MPRI 2014/15

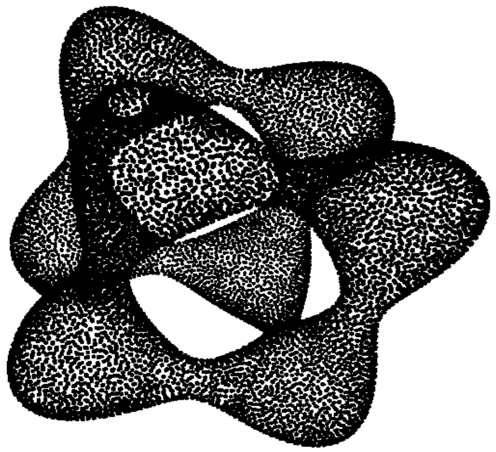
A sampling theory for compact sets

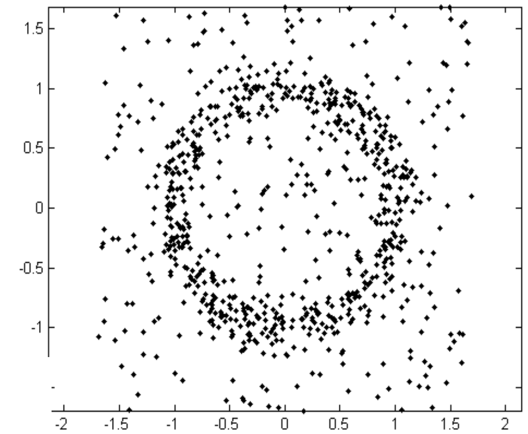
F. Chazal
Geometrica Group
INRIA Saclay

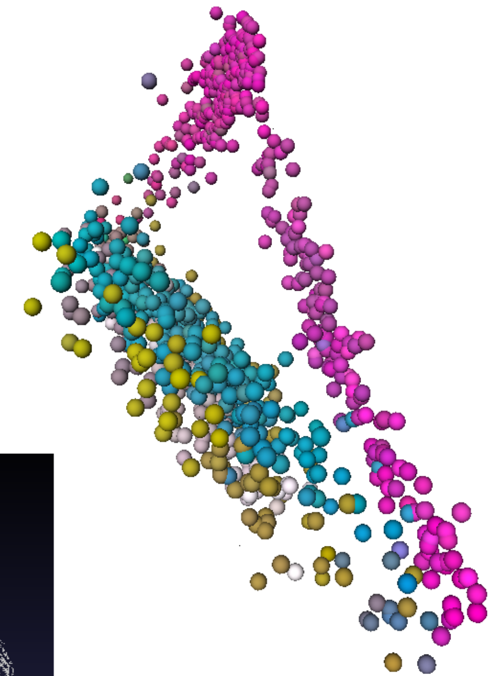
To download these slides:
http://geometrica.saclay.inria.fr/team/Fred.Chazal/Teaching/DistanceFunctions.pdf

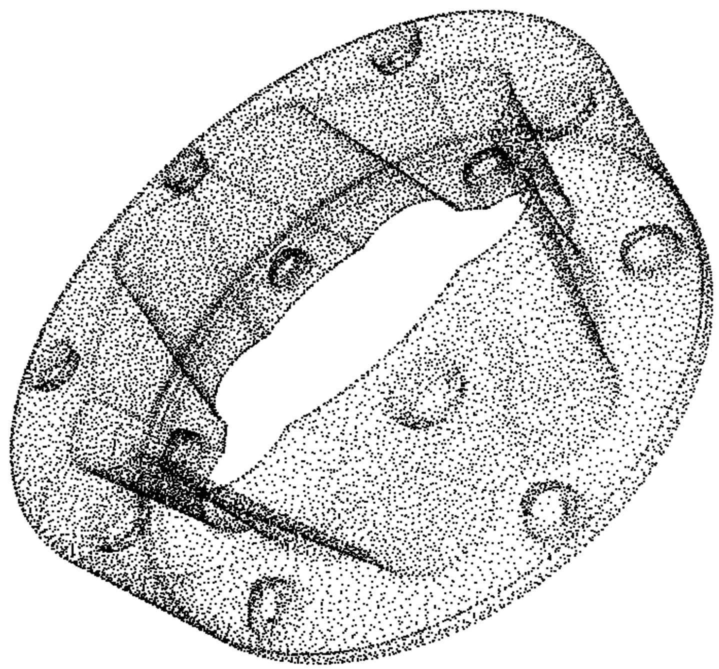
If you have any question:
frederic.chazal@inria.fr

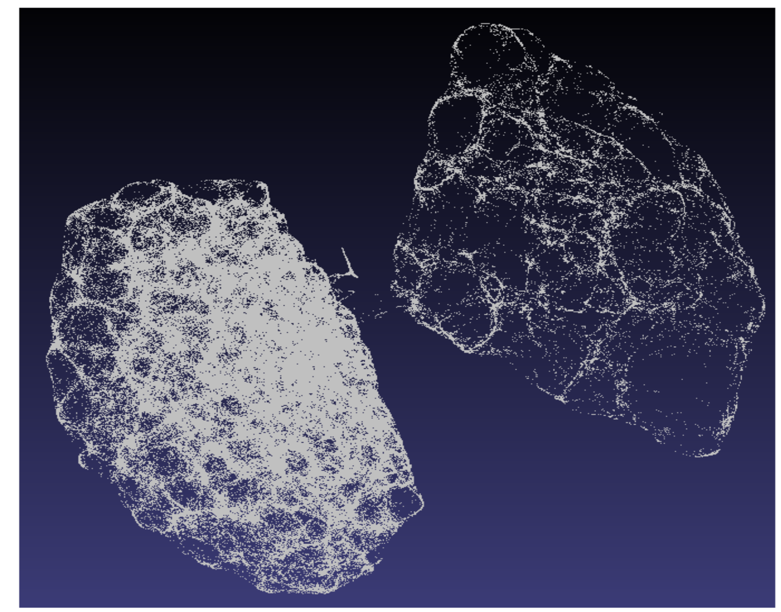
Introduction and motivations









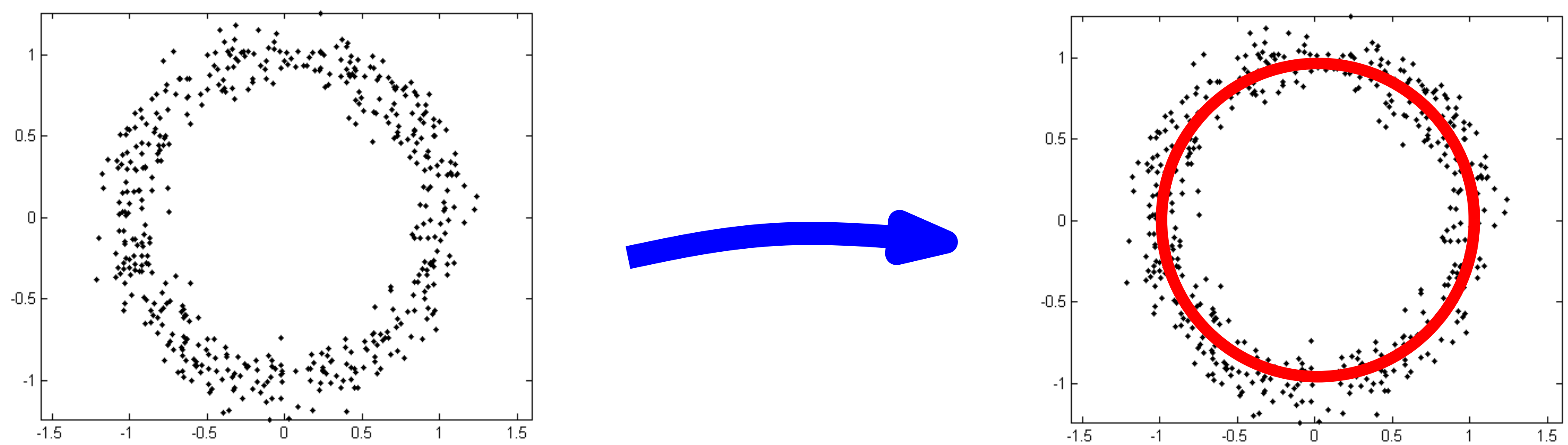


What can we say about the topology/geometry of spaces known only through a finite set of measurements?

What is the relevant topology/geometry of a point cloud data set?

Motivations: Reconstruction, Manifold Learning and NLDR, Clustering and Segmentation,...

Geometric Inference



Question: Given an approximation C of a geometric object K, is it possible to reliably estimate the topological and geometric properties of K, knowing only the approximation C?

Question *: Given a point cloud C (or some other more complicated set), is it possible to infer some robust topological or geometric information of C?

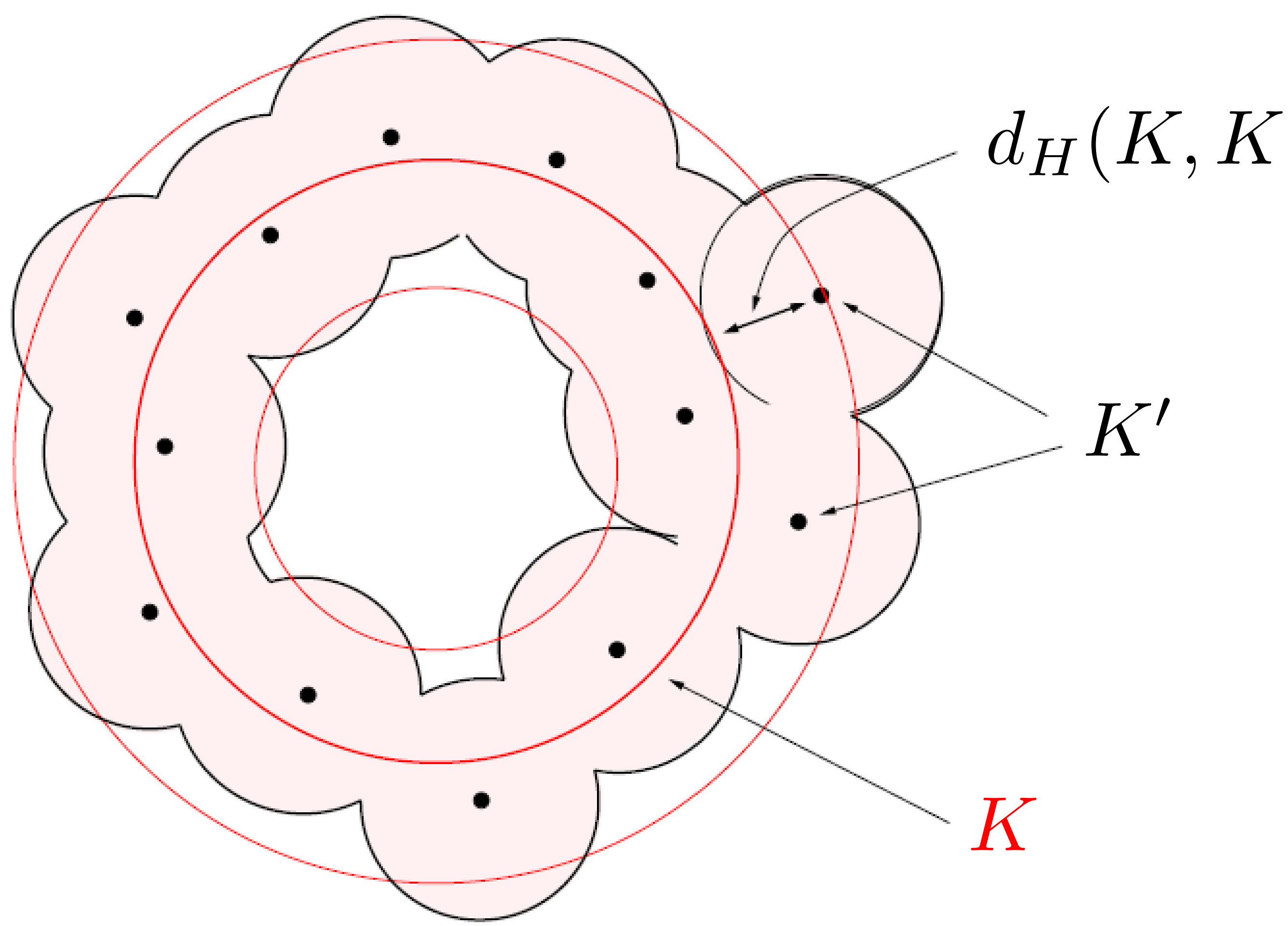
- The answer depends on:
  - the considered class of objects (no hope to get a positive answer in full generality),
  - a notion of distance between the objects (approximation).

Distance functions for geometric inference

Considered objects: compact subsets K of $\mathbb{R}^d$

Distance:
distance function to a compact $K \subset \mathbb{R}^d$: $d_K : x \to \inf_{p \in K} \|x - p\|$
Hausdorf distance between two compact sets:

$$d_H(K, K') = \sup_{x \in \mathbb{R}^d} |d_K(x) - d_{K'}(x)|$$



Distance functions for geometric inference

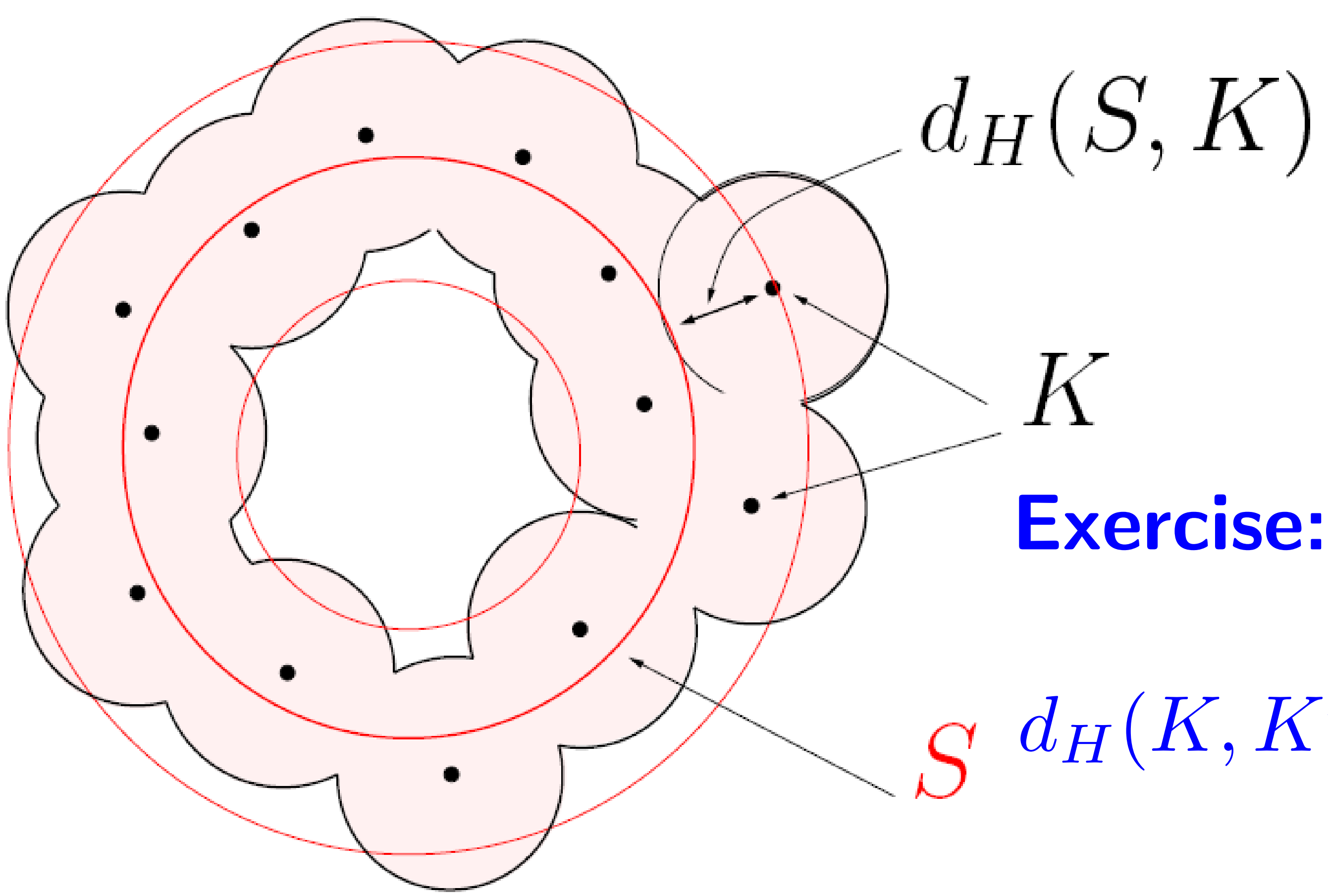
Considered objects: compact subsets K of $\mathbb{R}^d$

Distance:

distance function to a compact $K \subset \mathbb{R}^d$: $d_K : x \to \inf_{p \in K} \|x - p\|$

Hausdorf distance between two compact sets:

$$d_H(K, K') = \sup_{x \in \mathbb{R}^d} |d_K(x) - d_{K'}(x)|$$



Exercise: Show that

$$d_H(K, K') = \max \left(\sup_{y \in K'} d_K(y), \sup_{z \in K} d_{K'}(z) \right)$$

