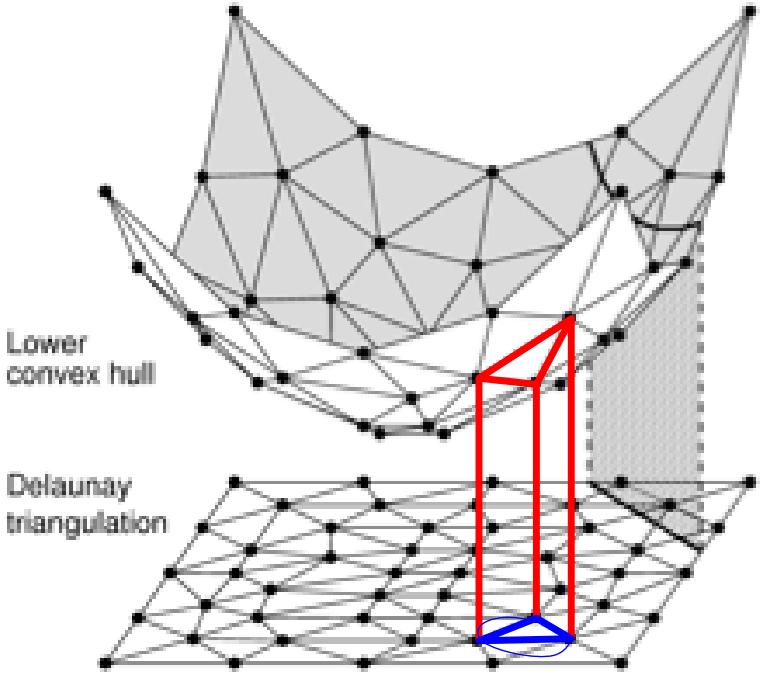
Duality





$$\mathcal{V}(P) = \cap_i h_{p_i}^+$$

$$\mathcal{D}(P) = \mathrm{conv}^-(\hat{P})$$

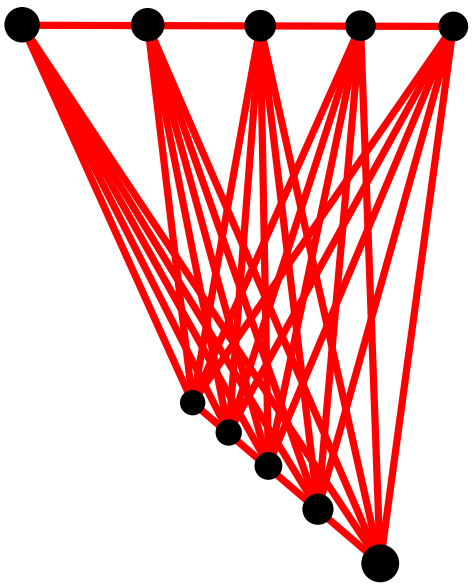
Voronoi diagrams, Delaunay triangulations and polytopes

If P is in general position wrt spheres :

$$\begin{array}{ccc} \mathcal{V}(P) = h^+_{p_1} \cap \ldots \cap h^+_{p_n} & \xrightarrow{\text{duality}} & \mathcal{D}(P) = \mathsf{conv}^-(\{\hat{p}_1, \ldots, \hat{p}_n\}) \\ \uparrow & & \downarrow \\ \text{Voronoi Diagram of } P & \xrightarrow{\text{nerve}} & \text{Delaunay Complex of } P \end{array}$$

Combinatorial complexity

The combinatorial complexity of the Delaunay triangulation diagram of n points of $\mathbb{R}^d$ is the same as the combinatorial complexity of a convex hull of n points of $\mathbb{R}^{d+1}$



