Topological Data Analysis — Exam

April 07, 2016

1 Bottleneck distance

Question 1. Show that the bottleneck distance satisfies the triangle inequality.

Question 2. Show that the bottleneck distance is only a pseudodistance, that is: exhibit a pair of distinct persistence diagrams whose distance is zero.

2 Euler characteristic

Given a topological space X and a field $\mathbf{k}$, the *Euler characteristic* is the quantity:

$$\chi(X;\mathbf{k}) = \sum_{i=0}^{+\infty} (-1)^i \dim \mathrm{H}_i(X;\mathbf{k}).$$

Question 3. Show that χ is a topological invariant, that is: for any spaces X, Y that are homotopy equivalent, $\chi(X;\mathbf{k}) = \chi(Y;\mathbf{k})$.
Hint: look at what happens to each individual homology group.

Now we want to prove the Euler-Poincaré theorem:

Theorem 1. *For any simplicial complex X and any field $\mathbf{k}$:*

$$\chi(X;\mathbf{k}) = \sum_{i=0}^{+\infty} (-1)^i n_i(X),$$

where $n_i(X)$ denotes the number of simplices of X of dimension i.

For this we will use topological persistence. Consider an arbitrary filtration of X:

$$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_m = X.$$

Assume without loss of generality that a single simplex σ_j is inserted at each step j:

$$\forall j = 1, \cdots, m,\ X_j \setminus X_{j-1} = \{\sigma_j\}.$$

Note that m is then equal to the number of simplices of X, that is:

$$m = \sum_{i=0}^{+\infty} n_i(X).$$

Let us apply the persistence algorithm to this simplicial filtration. Recall that we have the following property:

Lemma 1. *At each step j, the insertion of simplex σ_j either creates an independent d_j-dimensional cycle (i.e. increases the dimension of $\mathrm{H}_{d_j}(X_{j-1};\mathbf{k})$ by 1) or kills a (d_j-1)-dimensional cycle (i.e. decreases the dimension of $\mathrm{H}_{d_j-1}(X_{j-1};\mathbf{k})$ by 1), where d_j is the dimension of σ_j.*

Question 4. Using Lemma 1, prove Theorem 1.
Hint: proceed by induction on j.

Question 5. Deduce that the Euler characteristic of a triangulable space is independent of the choice of field $\mathbf{k}$.

3 Reeb graph and Mapper

Consider the function f depicted on the left-hand side of Figure 1. Note that its domain X (the gray area) is a subset of the plane, not a torus.

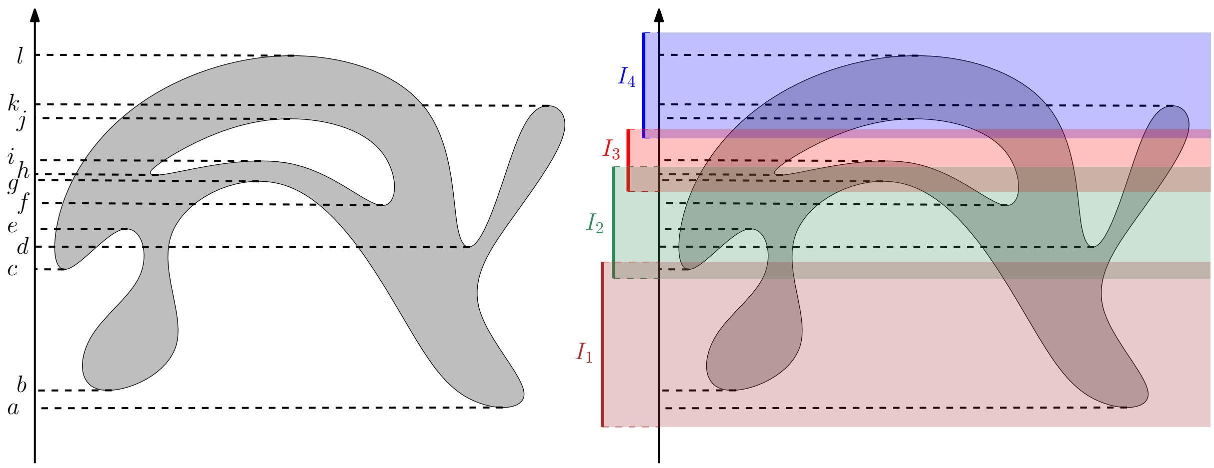

Figure 1: Left: the height function f on a planar domain. Right: the interval cover.