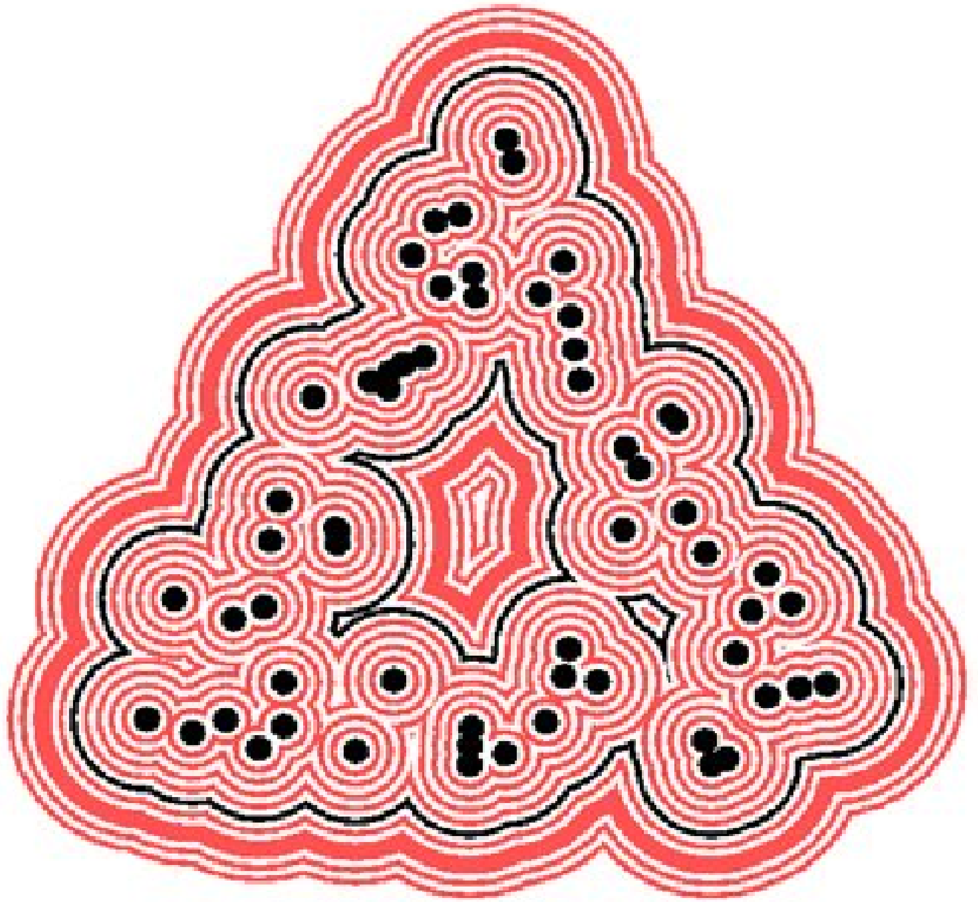
Distance functions for geometric inference

Considered objects: compact subsets K of $\mathbb{R}^d$

Distance:
distance function to a compact $K \subset \mathbb{R}^d$: $d_K : x \to \inf_{p \in K} \|x - p\|$
Hausdorf distance between two compact sets:

$$d_H(K, K') = \sup_{x \in \mathbb{R}^d} |d_K(x) - d_{K'}(x)|$$

- Replace K and C by d_K and d_C
- Compare the topology of the offsets $K^r = d_K^{-1}([0, r])$ and $C^r = d_C^{-1}([0, r])$

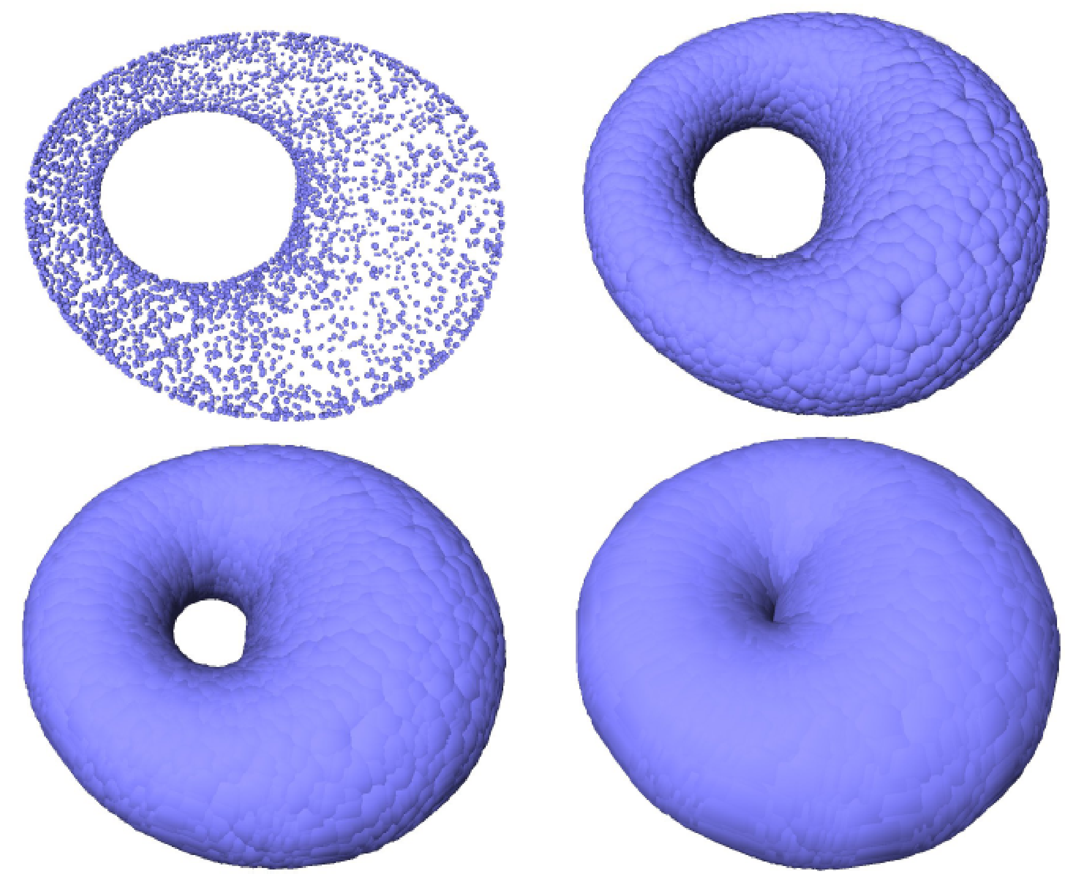


Distance functions for geometric inference

Considered objects: compact subsets K of $\mathbb{R}^d$

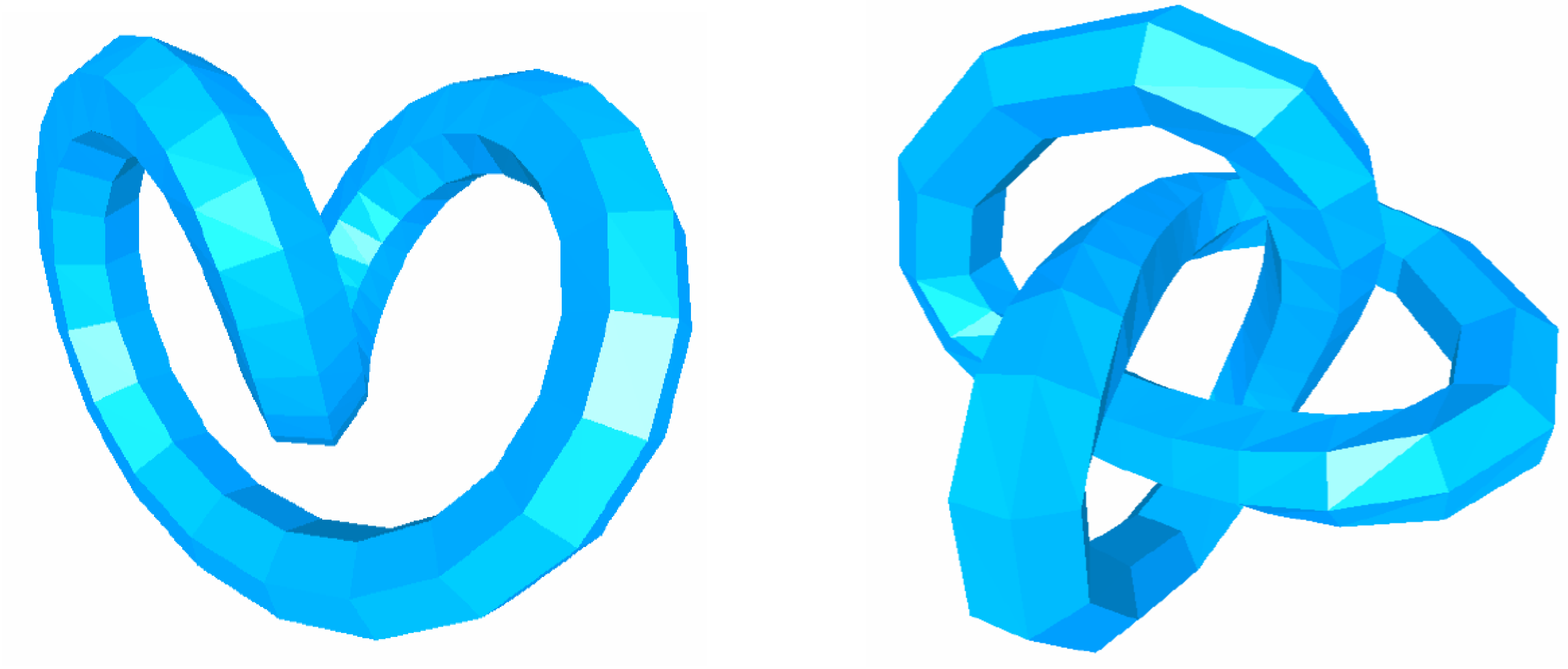
Distance:
distance function to a compact $K \subset \mathbb{R}^d$: $d_K : x \to \inf_{p \in K} \|x - p\|$
Hausdorf distance between two compact sets:

$$d_H(K, K') = \sup_{x \in \mathbb{R}^d} |d_K(x) - d_{K'}(x)|$$



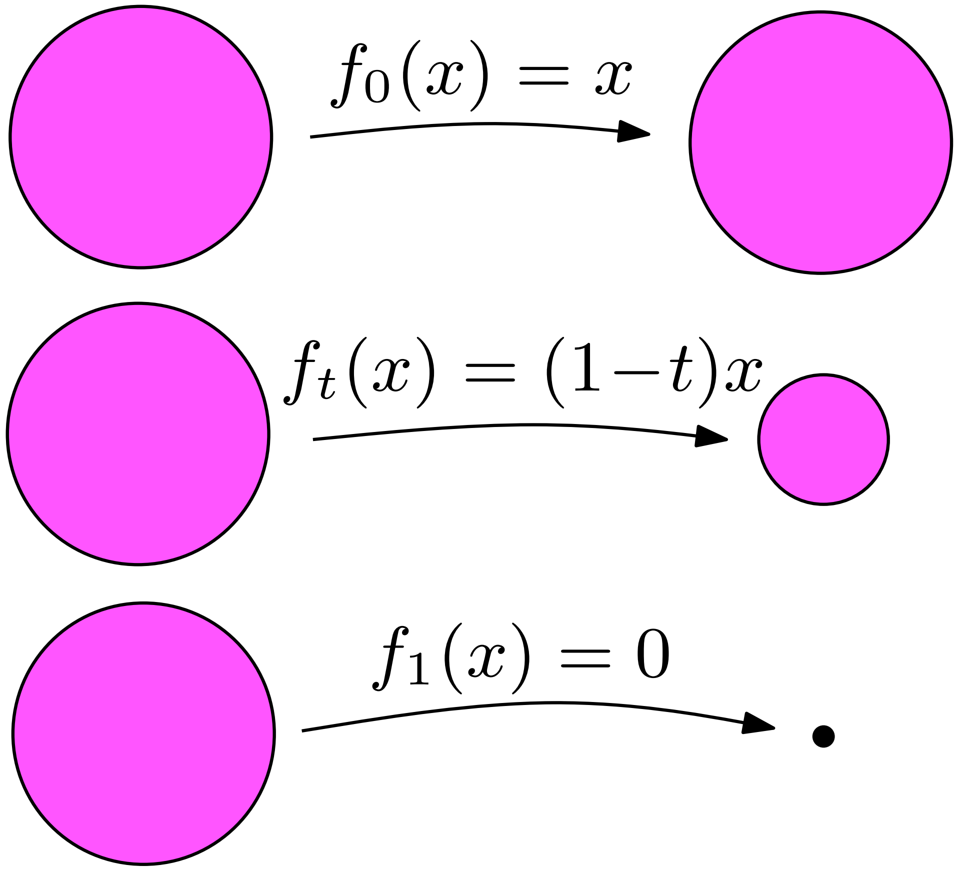
- Replace K and C by d_K and d_C
- Compare the topology of the offsets $K^r = d_K^{-1}([0, r])$ and $C^r = d_C^{-1}([0, r])$

Topology: homeomorphy and isotopy



- X and Y are homeomorphic if there exists a bijection $h : X \to Y$ s. t. h and h^{-1} are continuous.
- $X, Y \subset \mathbb{R}^d$ are isotopic if there exists a continuous map $F : X \times [0,1] \to \mathbb{R}^d$ s. t. $F(.,0) = Id_X$, $F(X,1) = Y$ and $\forall t \in [0,1]$, $F(.,t)$ is an homeomorphism on its image.
- $X, Y \subset \mathbb{R}^d$ are ambient isotopic if there exists a continuous map $F : \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$ s. t. $F(.,0) = Id_{\mathbb{R}^d}$, $F(X,1) = Y$ and $\forall t \in [0,1]$, $F(.,t)$ is an homeomorphim of $\mathbb{R}^d$.

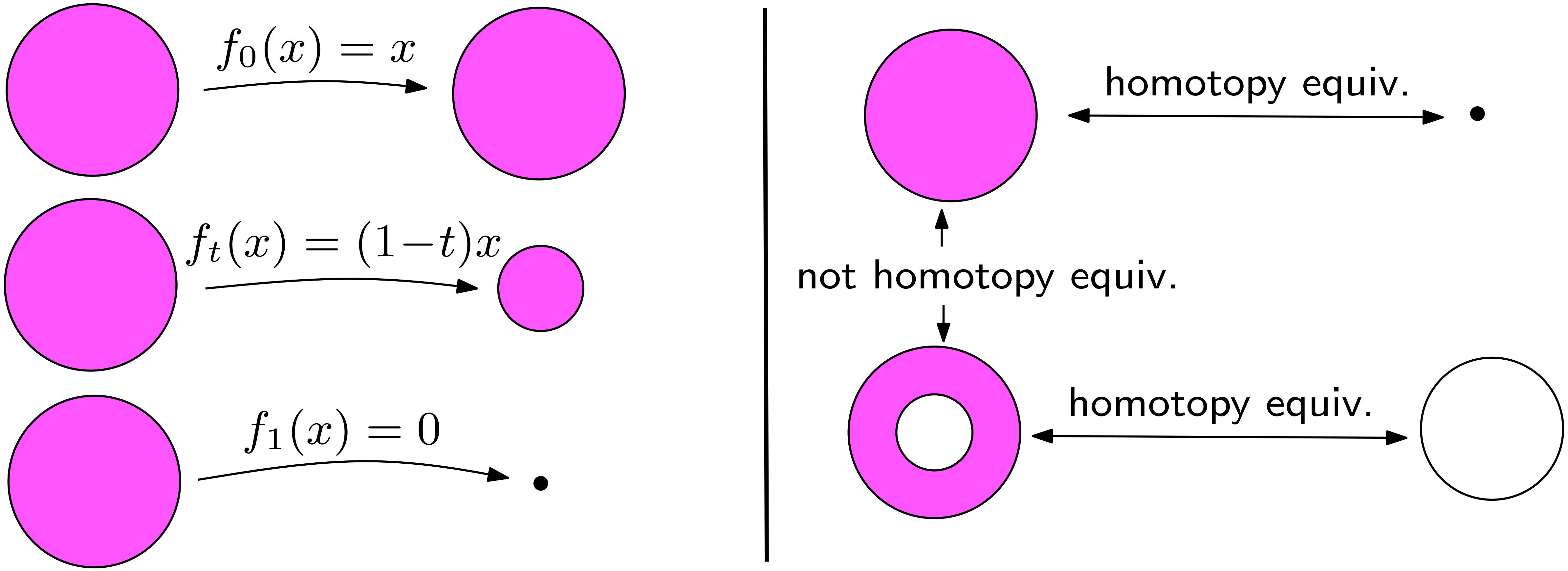
Topology: homotopy type



- Two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic if there exists a continuous map $H : [0,1] \times X \to Y$ s. t. $\forall x \in X$, $H(0,x) = f_0(x)$ and $H_1(1,x) = f_1(x)$.

- X and Y have the same homotopy type (or are homotopy equivalent) if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ s. t. $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y.

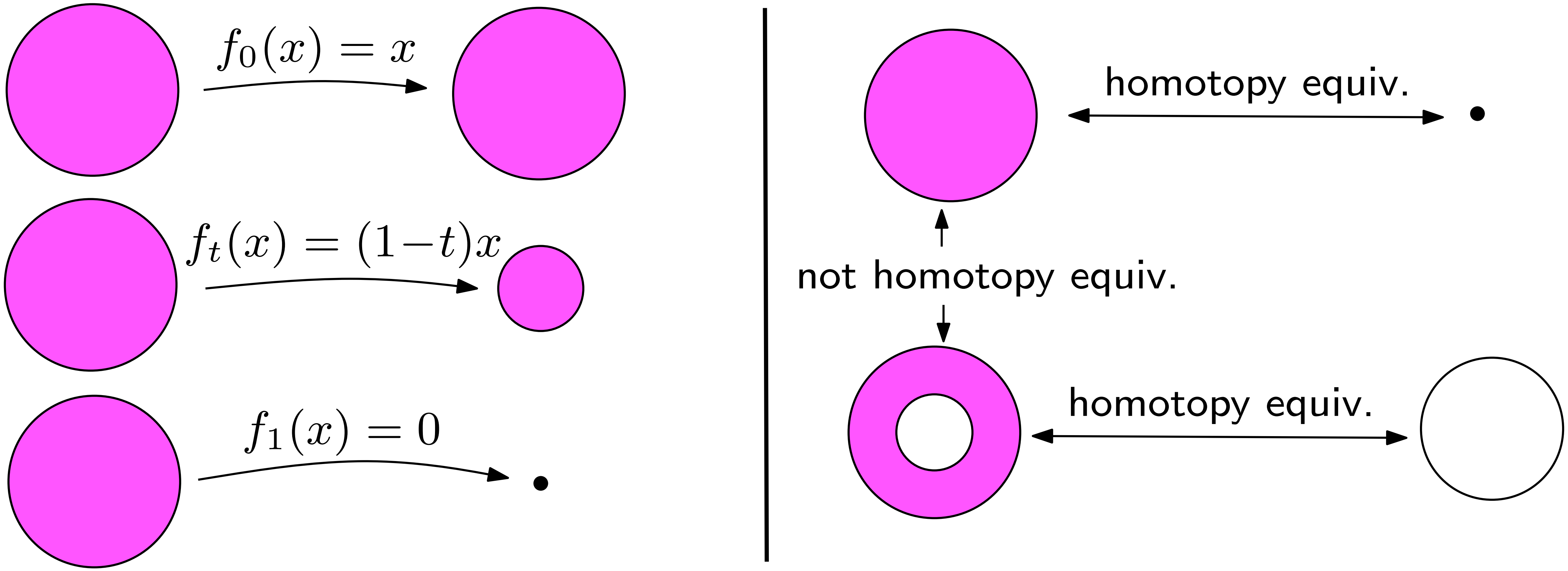
Topology: homotopy type



- Two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic if there exists a continuous map $H : [0,1] \times X \to Y$ s. t. $\forall x \in X$, $H(0,x) = f_0(x)$ and $H_1(1,x) = f_1(x)$.