$\Theta(n^{\lceil \frac{d}{2} \rceil})$

Quadratic in $\mathbb{R}^3$

Constructing $\mathrm{Del}(P)$, $P = \{p_1, ..., p_n\} \subset \mathbb{R}^d$

Algorithm

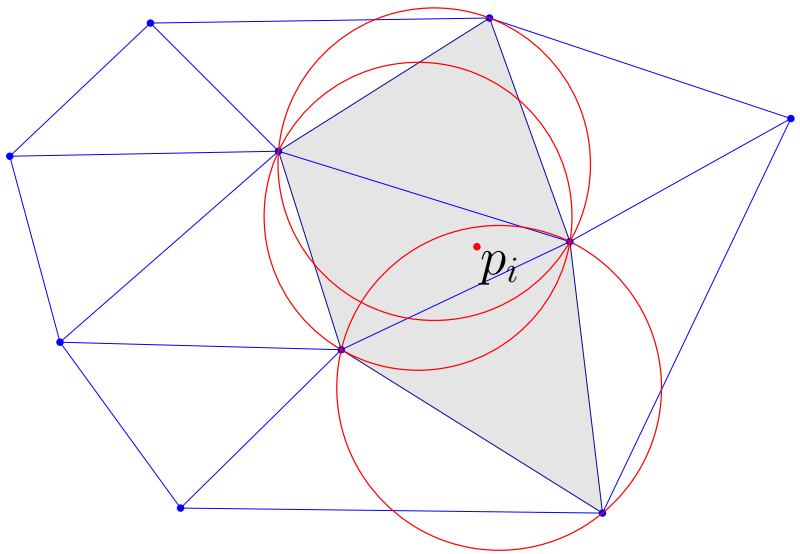
1. Lift the points of P onto the paraboloid $x_{d+1} = x^2$ of $\mathbb{R}^{d+1}$: $p_i \to \hat{p}_i = (p_i, p_i^2)$
2. Compute $\mathrm{conv}(\{\hat{p}_i\})$
3. Project the lower hull $\mathrm{conv}^-(\{\hat{p}_i\})$ onto $\mathbb{R}^d$

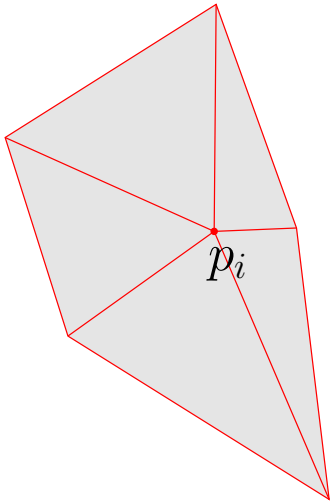
Complexity : $\Theta(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor})$

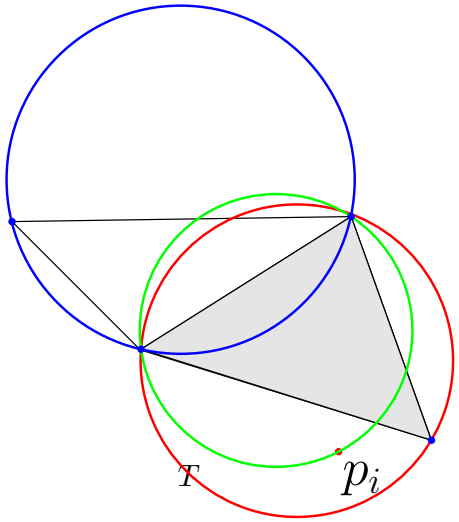
Direct algorithm : insertion of a new point p_i

1. Location : find all the d-simplices that conflict with p_i
   i.e. whose circumscribing ball contains p_i

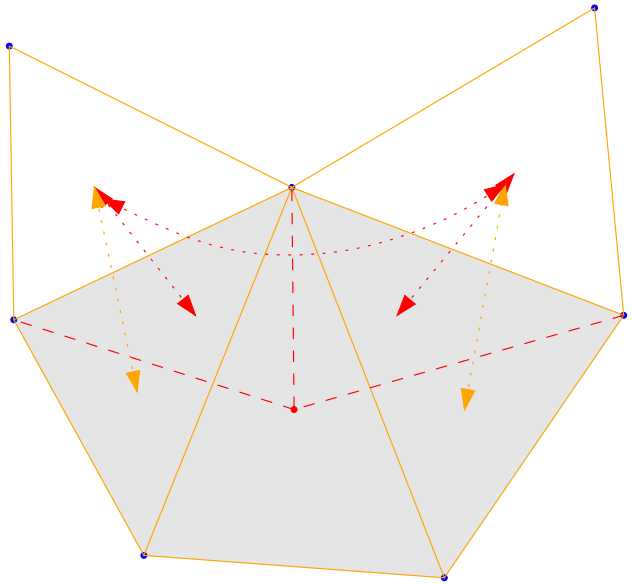
2. Update : construct the new d-simplices