Question 6. Compute the Reeb graph of f. You may simply draw it with labels.

Question 7. Compute the extended persistence diagram of the map f.
Hints: homology of dimension 2 and above is trivial; the values $a, \cdots, l$ are paired up to form the diagram points.

Question 8. Prune the diagram to obtain the extended persistence diagram of the quotient map $\tilde{f}$. What do you observe? Explain.

Question 9. Consider now the interval cover of $\operatorname{Im} f$ depicted on the right-hand side of Figure 1. Compute the corresponding Mapper.

Question 10. Relate the Mapper to the Reeb graph: in your opinion, to which points of the extended persistence diagram of $\tilde{f}$ do the topological features of the Mapper correspond?

4 The category of persistence modules

A *morphism* between two persistence modules $\mathbb{V}$ and $\mathbb{W}$ over T is a family Φ of linear maps $(\phi_t : V_t \to W_t)_{t \in T}$ such that the following diagram commutes for all $t \leq t' \in T$:

$$\begin{tikzcd}
V_t \arrow[r, "v_t^{t'}"] \arrow[d, "\phi_t"'] & V_{t'} \arrow[d, "\phi_{t'}"] \\
W_t \arrow[r, "w_t^{t'}"'] & W_{t'}
\end{tikzcd}$$

Φ is called an *isomorphism* if each map ϕ_t is itself an isomorphism $V_t \to W_t$.

Question 11. Show that the persistence modules over a fixed index set T, together with the morphisms between them, form a category, that is:

- for each persistence module $\mathbb{V}$ there exist isomorphisms $\mathbb{V} \to \mathbb{V}$,
- for any morphisms $\mathbb{U} \xrightarrow{\Phi} \mathbb{V}$ and $\mathbb{V} \xrightarrow{\Psi} \mathbb{W}$ there exists a morphism $\mathbb{U} \xrightarrow{\Psi \circ \Phi} \mathbb{W}$.

A *submodule* $\mathbb{W}$ of a persistence module $\mathbb{V}$ is composed of subspaces $W_t \subseteq V_t$ for all $t \in T$, and of maps $w_t^{t'} = v_t^{t'}|_{W_t}$ for all $t \leq t' \in T$. In particular, $v_t^{t'}(W_t) \subseteq W_{t'}$ for all $t \leq t' \in T$. A simple example of submodule is the null module $\mathbb{W} = 0$ (defined by $W_t = 0$ for all $t \in T$ and $w_t^{t'} = 0$ for all $t \leq t' \in T$), which is a submodule of any module $\mathbb{V}$ over T.

Question 12. Show that the null module is both an *initial* and a *terminal* object in the category, that is: for any module $\mathbb{V}$ there is a unique morphism $0 \longrightarrow \mathbb{V}$ and a unique morphism $\mathbb{V} \longrightarrow 0$.

Direct sums in the category are defined pointwise, that is: for any persistence modules $\mathbb{V}$ and $\mathbb{W}$ over T, the direct sum $\mathbb{V} \oplus \mathbb{W}$ is composed of the spaces $V_t \oplus W_t$ for all $t \in T$, and of the maps $v_t^{t'} \oplus w_t^{t'}$ for all $t \leq t' \in T$. We say that $\mathbb{V}$ and $\mathbb{W}$ are *summands* of the direct sum $\mathbb{V} \oplus \mathbb{W}$. In particular, we have $\mathbb{V} = 0 \oplus \mathbb{V} = \mathbb{V} \oplus 0$ for any persistence module $\mathbb{V}$, so $\mathbb{V}$ is always a summand of itself. A persistence module $\mathbb{V}$ is called *indecomposable* if its only summands are itself or the null module. The decomposition theorem that we saw in class asserts that the only indecomposable persistence modules are the so-called interval modules over T, and that (under some conditions on T or on the dimensions of the spaces) any module decomposes as a direct sum of interval modules.

Question 13. Prove the decomposition theorem in the case where $|T| = 1$.

In the general case however, the result is much more complicated to prove as the persistence modules over T do not share the same properties of vector spaces, in particular they are not *semisimple*:

Question 14. In the case where $|T| \geq 2$, exhibit a counterexample showing that a submodule of a persistence module $\mathbb{V}$ over T may not always be a summand of $\mathbb{V}$.