- X and Y have the same homotopy type (or are homotopy equivalent) if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ s. t. $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y.

Topology: homotopy type

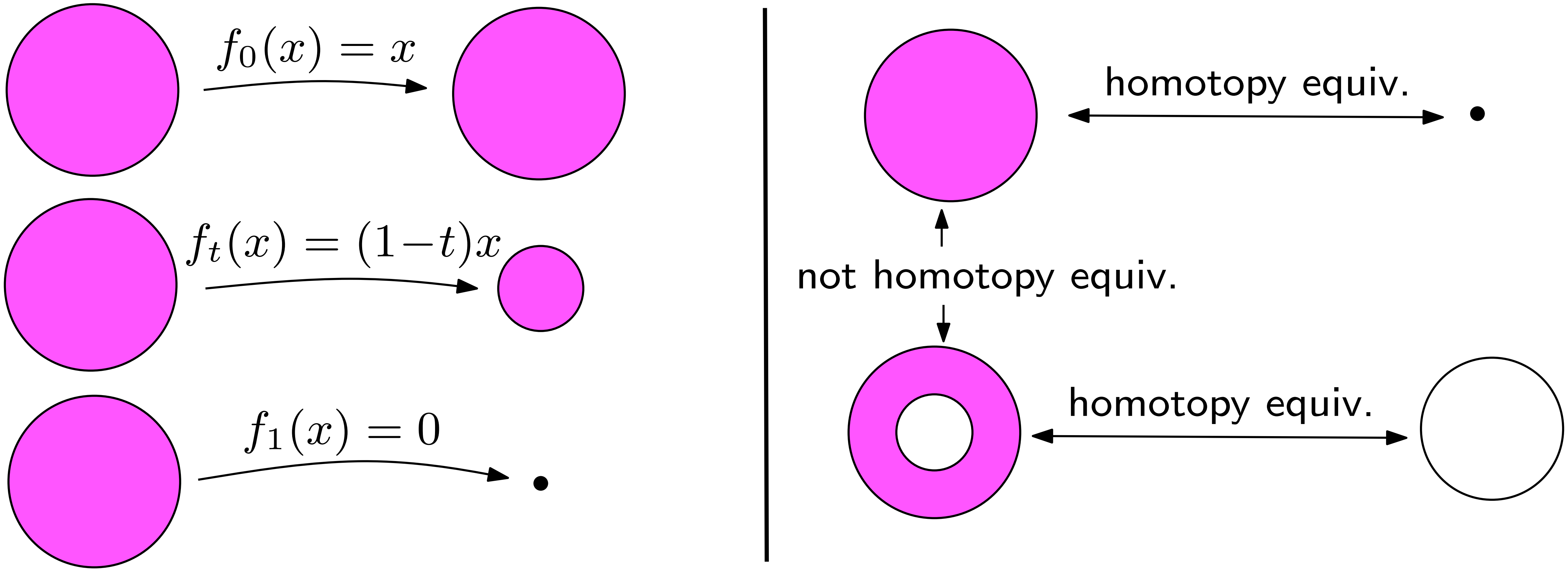


- Two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic if there exists a continuous map $H : [0,1] \times X \to Y$ s. t. $\forall x \in X$, $H(0,x) = f_0(x)$ and $H_1(1,x) = f_1(x)$.

- X and Y have the same homotopy type (or are homotopy equivalent) if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ s. t. $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y.

X and Y homotopy equivalent $\Rightarrow$ X and Y have isomorphic homotopy and homology groups.

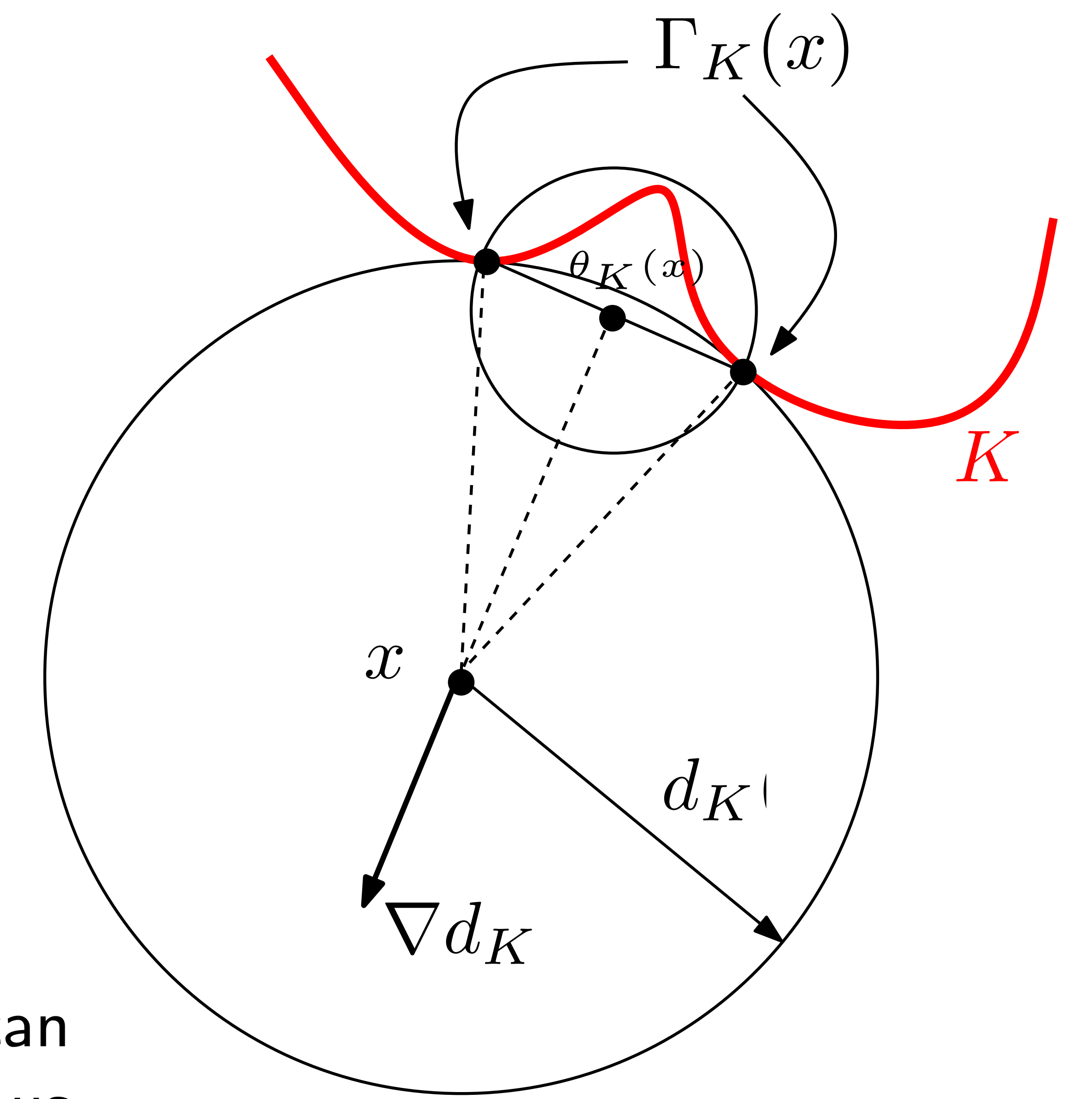
Topology: homotopy type



If $Y \subset X$ and if there exists a continuous map $H : [0,1] \times X \to X$ s.t.:
i) $\forall x \in X$, $H(0,x) = x$,
ii) $\forall x \in X$, $H(1,x) \in Y$
iii) $\forall y \in Y$, $\forall t \in [0,1]$, $H(t,y) \in Y$,
then X and Y are homotopy equivalent. If one replaces condition *iii)* by $\forall y \in Y$, $\forall t \in [0,1]$, $H(t,y) = y$ then H is a deformation retract of X onto Y.

The gradient of the distance function

- $\Gamma_K(x) = \{y \in K : d(x,y) = d_K(x)\}$
- $\theta_K(x)$: center and radius of the smallest ball enclosing $\Gamma_K(x)$

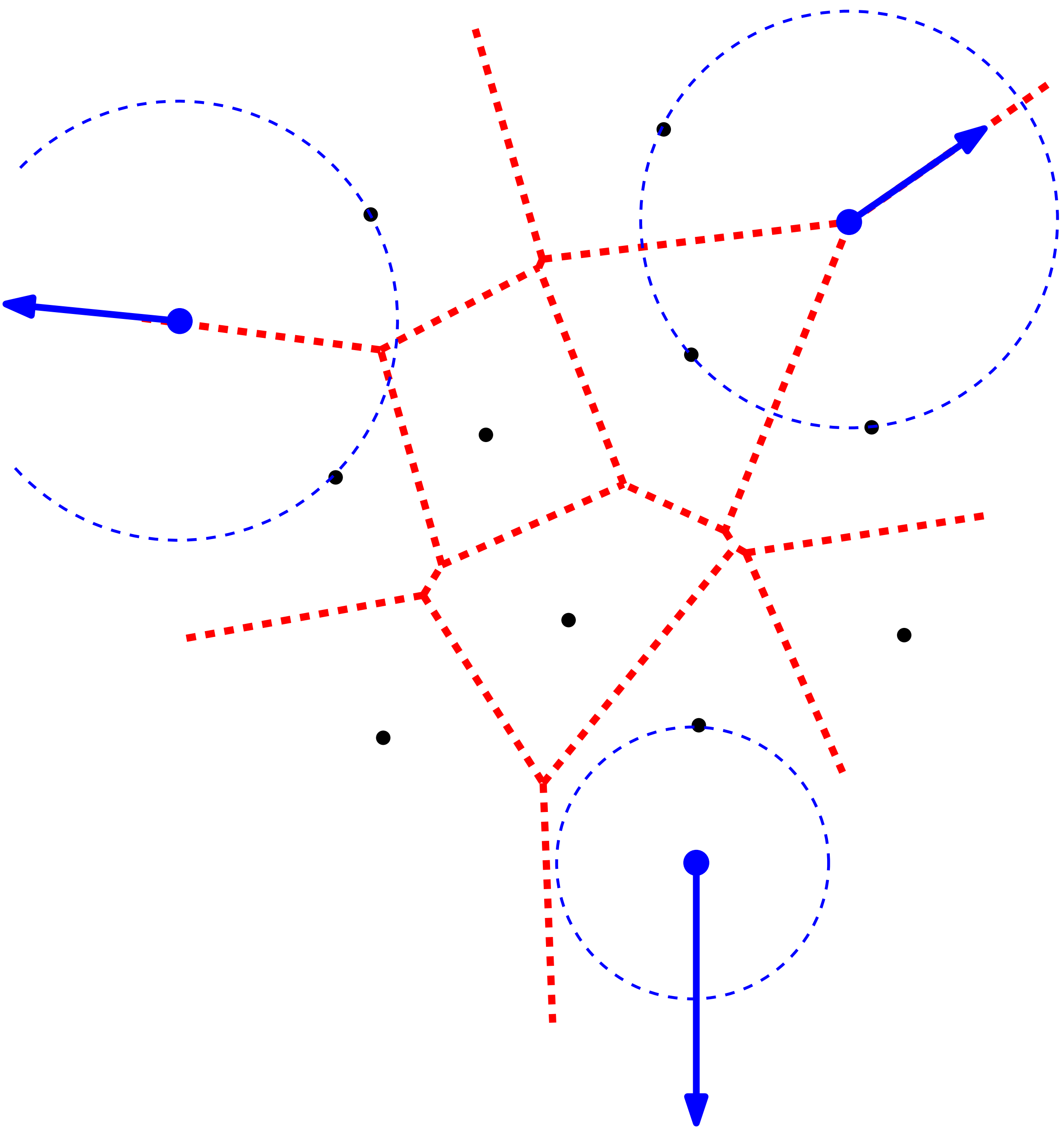


$$\boxed{\nabla d_K(x) = \frac{x - \theta_K(x)}{d_K(x)}}$$

Although not continuous, it can be integrated in a continuous flow.

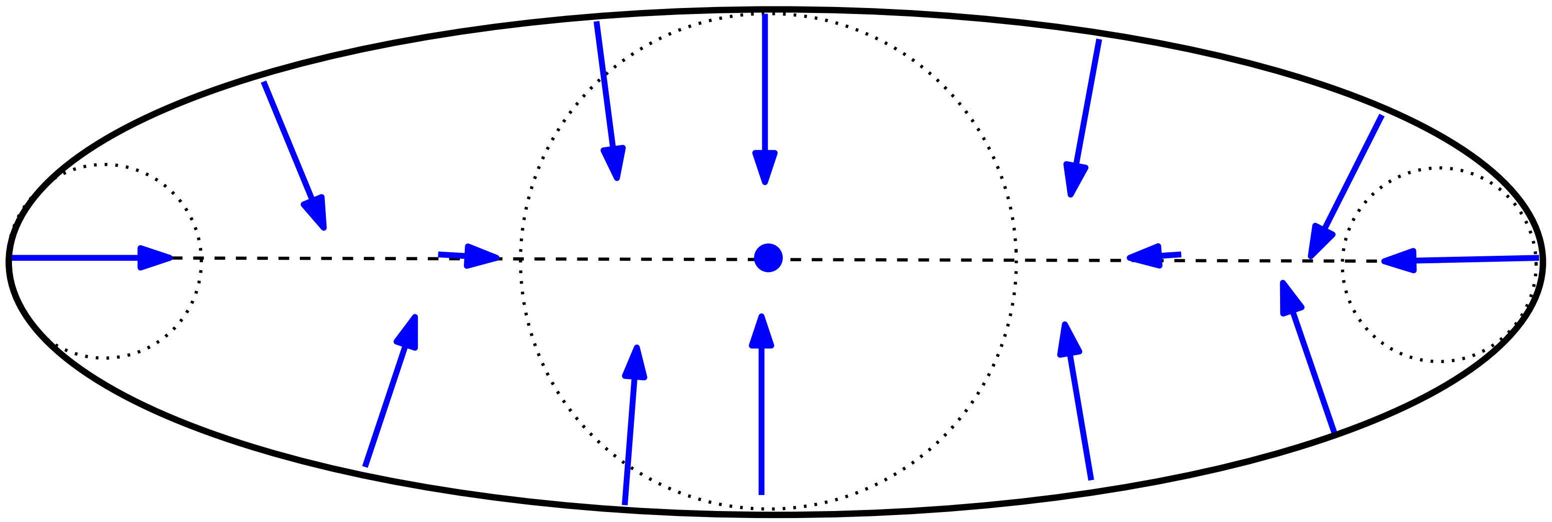
Definition: x is a *critical point* of d_K iff $\nabla d_K(x) = 0$

The gradient for a point cloud



The gradient of the distance function to a point cloud data set C is easy to compute if one knows how to compute the Voronoï diagram of C.

Integration of the gradient of d_K

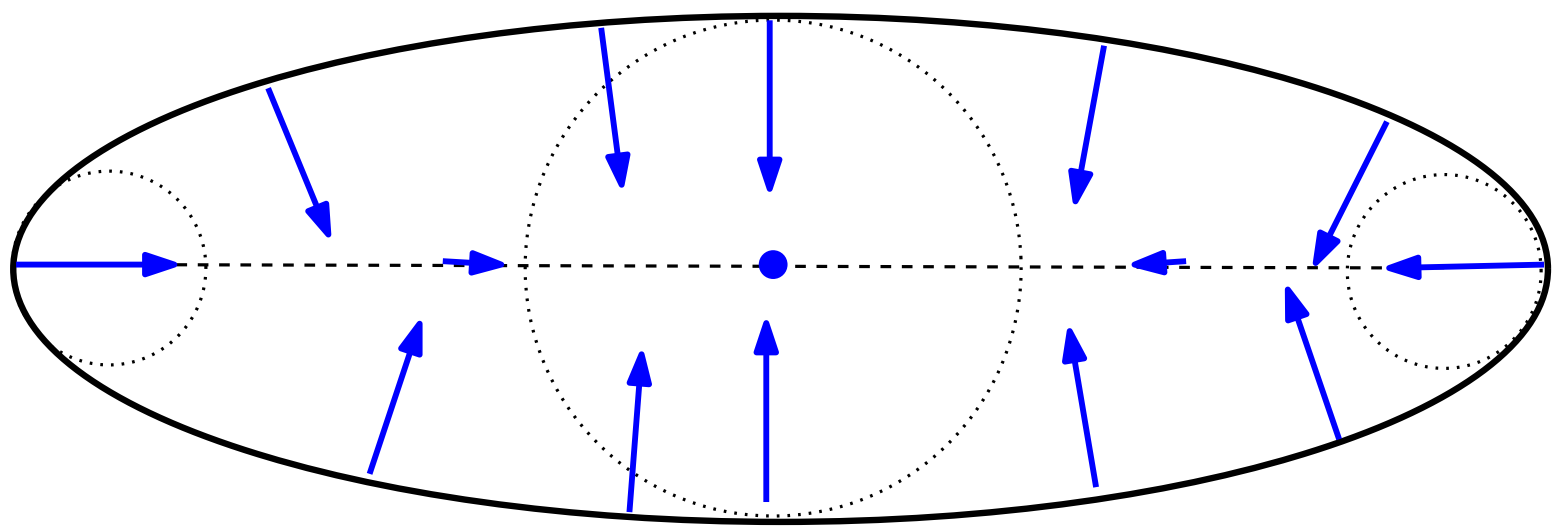


- Although ∇d_K is discontinuous, it can be integrated: there exists $\mathcal{C} : \mathbb{R}_+ \times (\mathbb{R}^d \setminus K) \to \mathbb{R}^d \setminus K$ a continuous function, right differentiable with respect to t s. t.

$$\frac{\partial \mathcal{C}}{\partial t}(t,x) = \nabla d_K(\mathcal{C}(t,x)) \text{ and } \mathcal{C}(t+s,x) = \mathcal{C}(s,\mathcal{C}(t,x))$$

- The function d_K is increasing along the trajectories of ∇d_K.