Updating the adjacency graph



We look at the d-simplices to be removed and at their neighbors

Each d-simplex is considered $\leq \frac{d(d+1)}{2}$ times

$$\begin{aligned}\text{Update cost} &= O(\text{\# created and deleted simplices}) \\ &= O(\text{\# created simplices})\end{aligned}$$

Some optimality properties of Delaunay triangulations

Among all possible triangulations of $\mathcal{P}$, $\mathrm{Del}(\mathcal{P})$

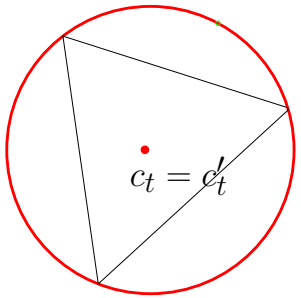
1. (2d) maximizes the smallest angle [Lawson]

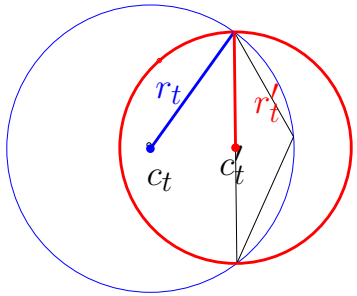
2. (2d) Linear interpolation of $\{(p_i, f(p_i))\}$ that minimizes [Rippa]

   $R(T) = \sum_i \int_{T_i} \left(\left(\frac{\partial \phi_i}{\partial x} \right)^2 + \left(\frac{\partial \phi_i}{\partial y} \right)^2 \right) dx\, dy$ (Dirichlet energy)

   $\phi_i =$ linear interpolation of the $f(p_j)$ over triangle $T_i \in T$

3. minimizes the radius of the maximal smallest ball enclosing a simplex) [Rajan]





Optimizing the angular vector ($d = 2$)

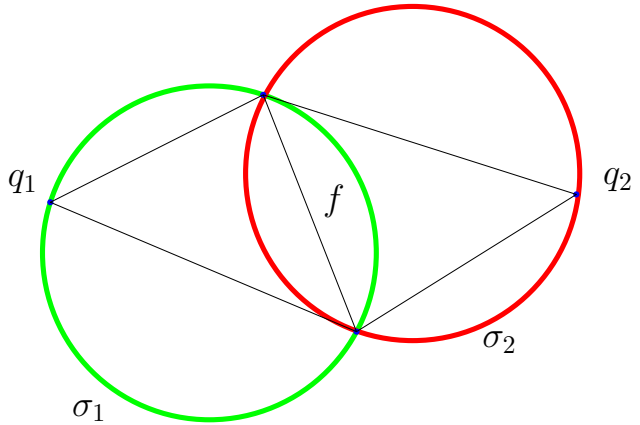
Angular vector of a triangulation $T(\mathcal{P})$

$$\mathrm{ang}\,(T(\mathcal{P})) = (\alpha_1, \ldots, \alpha_{3t}), \qquad \alpha_1 \leq \ldots \leq \alpha_{3t}$$

Optimality Any triangulation of a given point set $\mathcal{P}$ whose angular vector is maximal (for the lexicographic order) is a Delaunay triangulation of $\mathcal{P}$

