Witness Complex

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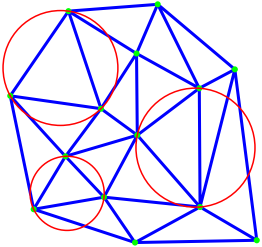
Delaunay triangulations

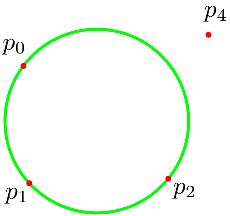
Finite set of points $P \in \mathbb{R}^d$



- $\sigma \in DT(P) \quad \Leftrightarrow \quad \exists c_\sigma : \quad \|c_\sigma - p\| \leq \|c_\sigma - q\| \quad \forall p \in \sigma \text{ and } \forall q \in P$
- It is embedded in $\mathbb{T}^d$ if P is generic wrt spheres [Delaunay 1934]
  no $d+2$ points on a same hypersphere

Motivations

- Define Delaunay-like complexes in any metric space
- Walk around the curse of dimensionality
  - The combinatorial complexity of DT depends exponentially on the ambient dimension d
  - The algebraic complexity depends on d



$$\texttt{insphere}(p_0,\ldots,p_{d+1}) = \texttt{orient}(\hat{p}_0,\ldots,\hat{p}_{d+1}) = \text{sign}\begin{vmatrix} 1 & \ldots & 1 \\ p_0 & \ldots & p_{d+1} \\ p_0^2 & \ldots & p_{d+1}^2 \end{vmatrix}$$

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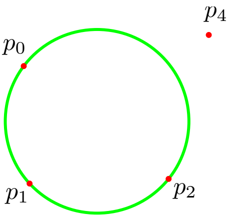

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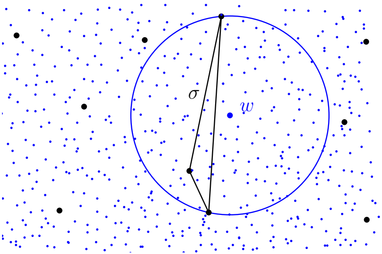
Witness Complex

[de Silva]

L a finite set of points (landmarks) — vertices of the complex

W a dense sample (witnesses) — pseudo circumcenters



Let σ be a (abstract) simplex with vertices in L, and let $w \in W$. We say that w is a witness of σ if

$$\|w - p\| \leq \|w - q\| \qquad \forall p \in \sigma \text{ and } \forall q \in L \setminus \sigma$$

The witness complex $\mathrm{Wit}(L, W)$ is the complex consisting of all simplexes σ such that for any simplex $\tau \subseteq \sigma$, τ has a witness in W

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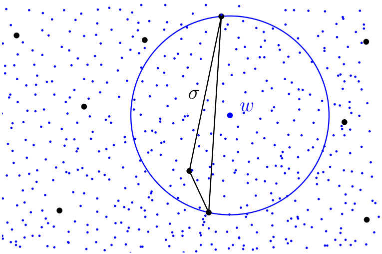

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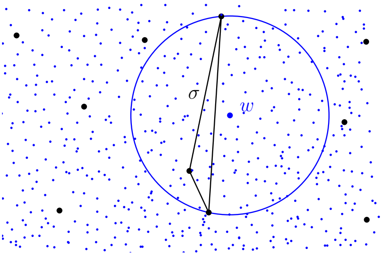

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Easy consequences of the definition

- The witness complex can be defined for any metric space and, in particular, for discrete metric spaces

- If $W' \subseteq W$, then $\text{Wit}(L, W') \subseteq \text{Wit}(L, W)$

- $\text{Del}(L) \subseteq \text{Wit}(L, \mathbb{R}^d)$

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Construction of the k-skeleton of $\mathrm{Wit}(L, W)$

Input: L, W, and $\forall w \in W$ the list $N(w) = (p_0(w), ..., p_k(w))$ of the k nearest landmarks of w, sorted by distance from w