Integration of the gradient of d_K



- The norm of the gradient is given by

radius of the smallest ball enclosing $\Gamma_K(x)$

$$\boxed{\|\nabla d_K(x)\|^2 = 1 - \frac{F_K(x)}{d_K(x)}}$$

- The trajectories of ∇d_K can be parametrized by arc length $s \to \mathcal{C}(t(s), x)$ and one has

$$\boxed{d_K(\mathcal{C}(t(l), x) = d_K(x) + \int_0^l \|\nabla d_K(\mathcal{C}(t(s), x))\| ds}$$

Critical points and offsets topology

For $\alpha \geq 0$, the *α-offset* of K is $K^\alpha = \{x \in \mathbb{R}^d : d_K(x) \leq \alpha\}$

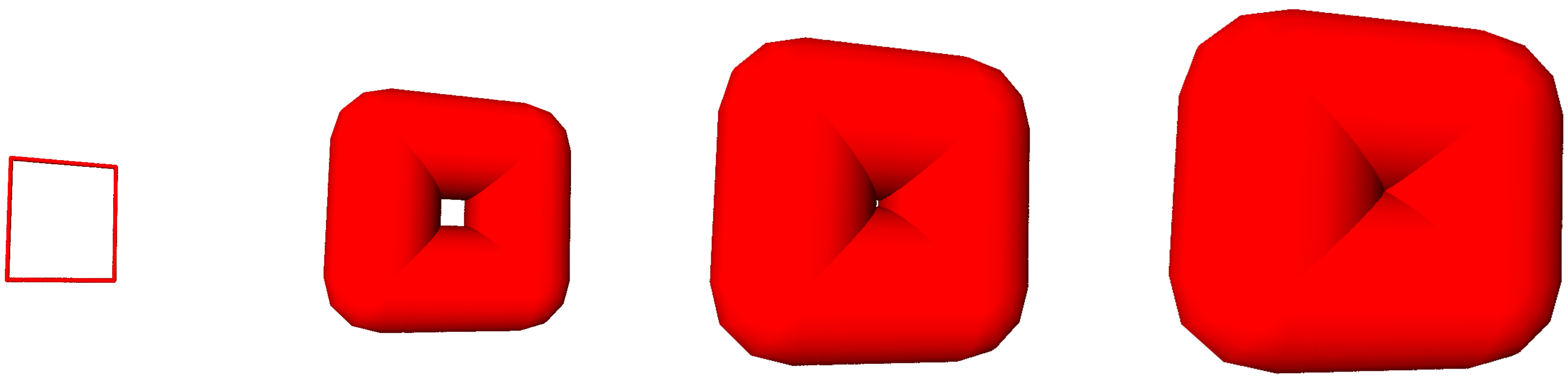
Theorem: [Grove, Cheeger,...] Let $K \subset \mathbb{R}^d$ be a compact set.

- Let r be a regular value of d_K. Then $d_K^{-1}(r)$ is a topological submanifold of $\mathbb{R}^d$ of codimension 1.

- Let $0 < r_1 < r_2$ be such that $[r_1, r_2]$ does not contain any critical value of d_K. Then all the level sets $d_K^{-1}(r)$, $r \in [r_1, r_2]$ are isotopic and

$$K^{r_2} \setminus K^{r_1} = \{x \in \mathbb{R}^d : r_1 < d_K(x) \leq r_2\}$$

  is homeomorphic to $d_K^{-1}(r_1) \times (r_1, r_2]$.



Weak feature size and stability

The *weak feature size* of a compact $K \subset \mathbb{R}^d$:

$$\mathsf{wfs}(K) = \inf\{c > 0 : c \text{ is a critical value of } d_K\}$$

Proposition: [C-Lieutier'05] Let $K, K' \subset \mathbb{R}^d$ be such that

$$d_H(K, K') < \varepsilon := \frac{1}{2}\min(\mathsf{wfs}(K), \mathsf{wfs}(K'))$$

Then for all $0 < r \leq 2\varepsilon$, K^r and K'^r are homotopy equivalent.

Weak feature size and stability

The *weak feature size* of a compact $K \subset \mathbb{R}^d$:

$$\mathsf{wfs}(K) = \inf\{c > 0 : c \text{ is a critical value of } d_K\}$$

Proposition: [C-Lieutier'05] Let $K, K' \subset \mathbb{R}^d$ be such that

$$d_H(K, K') < \varepsilon := \frac{1}{2} \min(\mathsf{wfs}(K), \mathsf{wfs}(K'))$$

Then for all $0 < r \leq 2\varepsilon$, K^r and K'^r are homotopy equivalent.

Proof: let $\delta > 0$ be s.t. $\delta + 2\varepsilon < \min(\mathsf{wfs}(K), \mathsf{wfs}(K'))$.

$$\begin{tikzcd}
K^\delta \arrow[r, "a_0"] \arrow[dr, "d_0"] & K^{\delta+\varepsilon} \arrow[r, "a_1"] \arrow[dr, "d_1"] & K^{\delta+2\varepsilon} \\
K'^\delta \arrow[r, "b_0"] \arrow[ur, "c_0"] & K'^{\delta+\varepsilon} \arrow[r, "b_1"] \arrow[ur, "c_1"] & K'^{\delta+2\varepsilon}
\end{tikzcd}$$

Weak feature size and stability

The *weak feature size* of a compact $K \subset \mathbb{R}^d$:

$$\text{wfs}(K) = \inf\{c > 0 : c \text{ is a critical value of } d_K\}$$

Proposition: [C-Lieutier'05] Let $K, K' \subset \mathbb{R}^d$ be such that

$$d_H(K, K') < \varepsilon := \frac{1}{2}\min(\text{wfs}(K), \text{wfs}(K'))$$

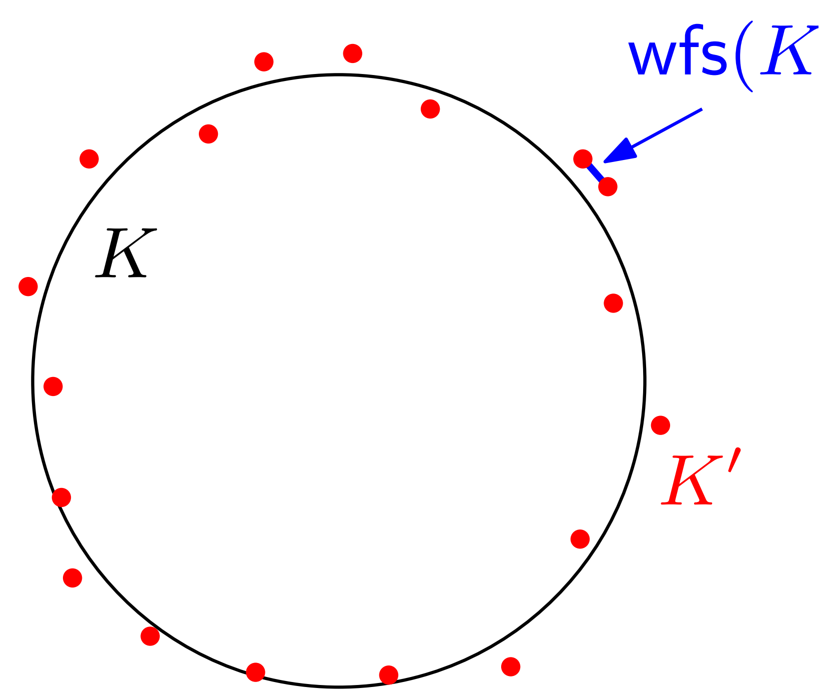
Then for all $0 < r \leq 2\varepsilon$, K^r and K'^r are homotopy equivalent.

Compact set with positive wfs:

 Stability properties

 Large class of compact sets (including sub-analytic sets)

 $K \to \text{wfs}(K)$ is not continuous (unstability of critical points).



Overcoming the discontinuity of wfs

Proposition: [C-Lieutier'05] Let $K, K' \subset \mathbb{R}^d$ be such that

$$d_H(K, K') < \varepsilon := \frac{1}{2} \min(\mathsf{wfs}(K), \mathsf{wfs}(K'))$$

Then for all $0 < r \leq 2\varepsilon$, K^r and K'^r are homotopy equivalent.

$K \to \mathsf{wfs}(K)$ is not continuous (unstability of critical points).

Option 1:
Restrict to a smaller class of compact sets with some stability properties of the critical points.

Option 2:
Try to get topological information about K without any assumption on $\mathsf{wfs}(K')$.

Overcoming the discontinuity of wfs

Proposition: [C-Lieutier'05] Let $K, K' \subset \mathbb{R}^d$ be such that

$$d_H(K, K') < \varepsilon := \frac{1}{2}\min(\mathsf{wfs}(K), \mathsf{wfs}(K'))$$

Then for all $0 < r \leq 2\varepsilon$, K^r and K'^r are homotopy equivalent.

$K \to \mathsf{wfs}(K)$ is not continuous (unstability of critical points).

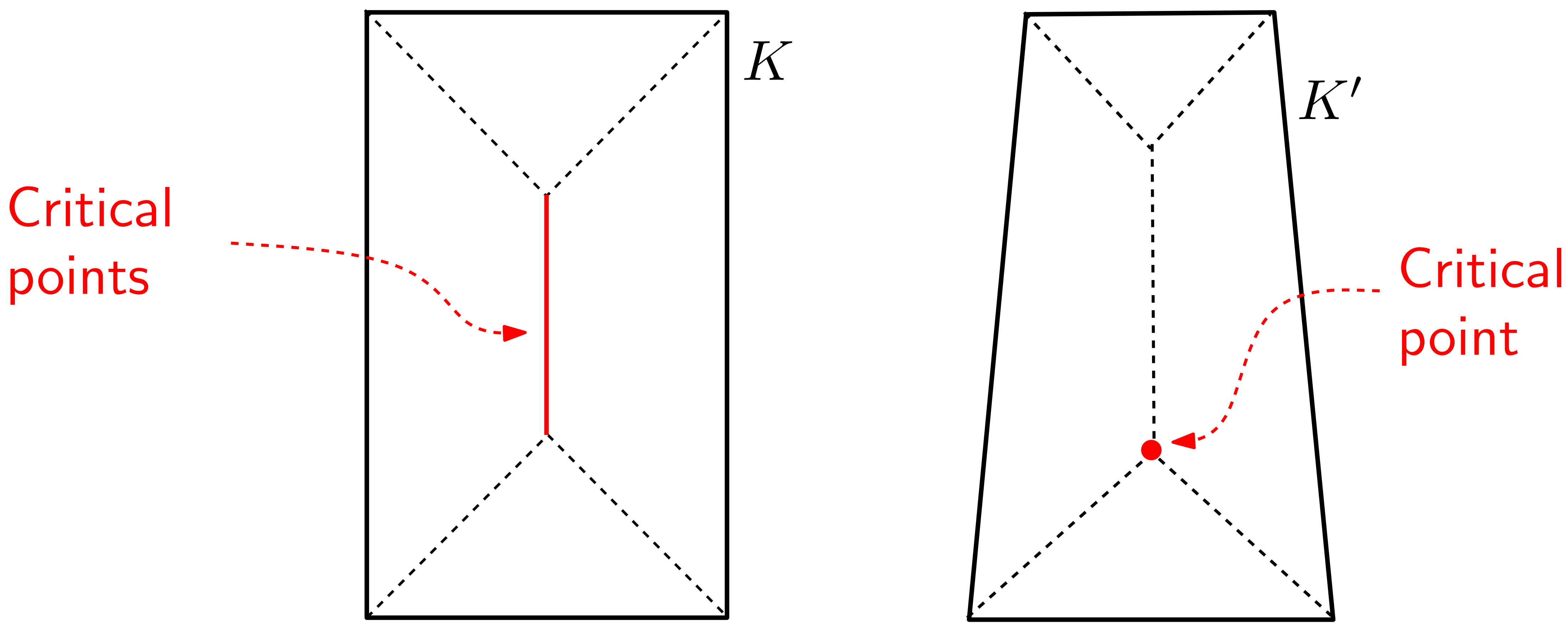
Option 1:
Restrict to a smaller class of compact sets with some stability properties of the critical points.

Notion of μ-critical points. Strong reconstruction results.

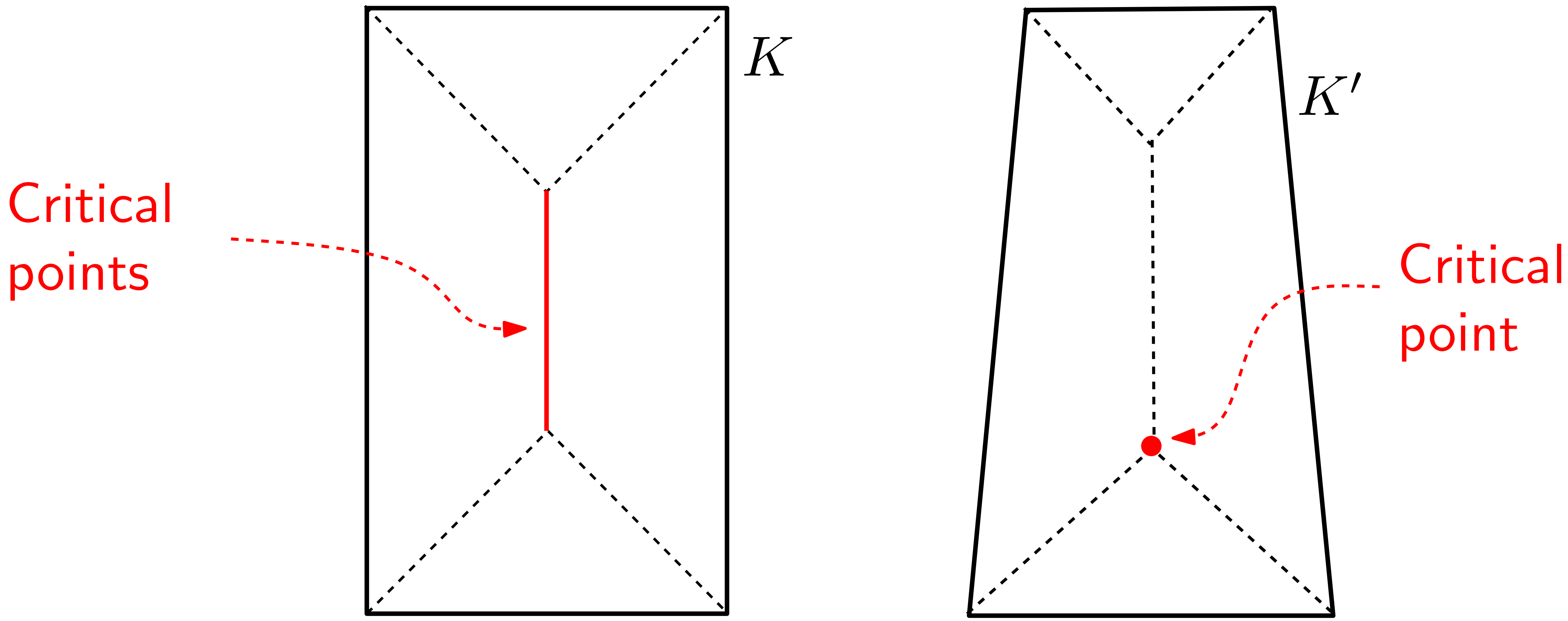
Option 2:
Try to get topological information about K without any assumption on $\mathsf{wfs}(K')$.

Persistence-based inference

Instability of critical points and μ-critical points

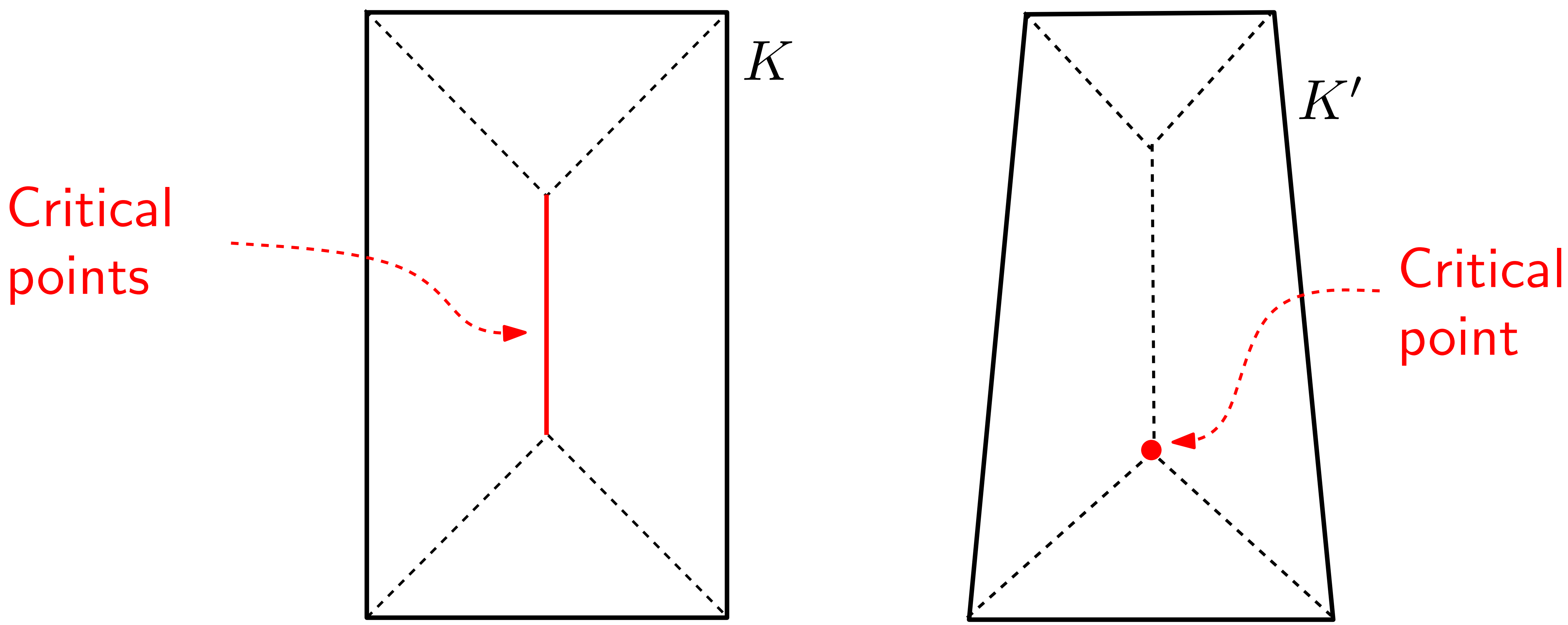


Instability of critical points and μ-critical points



A point $x \in \mathbb{R}^d$ is *μ-critical* for K if $\|\nabla d_K(x)\| \leq \mu$

Instability of critical points and μ-critical points



A point $x \in \mathbb{R}^d$ is *μ-critical* for K if $\|\nabla d_K(x)\| \leq \mu$

Theorem: [C-Cohen-Steiner-Lieutier'06] Let $K, K' \subset \mathbb{R}^d$ be two compact sets s. t. $d_H(K, K') \leq \varepsilon$. For any μ-critical point x for K, there exists a $(2\sqrt{\varepsilon/d_K(x)} + \mu)$-critical point for K' at distance at most $2\sqrt{\varepsilon d_K(x)}$ from x.