Good for matrix conditioning in FE methods

Local characterization of Delaunay complexes



Pair of regular simplices

$$\sigma_2(q_1) \geq 0 \quad \text{and} \quad \sigma_1(q_2) \geq 0$$

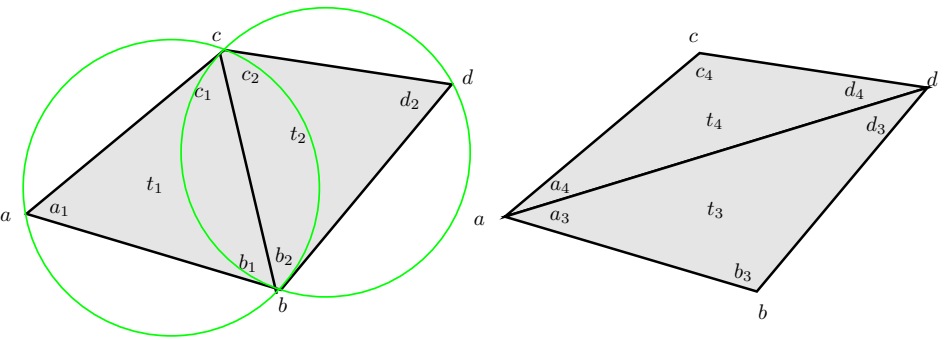
$\Leftrightarrow \hat{c}_1 \in h^+_{\sigma_2}$ and $\hat{c}_2 \in h^+_{\sigma_1}$

Theorem A triangulation $T(P)$ such that all pairs of simplexes are regular is a Delaunay triangulation $\mathrm{Del}(P)$

Proof The PL function whose graph G is obtained by lifting the triangles is locally convex and has a convex support

$$\Rightarrow \quad G = \mathrm{conv}^-(\hat{Q}) \quad \Rightarrow \quad T(Q) = \mathrm{Del}(Q)$$

Lawson's proof using flips



While $\exists$ a non regular pair (t_3, t_4)

/* $t_3 \cup t_4$ is convex */

replace (t_3, t_4) by (t_1, t_2)

Regularize $\Leftrightarrow$ improve $\mathrm{ang}\,(T(\mathcal{P}))$

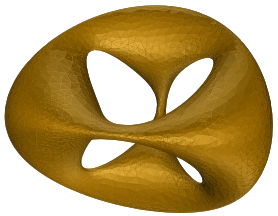
$\mathrm{ang}\,(t_1, t_2) \geq \mathrm{ang}\,(t_3, t_4)$

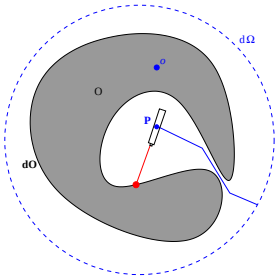
$$a_1 = a_3 + a_4,\ d_2 = d_3 + d_4,$$
$$c_1 \geq d_3,\quad b_1 \geq d_4,\quad b_2 \geq a_4,\quad c_2 \geq a_3$$

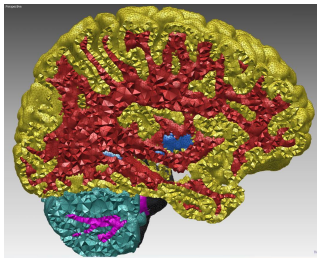
- The algorithm terminates since the number of triangulations of $\mathcal{P}$ is finite and $\mathrm{ang}(T(\mathcal{P}))$ cannot decrease
- The obtained triangulation is a Delaunay triangulation of $\mathcal{P}$ since all its edges are regular

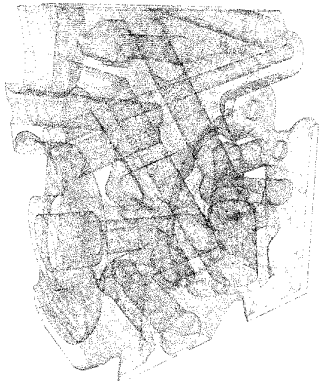
Meshing surfaces and 3D domains

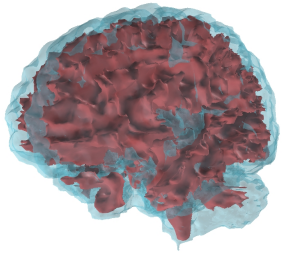
- visualization and graphics applications
- CAD and reverse engineering
- geometric modelling in medecine, geology, biology etc.
- autonomous exploration and mapping (SLAM)
- scientific computing : meshes for FEM











