Given: A set of (bad) events $A_1, ..., A_N$,
each happens with $\text{proba}(A_i) \leq p < 1$

Question : what is the probability that none of the events occur?

The case of independent events

$$\text{proba}(\neg A_1 \wedge ... \wedge \neg A_N) \geq (1-p)^N > 0$$

What if we allow a limited amount of dependency among the events?

LLL : symmetric version [Lovász & Erdös 1975]

Under the assumptions

1. $\text{proba}(A_i) \leq p < 1$
2. A_i depends of $\leq \Gamma$ other events A_j
3. $\text{proba}(A_i) \leq \frac{1}{e\,(\Gamma+1)}$ $\qquad e = 2.718...$

then

$$\text{proba}(\neg A_1 \wedge ... \wedge \neg A_N) > 0$$

Moser and Tardos' constructive proof of the LLL [2010]

$\mathcal{P}$ a finite set of mutually independent random variables

$\mathcal{A}$ a finite set of events that are determined by the values of $S \subseteq \mathcal{P}$

Two events are independent iff they share no variable

Algorithm

for all $P \in \mathcal{P}$ **do**
  $v_P \leftarrow$ a random evaluation of P;

while $\exists A \in \mathcal{A}$: A happens when $(P = v_P, P \in \mathcal{P})$ **do**

  pick an arbitrary happening event $A \in \mathcal{A}$;

  for all $P \in \text{variables}(A)$ **do**
    $v_P \leftarrow$ a new random evaluation of P;

return $(v_P)_{P \in \mathcal{P}}$;

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return $(v_P)_{P \in \mathcal{P}}$;

Moser and Tardos' theorem

if

1. $\text{proba}(A_i) \le p < 1$
2. A_i depends of $\le \Gamma$ other events A_j
3. $\text{proba}(A_i) \le \frac{1}{e\,(\Gamma+1)}$ $\qquad e = 2.718...$

then $\exists$ an assignment of values to the variables $\mathcal{P}$ such that no event in $\mathcal{A}$ happens

The randomized algorithm resamples an event $A \in \mathcal{A}$ at most $\boxed{\frac{1}{\Gamma}}$ expected times before it finds such an evaluation

The expected total number of resampling steps is at most $\boxed{\frac{N}{\Gamma}}$

Home work

- Read the proof of Moser & Tardos (or Spencer's nice note)
- Learn about the parallel and the derandomized versions
- Listen to a talk by Aravind Srinivasan on further extensions

  `https://video.ias.edu/csdm/2014/0407-AravindSrinivasan`