Proof of the stability theorem

Lemma 1: Let x be a μ-critical point for d_K. For any $y \in \mathbb{R}^d$,

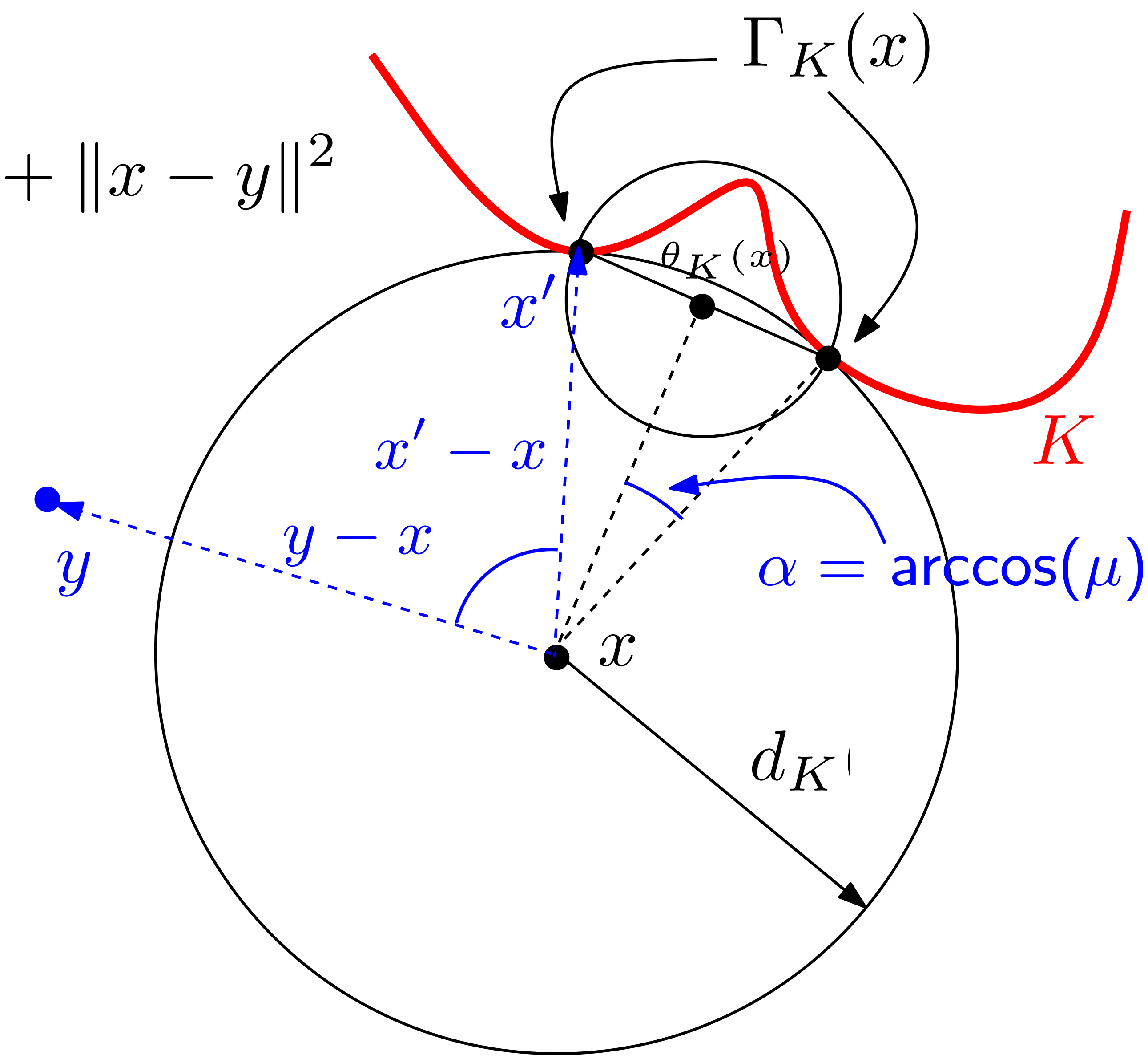
$$d_K(y)^2 \leq d_K(x)^2 + 2\mu d_K(x)\|x-y\| + \|x-y\|^2$$

Proof of the stability theorem

Lemma 1: Let x be a μ-critical point for d_K. For any $y \in \mathbb{R}^d$,

$$d_K(y)^2 \leq d_K(x)^2 + 2\mu d_K(x)\|x-y\| + \|x-y\|^2$$

Proof:



For any $x' \in \Gamma_K(x)$

$$\begin{aligned} d_K(y)^2 &\leq \|y-x'\|^2 \\ &= ((y-x)+(x-x'), (y-x)+(x-x')) \\ &= \|y-x\|^2 + \|x-x'\|^2 + 2(y-x, x-x') \\ &= d_K(x)^2 + 2d_K(x)\|x-y\|\cos(y-x, x-x') + \|x-y\|^2 \end{aligned}$$

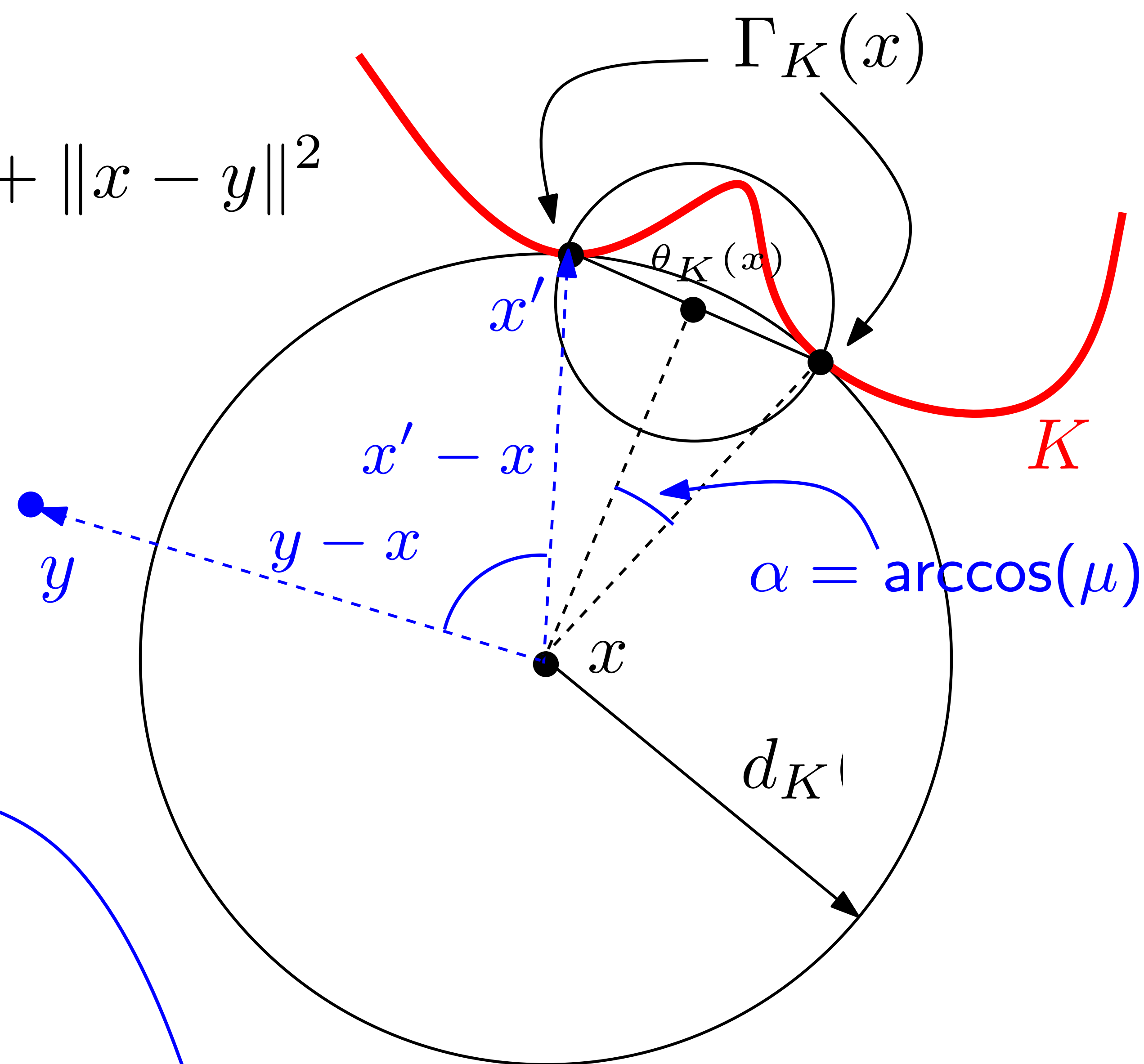
Proof of the stability theorem

Lemma 1: Let x be a μ-critical point for d_K. For any $y \in \mathbb{R}^d$,

$$d_K(y)^2 \leq d_K(x)^2 + 2\mu d_K(x)\|x-y\| + \|x-y\|^2$$

Proof:

Just need to prove that there exists $x' \in \Gamma_K(x)$ s.t. $\cos(y-x, x-x') \leq \mu$ i.e. $(y-x, x'-x) \leq \pi - \alpha$



For any $x' \in \Gamma_K(x)$

$$\begin{aligned} d_K(y)^2 &\leq \|y-x'\|^2 \\ &= ((y-x)+(x-x'), (y-x)+(x-x')) \\ &= \|y-x\|^2 + \|x-x'\|^2 + 2(y-x, x-x') \\ &= d_K(x)^2 + 2d_K(x)\|x-y\|\cos(y-x, x-x') + \|x-y\|^2 \end{aligned}$$

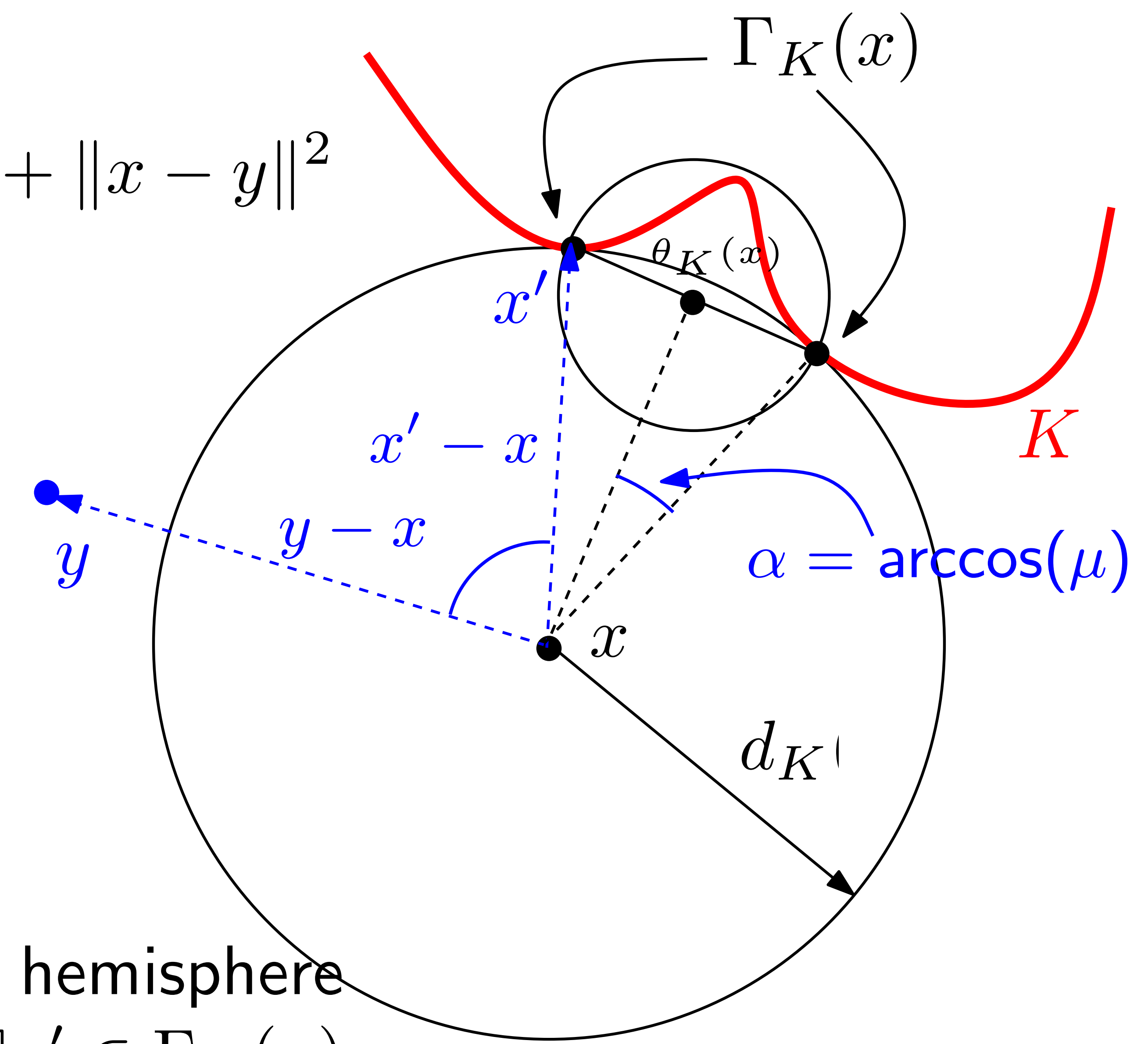
Proof of the stability theorem

Lemma 1: Let x be a μ-critical point for d_K. For any $y \in \mathbb{R}^d$,

$$d_K(y)^2 \leq d_K(x)^2 + 2\mu d_K(x)\|x-y\| + \|x-y\|^2$$

Proof:

Just need to prove that there exists $x' \in \Gamma_K(x)$ s.t. $\cos(y-x, x-x') \leq \mu$ i.e. $(y-x, x'-x) \leq \pi - \alpha$

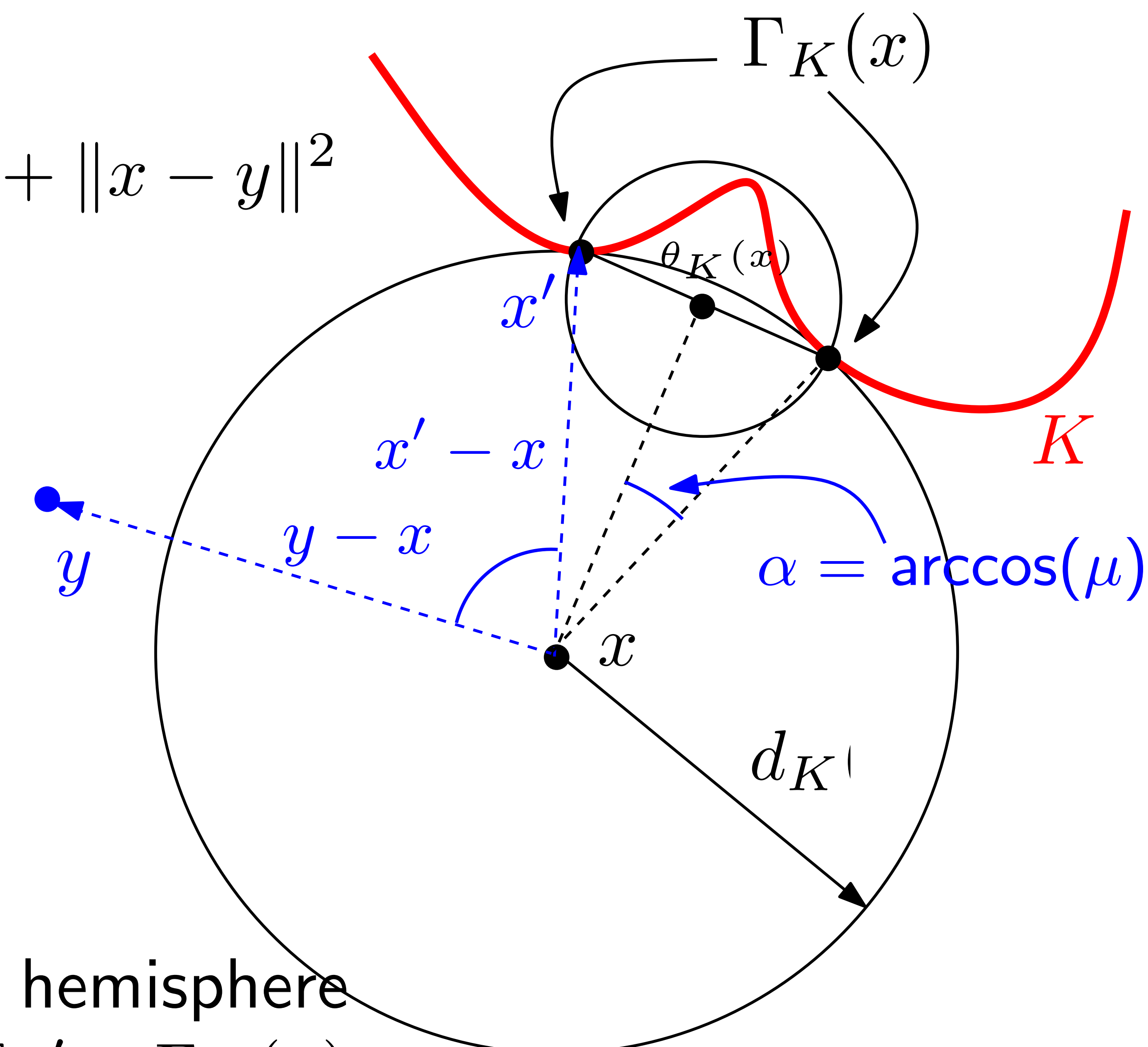


- If $\mu = 0$: $\theta_K(x) = x$ and any closed hemisphere of $S(x, d_K(x))$ intersects $\Gamma_K(x) \Rightarrow \exists x' \in \Gamma_K(x)$ s.t. $(y-x, x'-x) \leq \pi - \alpha$.