Mesh generation : from art to science

Grid methods

Lorensen & Cline [87] : marching cube
Lopez & Brodlie [03] : topological consistency
Plantiga & Vegter [04] : certified topology using interval arithmetic

Morse theory

Stander & Hart [97]
B., Cohen-Steiner & Vegter [04] : certified topology

Delaunay refinement

Hermeline [84]
Ruppert [95]
Shewchuk [02]
Chew [93]
B. & Oudot [03,04]
Cheng et al. [04]

Main issues

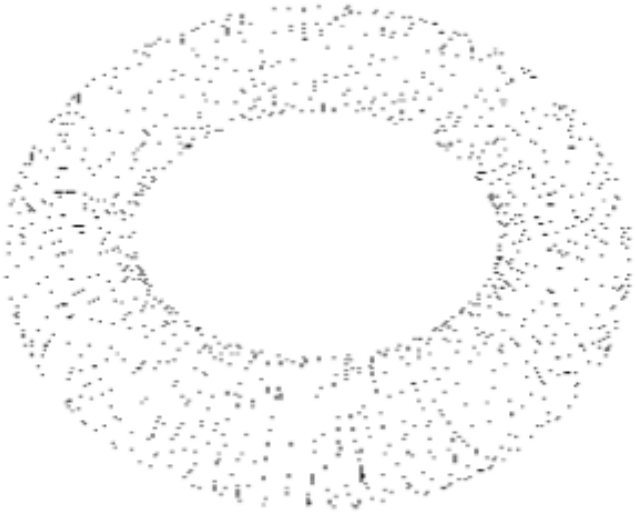
Sampling

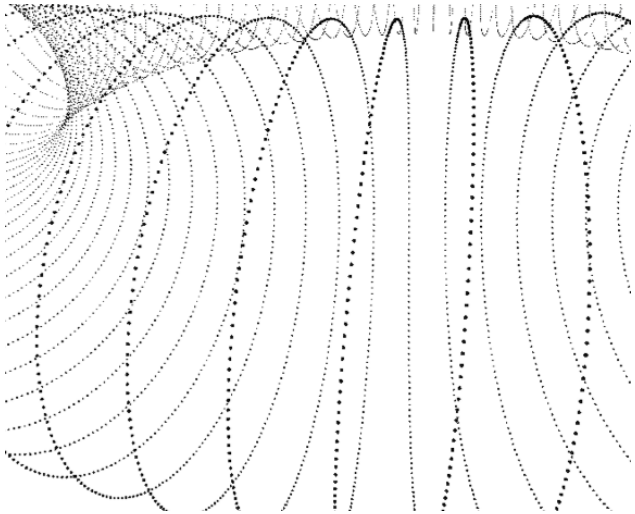
- How do we choose points in the domain ?
- What information do we need to know/measure about the domain ?