$\mathrm{Wit}(L, W) := \emptyset$
$W' := W$

for $i = 0, ..., k$ **do**

  for each $w \in W'$ **do**

    if the i-simplex $\sigma(w) = (p_0(w), p_2(w), ..., p_i(w)) \notin \mathrm{Wit}(L, W)$
    then
      if the $(i-1)$-faces of $\sigma(w)$ are all in $\mathrm{Wit}(L, W)$ **then**
        add $\sigma(w)$ to $\mathrm{Wit}(L, W)$
      else
        $W' := W' \setminus \{w\}$

Output: the witness complex $\mathrm{Wit}(L, W)$

Construction of the k-skeleton of the witness complexes

Constructing the lists $N(w)$: $O(|W| \log |W| + k\,|W|)$ [Callahan & Kosaraju 1995]

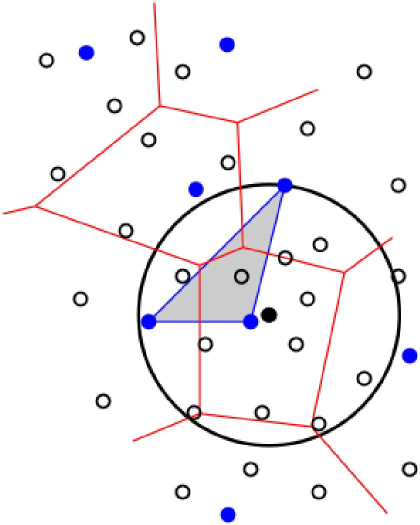
Construction of $\mathrm{Wit}(L, W)$: $O\left((|\mathrm{Wit}^k(L, W)| + |W|)\, k^2 \, \log |L|\right)$ (using ST)

$$|\mathrm{Wit}^k(L, W) \leq k\,|W|$$

Algebraic complexity : comparisons of (squared) distances : degree 2

Implementation and experimental results : see the Gudhi library !

Delaunay and Witness complexes

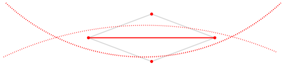


Identity of $\mathrm{Del}_{|\Omega}(L)$ and $\mathrm{Wit}(L, \mathbb{R}^d)$ [de Silva 2008]

Theorem : $\mathrm{Wit}(L, W) \subseteq \mathrm{Wit}(L, \mathbb{R}^d) = \mathrm{Del}(L)$

Remarks

- Faces of all dimensions have to be witnessed



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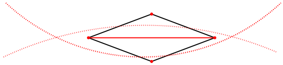

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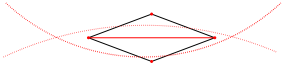

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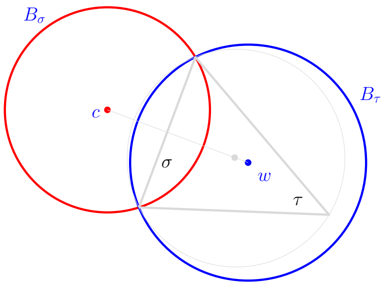
Proof of de Silva's theorem

$\tau = [p_0, ..., p_k]$ is a k-simplex of $\mathrm{Wit}(L)$ witnessed by a ball B_τ (i.e. $B_\tau \cap L = \tau$)

We prove that $\tau \in \mathrm{Del}(L)$ by induction on k

Clearly true for $k = 0$



Hyp. : true for $k' \le k - 1$

$B := B_\tau$

$\sigma := \partial B \cap \tau, \quad l := |\sigma|$

// $\sigma \in \mathrm{Del}(L)$ by the hyp.

while $l + 1 = \dim \sigma < k$ do

$B \leftarrow$ the ball centered on $[cw]$ s.t.