Proof of the stability theorem

Lemma 1: Let x be a μ-critical point for d_K. For any $y \in \mathbb{R}^d$,

$$d_K(y)^2 \leq d_K(x)^2 + 2\mu d_K(x)\|x-y\| + \|x-y\|^2$$



Proof:

Just need to prove that there exists $x' \in \Gamma_K(x)$ s.t. $\cos(y-x, x-x') \leq \mu$ i.e. $(y-x, x'-x) \leq \pi - \alpha$

- If $\mu = 0$: $\theta_K(x) = x$ and any closed hemisphere of $S(x, d_K(x))$ intersects $\Gamma_K(x) \Rightarrow \exists x' \in \Gamma_K(x)$ s.t. $(y-x, x'-x) \leq \pi - \alpha$.

- If $\mu \neq 0$: compactness of $\Gamma_K(x) \to$ there exists a circular cone with apex x and angle $\alpha' < \alpha$ containing $\Gamma_K(x)$.

Proof of the stability theorem

Lemma 2: Let $K, K' \subset \mathbb{R}^d$ be two compact sets s.t. $d_H(K, K') \leq \varepsilon$. For any μ-critical point x for K and for any $\rho > 0$, there exists a μ'-critical point for K' at distance at most ρ from x with

$$\mu' \leq \mu + \frac{\rho}{2d_K(x)} + 2\frac{\varepsilon}{\rho}$$

Proof of the stability theorem

Lemma 2: Let $K, K' \subset \mathbb{R}^d$ be two compact sets s.t. $d_H(K, K') \leq \varepsilon$. For any μ-critical point x for K and for any $\rho > 0$, there exists a μ'-critical point for K' at distance at most ρ from x with

$$\mu' \leq \mu + \frac{\rho}{2d_K(x)} + 2\frac{\varepsilon}{\rho}$$

Proof:

- Let $s \to \mathcal{C}(s)$ be the trajectory of $\nabla d_{K'}$ starting at x and parametrized by arc length.
- if $\mathcal{C}$ meets a critical point of K' before $y = \mathcal{C}(\rho)$: ok!

Proof of the stability theorem

Lemma 2: Let $K, K' \subset \mathbb{R}^d$ be two compact sets s.t. $d_H(K, K') \leq \varepsilon$. For any μ-critical point x for K and for any $\rho > 0$, there exists a μ'-critical point for K' at distance at most ρ from x with

$$\mu' \leq \mu + \frac{\rho}{2d_K(x)} + 2\frac{\varepsilon}{\rho}$$

Proof:

- Let $s \to \mathcal{C}(s)$ be the trajectory of $\nabla d_{K'}$ starting at x and parametrized by arc length.

- if $\mathcal{C}$ meets a critical point of K' before $y = \mathcal{C}(\rho)$: ok!
- Otherwise,

$$d_{K'}(y) - d_{K'}(x) = \int_0^\rho \|\nabla_{K'}(C(s))\| ds$$

and there exist a point p on $\mathcal{C}$ between $s = 0$ and $s = \rho$ such that:

$$\|\nabla d_{K'}(p)\| \leq \frac{d_{K'}(y) - d_{K'}(x)}{\rho}$$

Proof of the stability theorem

Lemma 2: Let $K, K' \subset \mathbb{R}^d$ be two compact sets s.t. $d_H(K, K') \leq \varepsilon$. For any μ-critical point x for K and for any $\rho > 0$, there exists a μ'-critical point for K' at distance at most ρ from x with

$$\mu' \leq \mu + \frac{\rho}{2d_K(x)} + 2\frac{\varepsilon}{\rho}$$

Proof:

- Applying Lemma 1 to x and $y = \mathcal{C}(\rho)$ gives

$$d_K(y) \leq \sqrt{d_K(x)^2 + 2\mu d_K(x)||x - y|| + ||x - y||^2}$$

- Since $\varepsilon = d_H(K, K')$, we have that for all $z \in \mathbb{R}^d$, $|d_K(z) - d_{K'}(z)| \leq \varepsilon$.

Proof of the stability theorem

Lemma 2: Let $K, K' \subset \mathbb{R}^d$ be two compact sets s.t. $d_H(K, K') \leq \varepsilon$. For any μ-critical point x for K and for any $\rho > 0$, there exists a μ'-critical point for K' at distance at most ρ from x with

$$\mu' \leq \mu + \frac{\rho}{2d_K(x)} + 2\frac{\varepsilon}{\rho}$$

Proof:

It follows that

$$\begin{aligned} d_{K'}(y) - d_{K'}(x) &\leq \sqrt{d_K(x)^2 + 2\mu d_K(x)||x-y|| + ||x-y||^2} \\ &\quad -d_K(x) + 2\varepsilon \\ &\leq d_K(x)[\sqrt{1 + \frac{2\mu||x-y||}{d_K(x)} + \frac{||x-y||^2}{d_K(x)^2}} - 1] \\ &\quad + 2\varepsilon \\ &\leq \mu||x-y|| + \frac{||x-y||^2}{2d_K(x)} + 2\varepsilon \end{aligned}$$

Proof of the stability theorem

Lemma 2: Let $K, K' \subset \mathbb{R}^d$ be two compact sets s.t. $d_H(K, K') \leq \varepsilon$. For any μ-critical point x for K and for any $\rho > 0$, there exists a μ'-critical point for K' at distance at most ρ from x with

$$\mu' \leq \mu + \frac{\rho}{2d_K(x)} + 2\frac{\varepsilon}{\rho}$$

Proof:

It follows that

$$\|\nabla_{K'}(p)\| \leq \frac{d_{K'}(y) - d_{K'}(x)}{\rho}$$

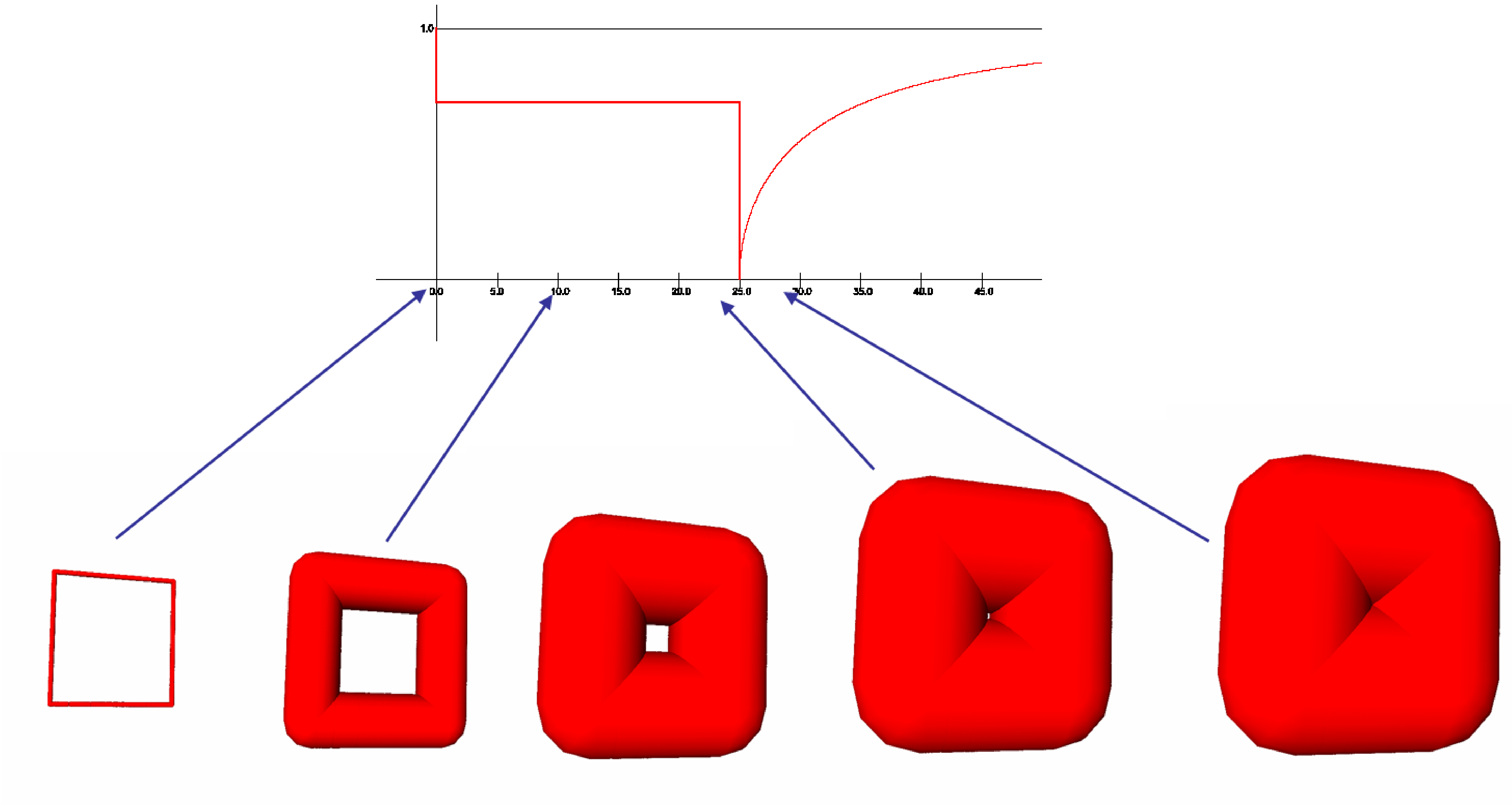
$$d_{K'}(y) - d_{K'}(x) \leq \sqrt{d_K(x)^2 + 2\mu d_K(x)||x-y|| + ||x-y||^2}$$
$$-d_K(x) + 2\varepsilon$$

$$\leq d_K(x)[\sqrt{1 + \frac{2\mu||x-y||}{d_K(x)} + \frac{||x-y||^2}{d_K(x)^2}} - 1]$$
$$+ 2\varepsilon$$

$$\leq \mu||x-y|| + \frac{||x-y||^2}{2d_K(x)} + 2\varepsilon$$

smaller than ρ since $y = \mathcal{C}(\rho)$ and $\mathcal{C}$ is parametrized by arc length.

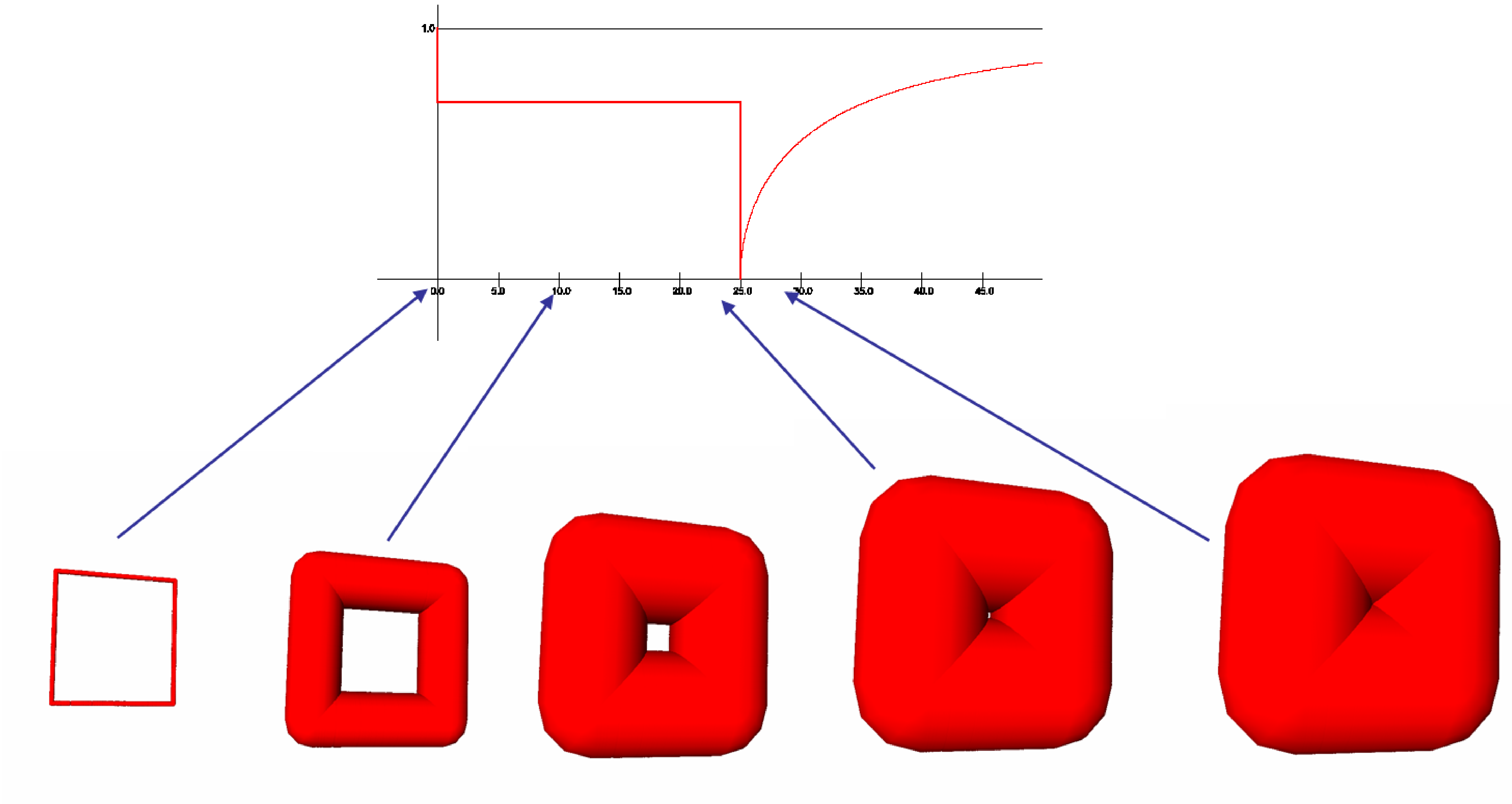
The critical function of a compact set



Definition: The critical function $\chi_K : (0, +\infty) \to \mathbb{R}_+$ of a compact set K is the function defined by

$$\chi_K(r) = \inf_{x \in d_K^{-1}(r)} \|\nabla_K(x)\|$$

The critical function of a compact set

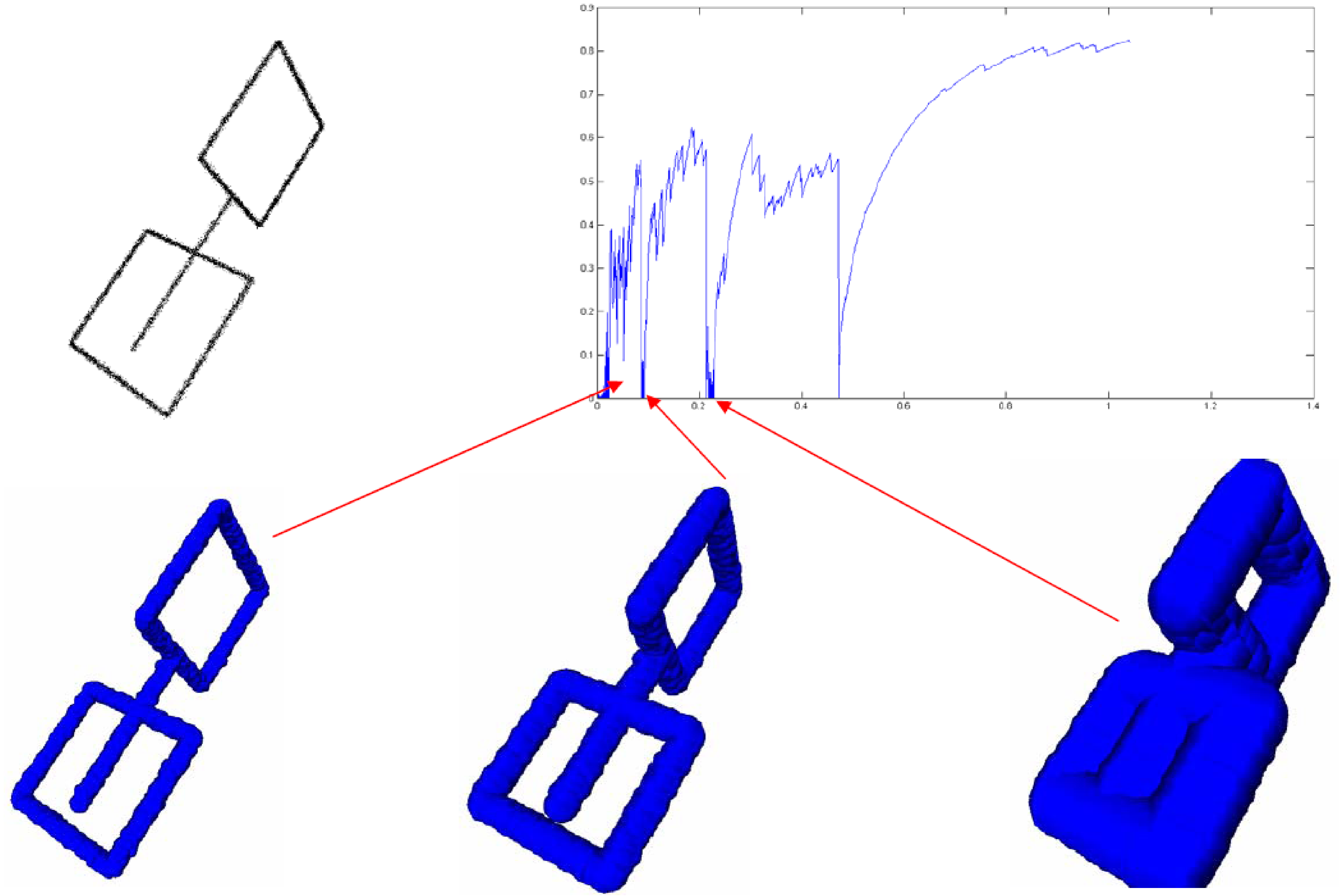


Definition: The critical function $\chi_K : (0, +\infty) \to \mathbb{R}_+$ of a compact set K is the function defined by