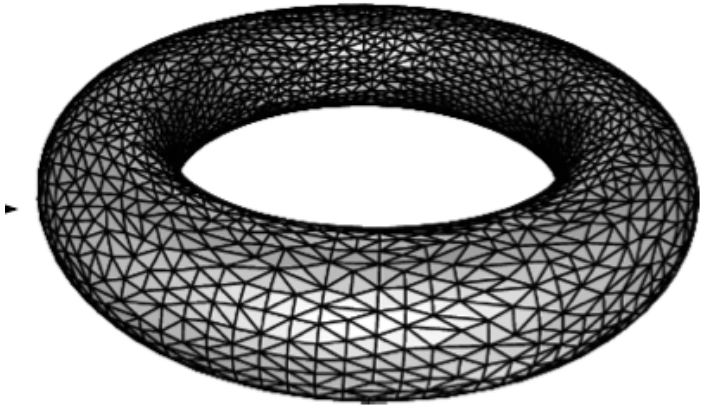
Meshing

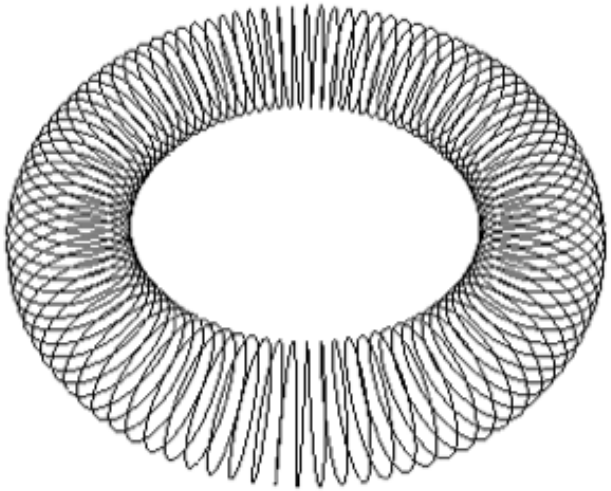
- How do we connect the points ?
- Under what sampling conditions can we compute a good approximation of the domain ?

Scale and dimension





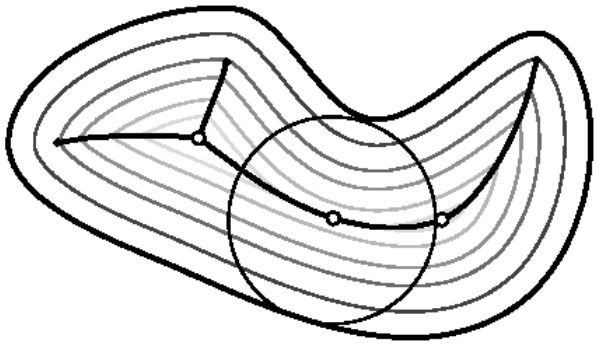




Sampling conditions [Federer 1958], [Amenta & Bern 1998]

Medial axis of $\mathbb{M}$: $\mathrm{axis}(\mathbb{M})$

set of points with at least two closest points on $\mathbb{M}$



Local feature size and reach

$\forall x \in \mathbb{M}, \quad \mathrm{lfs}(x) = d(x, \mathrm{axis}(\mathbb{M}))$

$\mathrm{rch}(\mathbb{M}) = \inf_{x \in \mathbb{M}} \mathrm{lfs}(x)$

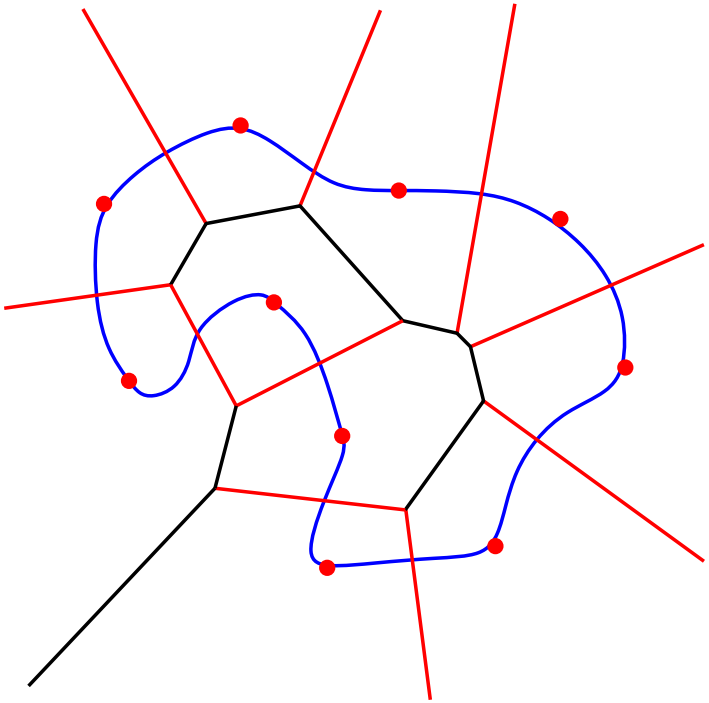
$(\epsilon, \bar{\eta})$-net of $\mathbb{M}$