- $\sigma \subset \partial B$,
- B witnesses τ
- $|\partial B \cap \tau| = l + 1$

$(B \text{ witnesses } \tau) \wedge (\tau \subset \partial B) \Rightarrow \tau \in \mathrm{Del}(L)$

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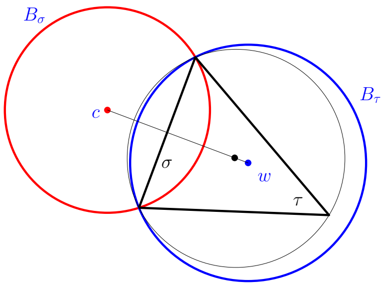

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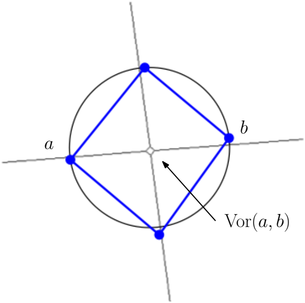
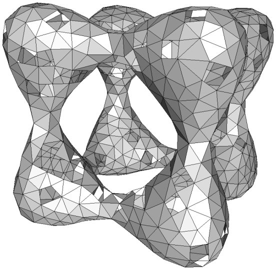
Case of sampled domains : $\mathrm{Wit}(L, W) \neq \mathrm{Del}(L)$

W a finite set of points $\subset \Omega \subset \mathbb{R}^d$

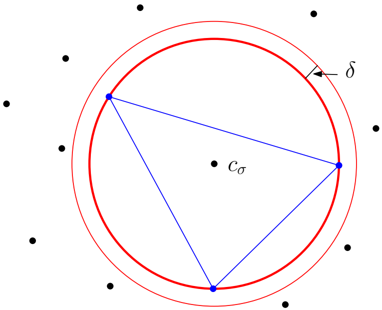
$\mathrm{Wit}(L, W) \neq \mathrm{Del}(L)$, even if W is a dense sample of Ω





$$[ab] \in \mathrm{Wit}(L, W) \quad \Leftrightarrow \quad \exists p \in W, \quad \mathrm{Vor}_2(a, b) \cap W \neq \emptyset$$

Protection



δ-protection We say that a Delaunay simplex $\sigma \subset L$ is δ-protected if

$$\|c_\sigma - q\| > \|c_\sigma - p\| + \delta \qquad \forall p \in \sigma \text{ and } \forall q \in L \setminus \sigma.$$

Protection implies $\mathrm{Wit}(L, W) = \mathrm{Del}(L)$

Lemma Let Ω be a convex subset of $\mathbb{R}^d$ and let W and L be two finite sets of points in Ω. If W is ε-dense in Ω and if all simplices (of all dimensions) of $\mathrm{Del}_{|\Omega}(L)$ are δ-protected with $\delta \geq 2\varepsilon$, then $\mathrm{Wit}(L, W)) = \mathrm{Del}_{|\Omega}(L)$

Proof It suffices to prove $\mathrm{Wit}(L, W) \supseteq \mathrm{Del}_{|\Omega}(L)$

$\forall \sigma \in \mathrm{Del}_{|\Omega}(L)$, $\exists c \in \Omega$ such that

$$\forall p \in \sigma,\ \forall q \in L \setminus \sigma, \qquad \|c-p\| \leq \|c-q\| - \delta$$

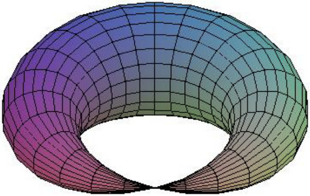
W ε-dense in $\Omega \quad \Rightarrow \quad \exists w \in W$ such that $\|w-c\| \leq \varepsilon$

$$\begin{aligned}
\forall p \in \sigma,\ \forall q \in L \setminus \sigma \quad \|w-p\| &\leq \|w-c\| + (\|c-q\| - \delta) \\
&\leq \|w-q\| + 2\|w-c\| - \delta \\
&\leq \|w-q\| + 2\varepsilon - \delta
\end{aligned}$$

Good links

A simplicial complex K is a k-pseudomanifold complex without boundary if

1. K is a pure k-complex
2. every $(k-1)$-simplex is the face of exactly two k-simplices



We say that a complex $K \subset \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with vertex set L has good links if

$$\forall p \in L, \text{ link } (p, K) \text{ is a } (d-1)\text{-pseudomanifold}$$