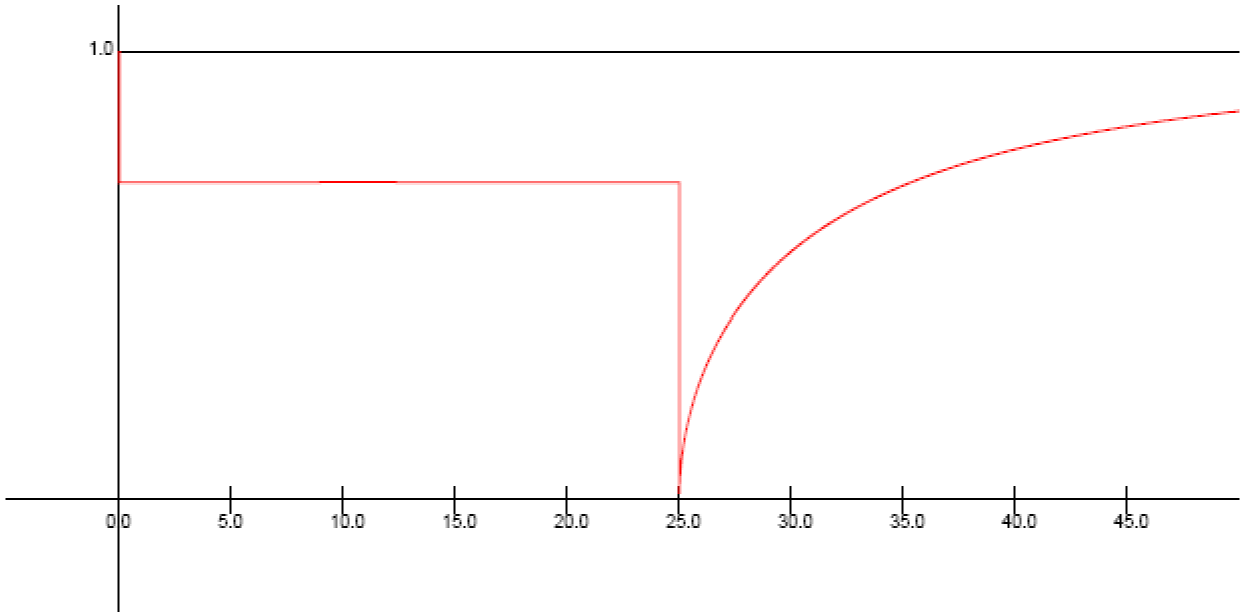
$$\chi_K(r) = \inf_{x \in d_K^{-1}(r)} \|\nabla_K(x)\|$$

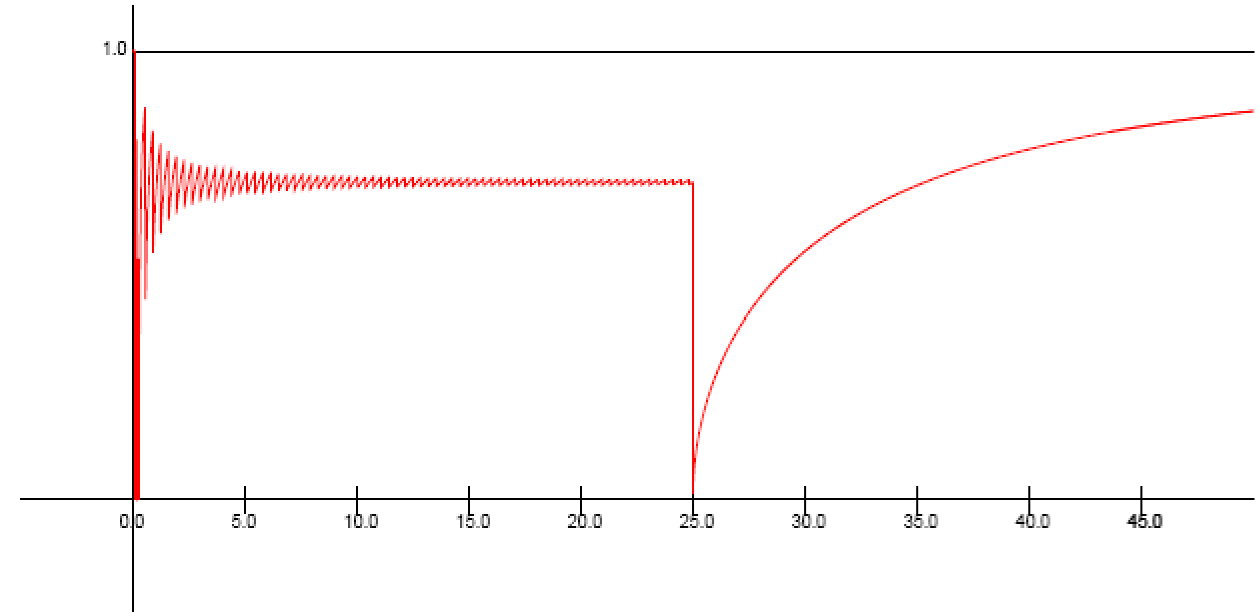
Easy to compute from Vor(K) when K is a finite point cloud!

The critical function of a compact set



Stability of the critical function





Theorem:[critical function stability theorem CCSL'06] Let K and K' be two compact subsets of $\mathbb{R}^d$ s. t. $d_H(K, K') \leq \varepsilon$. For all $r \geq 0$, we have:

$$\inf\{\chi_{K'}(u) \,|\, u \in I(r,\varepsilon)\} \leq \chi_K(r) + 2\sqrt{\frac{\varepsilon}{r}}$$

where $I(r,\varepsilon) = [r - \varepsilon, r + 2\chi_K(r)\sqrt{\varepsilon r} + 3\varepsilon]$

Stability of the critical function

Proof: this is an easy consequence of the critical point stability theorem.

Let $r \geq 0$ and $x \in d_K^{-1}(r)$ be such that $\|\nabla d_K(x)\| = \chi_K(r)$.

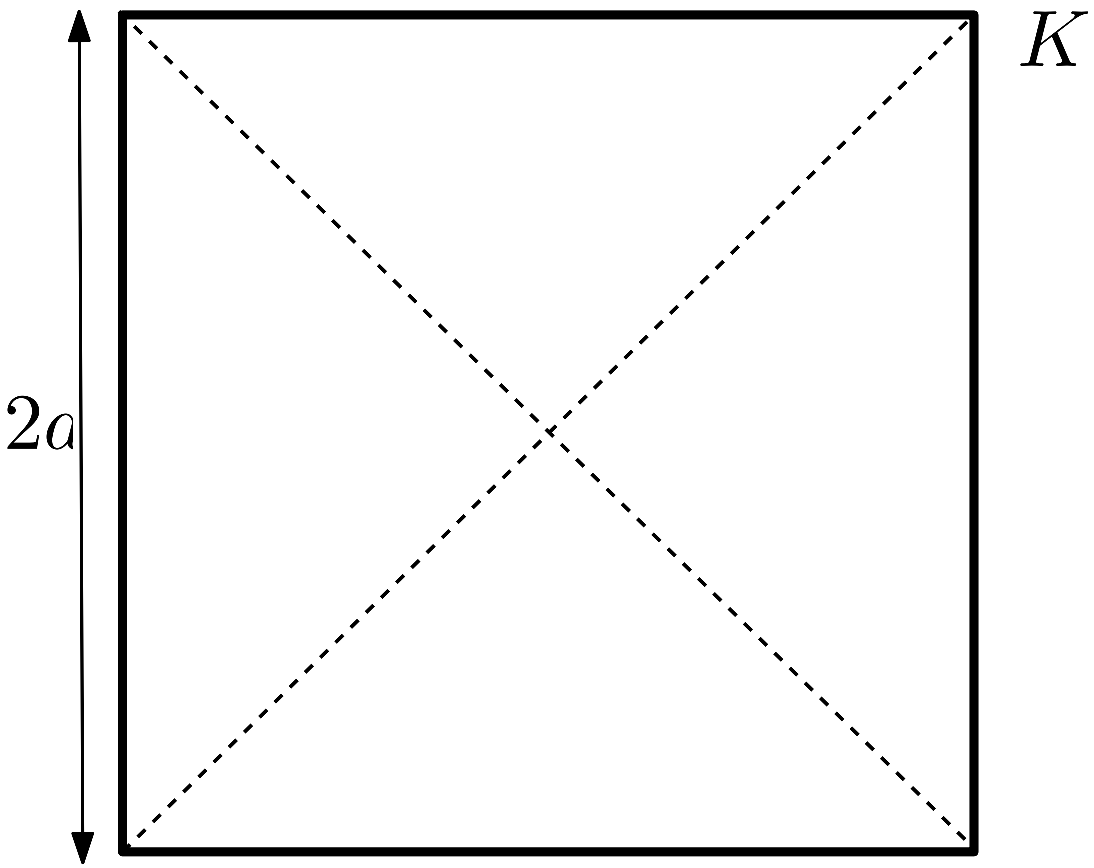
- Critical point stability theorem $\Rightarrow$ there exists a $(2\sqrt{\varepsilon/r} + \chi_K(r))$-critical point p for K' s.t. $d(p,x) \leq 2\sqrt{\varepsilon r}$ and $d_{K'}(p) \geq d_{K'}(x)$.

- Applying lemma 1 to x, p and K gives

$$\begin{aligned} d_K(p) &\leq \sqrt{r^2 + 4\chi_K(r) d\sqrt{\varepsilon r} + 4\varepsilon r} \\ &\leq r\sqrt{1 + 4\chi_K(r)\sqrt{\varepsilon/r} + 4\varepsilon/r} \\ &\leq r + 2\chi_K(r)\sqrt{\varepsilon r} + 2\varepsilon \end{aligned}$$

- to conclude the proof use that $d_{K'}(p) \geq d_{K'}(x)$ and $|d_{K'}(p) - d_K(p)| < \varepsilon$.

Reach(es)



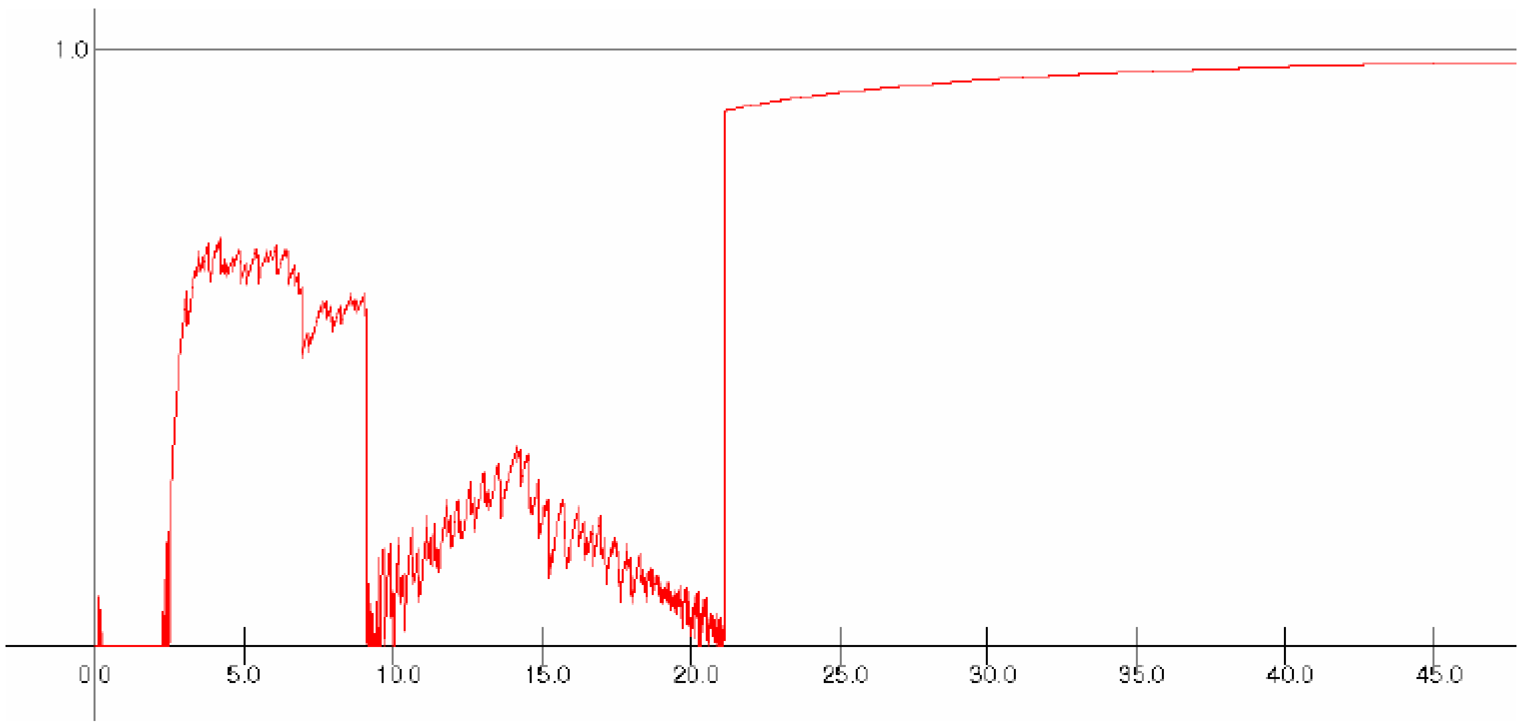
- $r_\mu(K) = 0$ if $\mu \geq \sqrt{2}/2$
- $r_\mu(K) = a$ if $\mu < \sqrt{2}/2$
- $\mathrm{wfs}(K) = a$

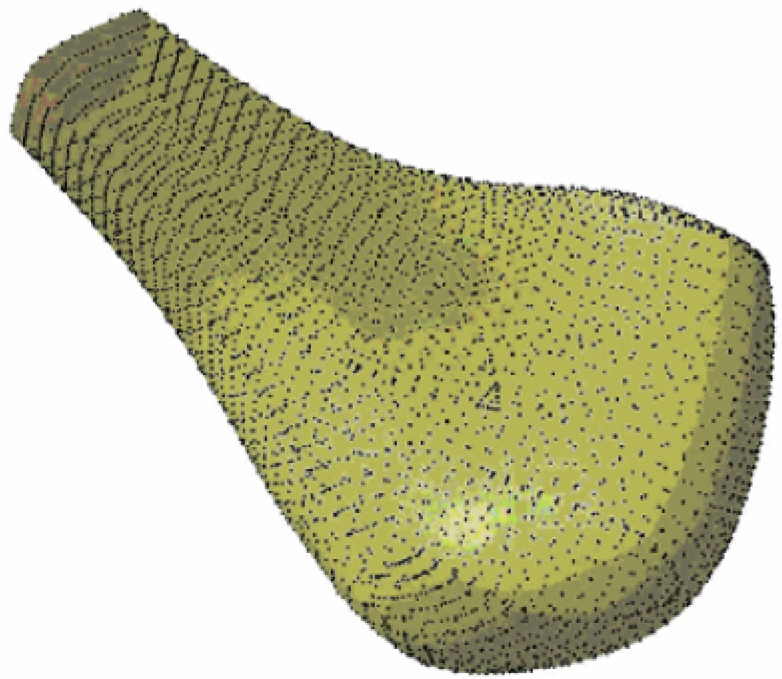
μ-*reach* of a compact $K \subset \mathbb{R}^d$:

$$r_\mu(K) = \inf\{d_K(x) : \|\nabla d_K(x)\| < \mu\}$$

- $\forall \mu \in (0, 1)$, $r_\mu(K) \leq$ wfs(K)
- for $\mu = 1$, $r_\mu(K)$ is the reach introduced by Federer in Geometric Measure Theory

Separation of critical values

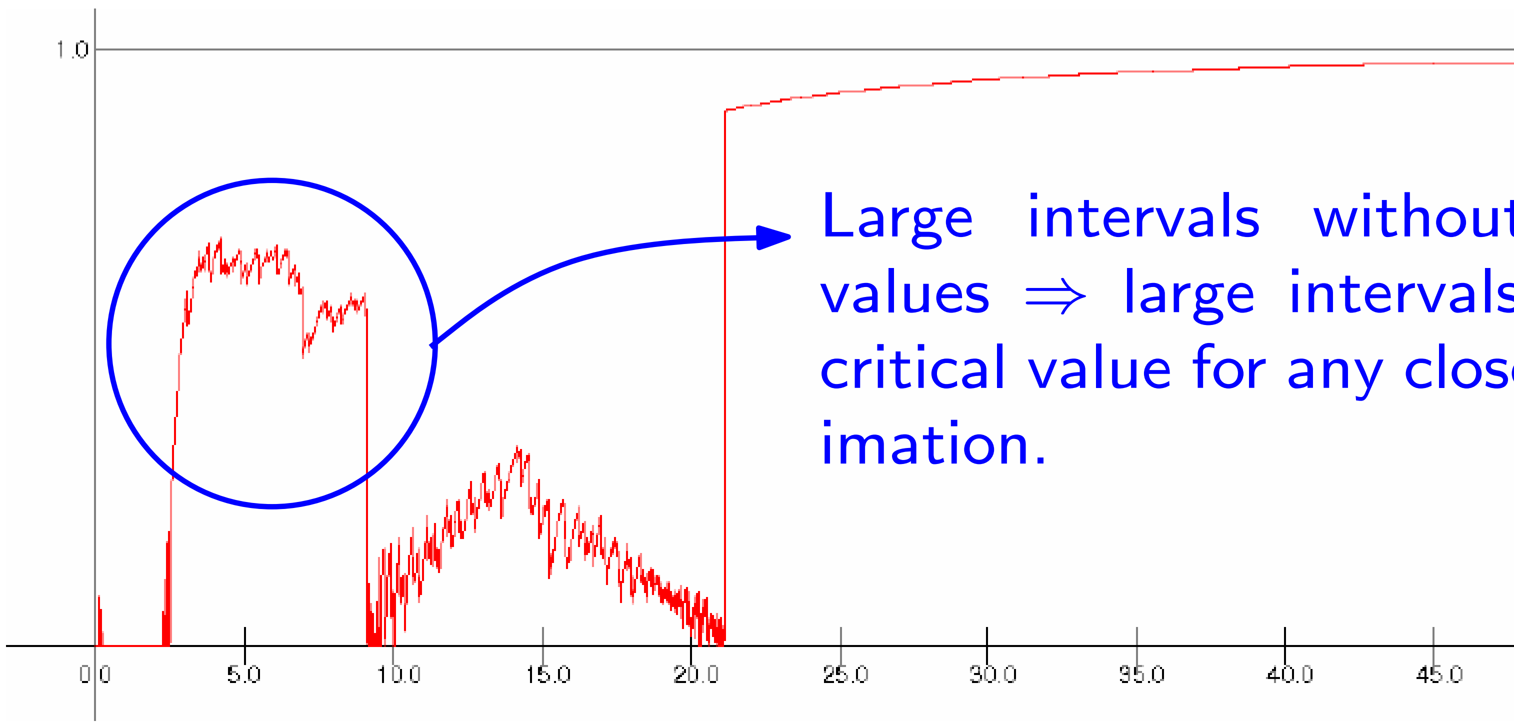




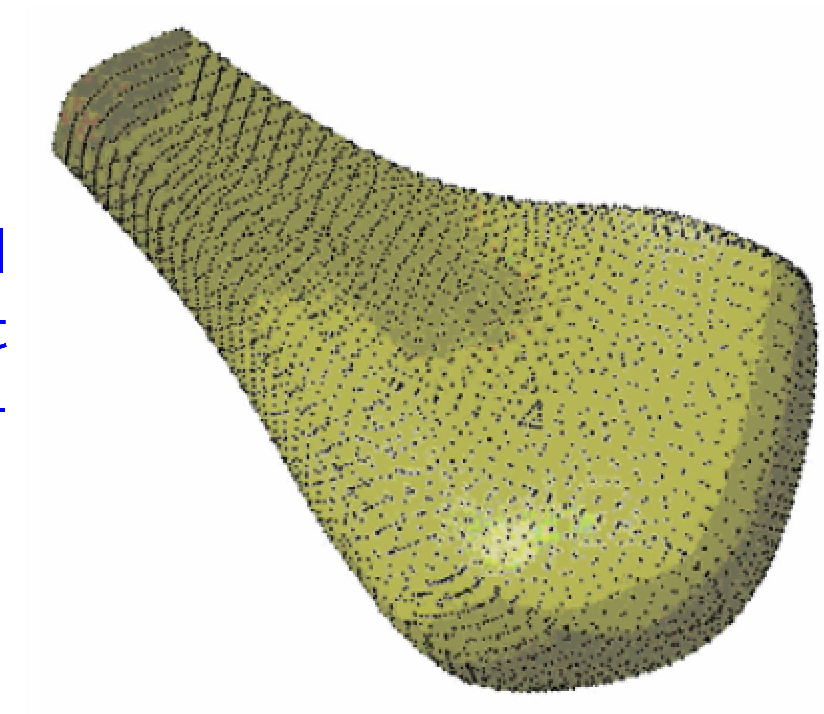
Theorem: [CCSL'06] Let K and K' be two compact subsets of $\mathbb{R}^d$, ε be the Hausdorff distance between K and K', and μ be a non-negative number. The distance function d_K has no critical values in the interval $]4\varepsilon/\mu^2\,,\, r_\mu(K') - 3\varepsilon[$. Besides, for any $\mu' < \mu$, χ_K is larger than μ' on the interval

$$]\frac{4\varepsilon}{(\mu-\mu')^2}\,,\, r_\mu(K') - 3\sqrt{\varepsilon r_\mu(K')}[$$