1. $\mathcal{P} \subset \mathbb{M}, \forall x \in \mathbb{M} : \quad d(x, \mathcal{P}) \leq \epsilon \, \mathrm{lfs}(x)$
2. $\forall p, q \in \mathcal{P}, \quad \|p - q\| \geq \bar{\eta}\varepsilon \, \min(\mathrm{lfs}(p), \mathrm{lfs}(q))$

Restricted Delaunay triangulation [Chew 93]

Definition

The restricted Delaunay triangulation $\mathrm{Del}_X(\mathcal{P})$ to $X \subset \mathbb{R}^d$ is the nerve of $\mathrm{Vor}(\mathsf{P}) \cap X$



If P is an ε-sample, any ball centered on X that circumscribes a facet f of $\mathrm{Del}_X(\mathcal{P})$ has a radius $\leq \varepsilon\,\mathrm{lfs}(c_f)$

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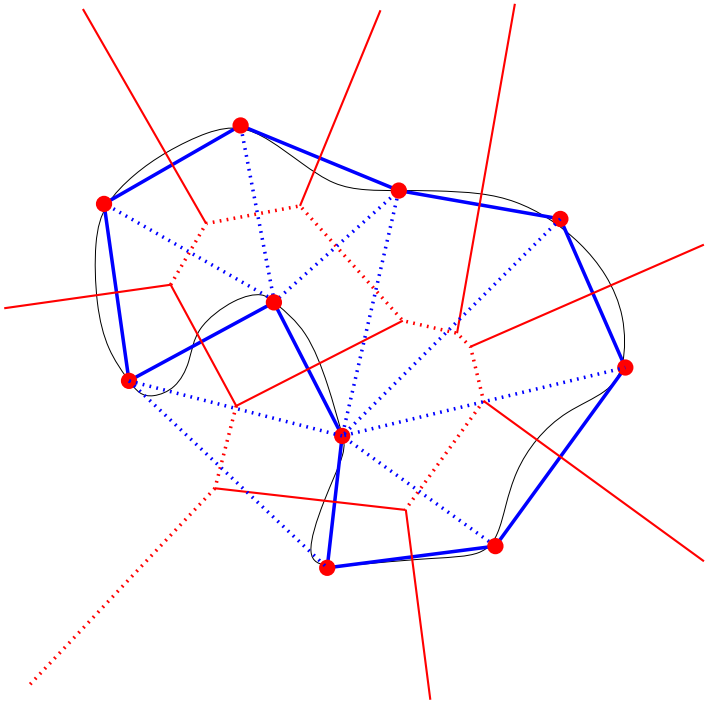

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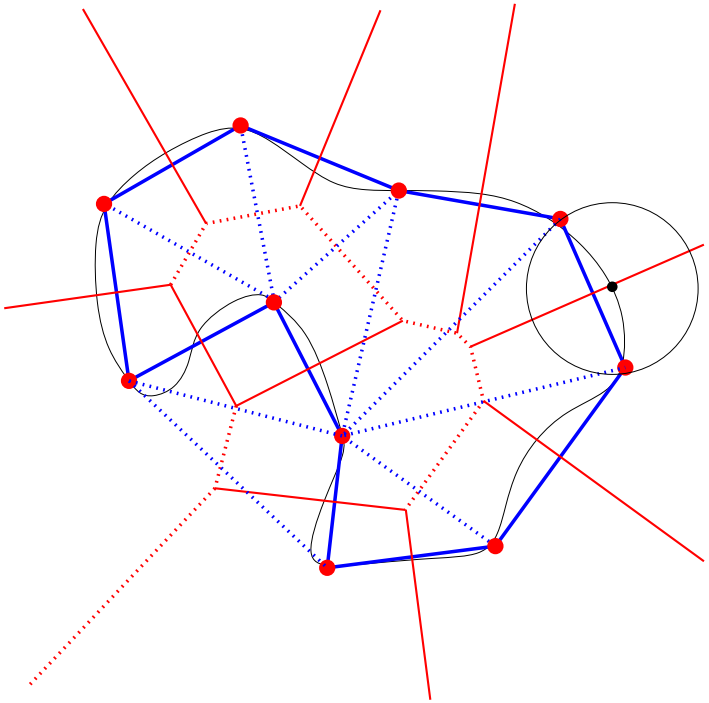