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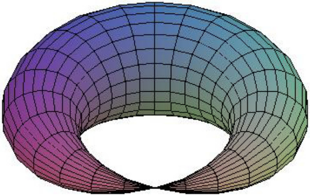

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Good links implies $\mathrm{Wit}(L, W) = \mathrm{Del}(L)$

Lemma

If K is a triangulation of $\mathbb{T}^d$ and $K' \subseteq K$ a simplicial complex with the same vertex set

then $\quad K' = K \quad \Leftrightarrow \quad K'$ has good links

Corollary

If all vertices of $\mathrm{Wit}(L, W)$ have good links, $\mathrm{Wit}(L, W) = \mathrm{Del}(L)$

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Turning witness complexes into Delaunay complexes

Input: L, W, ρ (perturbation radius)

Init : $L' := L$; compute $\mathrm{Wit}(L', W)$

while a vertex p' of $\mathrm{Wit}(L', W)$ has a bad link **do**

perturb p' and the points of $I(p')$

update $\mathrm{Wit}(L', W)$

Output: $\mathrm{Wit}(L', W) = \mathrm{Del}(L')$

Analysis : adapt Moser-Tardos algorithmic version of the Lovász Local Lemma (last lecture)

Main result

Under the condition

$$\frac{\mu}{4} \geq \rho \geq \frac{24d\varepsilon}{\bar{\mu}J} \quad \text{where} \;\; J^{-1} = \left(\frac{2}{\bar{\mu}}\right)^{O(d^2)}$$

the algorithm terminates and returns $\mathrm{Wit}(L', W) = \mathrm{Del}(L')$

The expected complexity is linear in $|L|$

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The expected complexity is linear in $|L|$

Relaxed witness complex

Alpha-witness Let σ be a simplex with vertices in L. We say that a point $w \in W$ is an α-witness of σ if

$$\|w - p\| \leq \|w - q\| + \alpha \quad \forall p \in \sigma \quad \text{and} \quad \forall q \in L \setminus \sigma$$

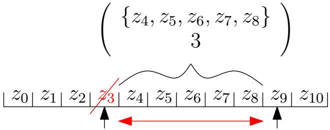
Alpha-relaxed witness complex The α-relaxed witness complex $\mathrm{Wit}^\alpha(L, W)$ is the maximal simplicial complex with vertex set L whose simplices have an α-witness in W

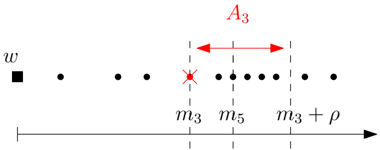
$\mathrm{Wit}^0(L, W) = \mathrm{Wit}(L, W)$

Filtration : $\alpha \leq \beta \quad \Rightarrow \quad \mathrm{Wit}^\alpha(L, W) \subseteq \mathrm{Wit}^\beta(L, W)$

Construction

$$A_i = \{z \in L : \quad m_i \leq d(w,z) \leq m_i + \alpha\}$$





For $i \leq j+1$, w α-witnesses all the j-simplices that contain $\{z_0, \cdots, z_{i-1}\}$ and a $(j+1-i)$-subset of A_i (provided $|A_i| \geq j+1-i$)

All j-simplices that are α-witnessed by w are obtained this way, and exactly once, when i ranges from 0 to $j+1$

Lemma Assume that W is ε-dense in Ω and let $\alpha \geq 2\varepsilon$. Then

$$\mathrm{Wit}(L, W) \subseteq \mathrm{Del}_{|\Omega}(L) \subseteq \mathrm{Wit}^{\alpha}(L, W)$$

Proof

σ : a d-simplex of $\mathrm{Del}(L)$, c_σ its circumcenter

W ε-dense in Ω $\exists w \in W$ s.t. $\|c_\sigma - w\| \leq \varepsilon$

For any $p \in \sigma$ and $q \in L \setminus \sigma$, we then have

$$\begin{aligned}
\forall p \in \sigma \ \text{ and } \ q \in L \setminus \sigma \qquad \|w - p\| &\leq \|c_\sigma - p\| + \|c_\sigma - w\| \\
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