Separation of critical values



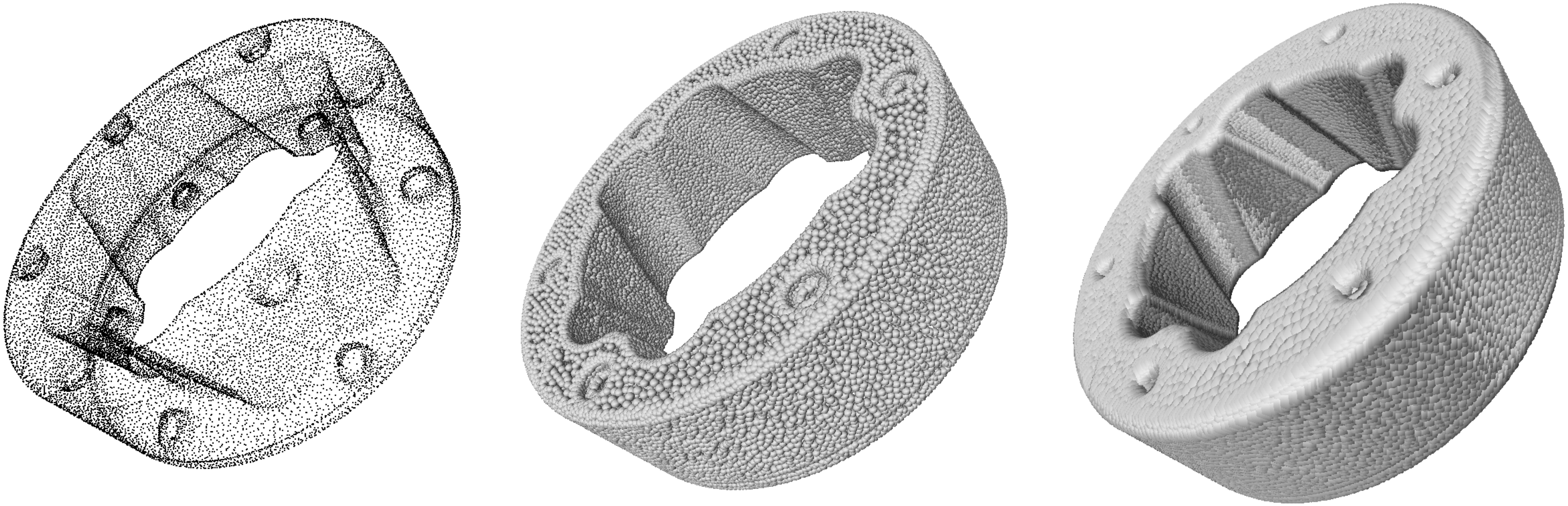
Large intervals without critical values $\Rightarrow$ large intervals without critical value for any close approximation.



Theorem: [CCSL'06] Let K and K' be two compact subsets of $\mathbb{R}^d$, ε be the Hausdorff distance between K and K', and μ be a non-negative number. The distance function d_K has no critical values in the interval $]4\varepsilon/\mu^2\,,\, r_\mu(K')-3\varepsilon[$. Besides, for any $\mu' < \mu$, χ_K is larger than μ' on the interval

$$]\frac{4\varepsilon}{(\mu-\mu')^2}\,,\, r_\mu(K') - 3\sqrt{\varepsilon r_\mu(K')}[$$

A reconstruction theorem

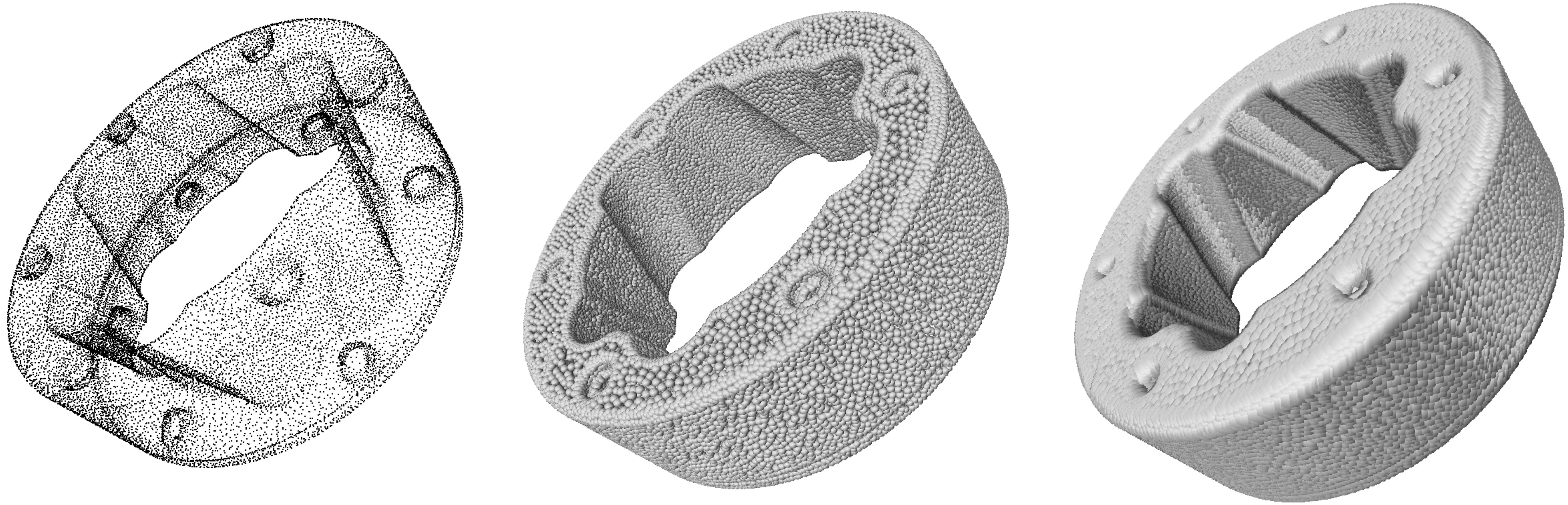


A reconstruction theorem: [C-Cohen-Steiner-Lieutier'06]
Let $K \subset \mathbb{R}^d$ be a compact set s.t. $r_\mu = r_\mu(K) > 0$ for some $\mu > 0$. Let $K \subset \mathbb{R}^d$ be such that $d_H(K, K') < \kappa r_\mu(K)$ with $\kappa < \min(\frac{\sqrt{5}}{2} - 1, \frac{\mu^2}{16+2\mu^2})$
Then for any d, d' s.t.

$$0 < d < \mathrm{wfs}(K) \;\; \text{and} \;\; \frac{4\kappa r_\mu}{\mu^2} \leq d' < r_\mu - 3\kappa r_\mu$$

the hypersurfaces $d_{K'}^{-1}(d')$ and $d_K^{-1}(d)$ are isotopic.

A reconstruction theorem



Reconstruction theorem: (Weak version)
Let $K \subset \mathbb{R}^d$ be a compact set s.t. $r_\mu = r_\mu(K) > 0$ for some $\mu > 0$. Let $K \subset \mathbb{R}^d$ be such that $d_H(K, K') = \varepsilon < \kappa r_\mu(K)$ with $\kappa < \frac{\mu^2}{5\mu^2+12}$ Then for any d, d' s.t.

$$0 < d < \text{wfs}(K) \ \text{ and } \ \frac{4\kappa r_\mu}{\mu^2} \leq d' < r_\mu - 3\kappa r_\mu$$

the offsets ${K'}^{d'}$ and K^d are homotopy equivalent.

Proof of the reconstruction theorem

- Separation of critical values: $d_{K'}$ does not have any critical value in $(\frac{4\varepsilon}{\mu^2}, r_\mu(K) - 3\varepsilon)$

  $\Rightarrow$ it is enough to prove the theorem for $d' = 4\varepsilon/\mu^2$.

- We have $\mathrm{wfs}(K'^{d'}) \geq r_\mu(K) - 3\varepsilon - 4\varepsilon/\mu^2$ and

$$d_H(K, K'^{d'}) \leq \frac{4\varepsilon}{\mu^2} + \varepsilon$$

- The conclusion of the theorem holds as soon as

$$d_H(K, K'^{d'}) < \frac{1}{2}\min(\mathrm{wfs}(K'^{d'}), \mathrm{wfs}(K))$$

Proof of the reconstruction theorem

- Separation of critical values: $d_{K'}$ does not have any critical value in $(\frac{4\varepsilon}{\mu^2}, r_\mu(K) - 3\varepsilon)$

  $\Rightarrow$ it is enough to prove the theorem for $d' = 4\varepsilon/\mu^2$.

- We have wfs$(K'^{d'}) \geq r_\mu(K) - 3\varepsilon - 4\varepsilon/\mu^2$ and

$$d_H(K, K'^{d'}) \leq \frac{4\varepsilon}{\mu^2} + \varepsilon$$

- The conclusion of the theorem holds as soon as

$$d_H(K, K'^{d'}) < \frac{1}{2} \min(\mathrm{wfs}(K'^{d'}), \mathrm{wfs}(K))$$

This is exactly the assumption made on κ!

Proof of the reconstruction theorem

- Separation of critical values: $d_{K'}$ does not have any critical value in $(\frac{4\varepsilon}{\mu^2}, r_\mu(K) - 3\varepsilon)$

  $\Rightarrow$ it is enough to prove the theorem for $d' = 4\varepsilon/\mu^2$.

- We have $\mathrm{wfs}(K'^{d'}) \geq r_\mu(K) - 3\varepsilon - 4\varepsilon/\mu^2$ and

$$d_H(K, K'^{d'}) \leq \frac{4\varepsilon}{\mu^2} + \varepsilon$$

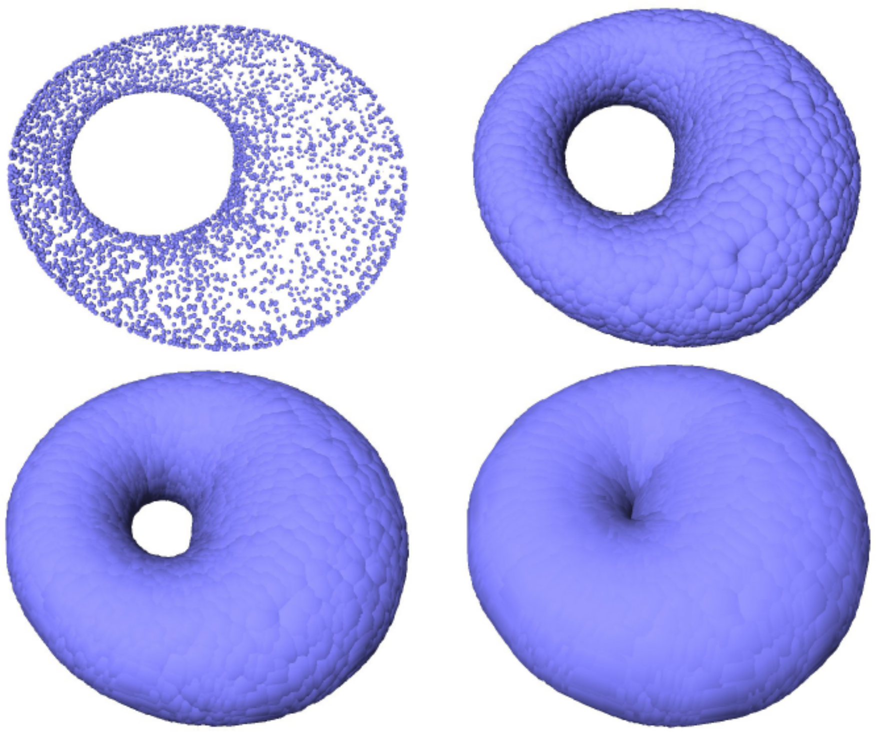
- The conclusion of the theorem holds as soon as

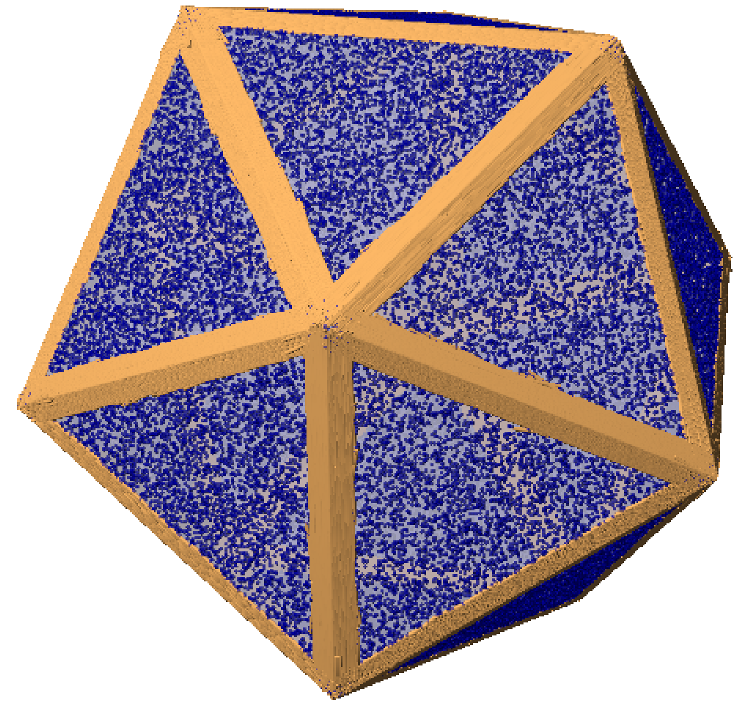
$$d_H(K, K'^{d'}) < \frac{1}{2}\min(\mathrm{wfs}(K'^{d'}), \mathrm{wfs}(K))$$

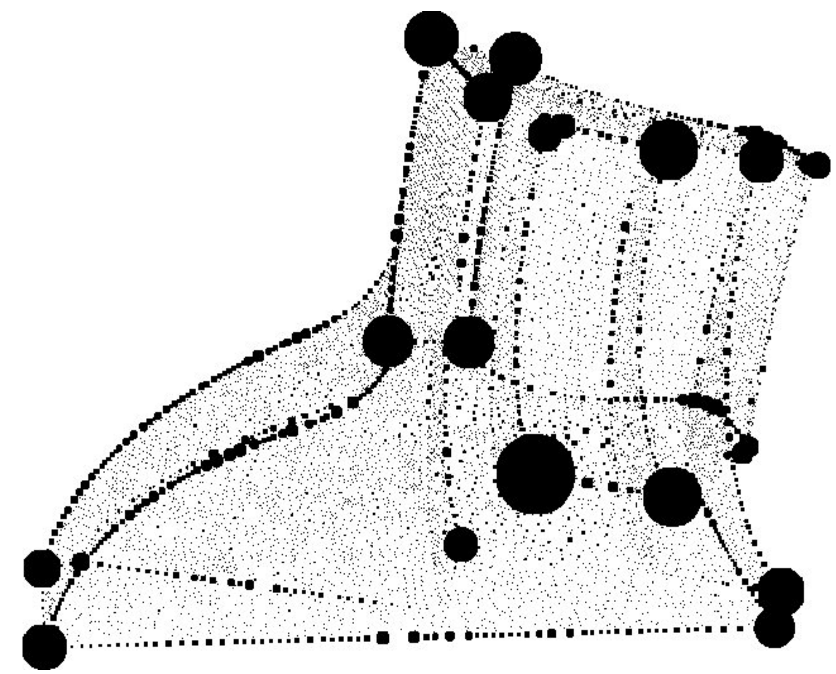
This is exactly the assumption made on κ!

Remark: to get the isotopy one needs stability results for ∇d_K...

Distance-based geometric inference







Topological/geometric properties of the offsets of K are stable with respect to Hausdorff approximation:

1. Topological stability of the offsets of K (CCSL'06, NSW'06).

2. Approximate normal cones (CCSL'08).

3. Boundary measures (CCSM'07), curvature measures (CCSLT'09), Voronoi covariance measures (GMO'09).

Take-home messages

- Distance functions provide a powerful framework for robust geometric inference with theoretical guarantees:
  - for a wide class of (non smooth) shapes
  - in any dimension.
- In practice (for point clouds) the algorithms rely on the Voronoï diagram or the Delaunay triangulation $\Rightarrow$ ok in 2D and 3D!
- But no efficient reconstruction algorithm in higher dimension...