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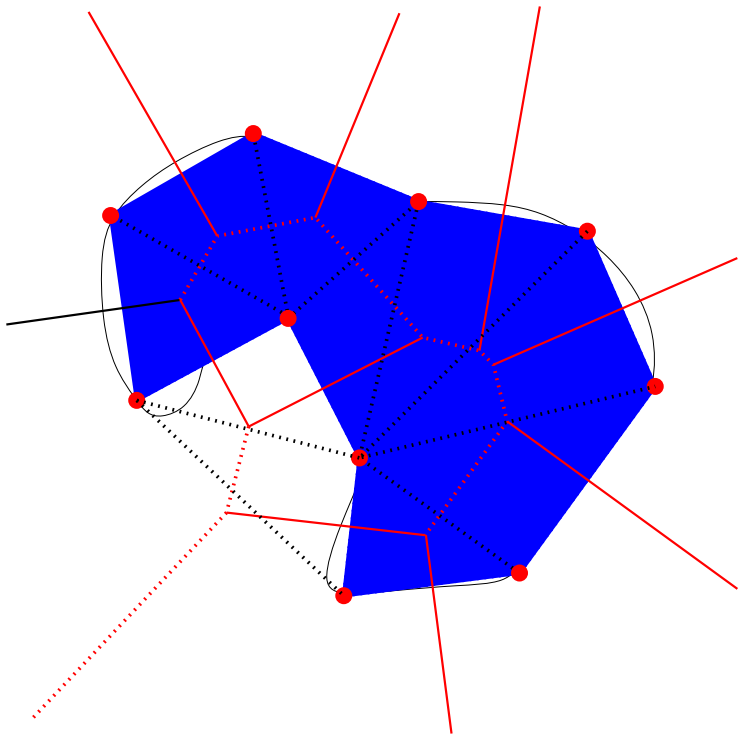
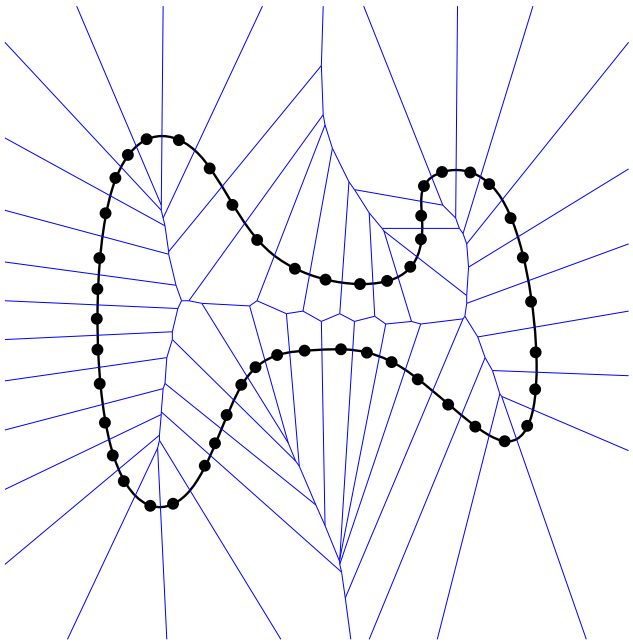


Restricted Delaunay triang. of $(\varepsilon, \bar{\eta})$-nets

[Amenta et al. 1998-], [B. & Oudot 2005]



If

- $\mathcal{S} \subset \mathbb{R}^3$ is a compact surface of positive reach without boundary
- $\mathcal{P}$ is an $(\varepsilon, \bar{\eta})$-net with $\bar{\eta} \geq cst$ and ε small enough

then

- $\mathrm{Del}_{|\mathcal{S}}(\mathcal{S})$ provides good estimates of normals
- There exists a homeomorphism $\phi : \mathrm{Del}_{|\mathcal{S}}(\mathcal{P}) \to \mathcal{S}$
- $\sup_x(\|\phi(x) - x\|) = O(\varepsilon^2)$

Schwarz lantern