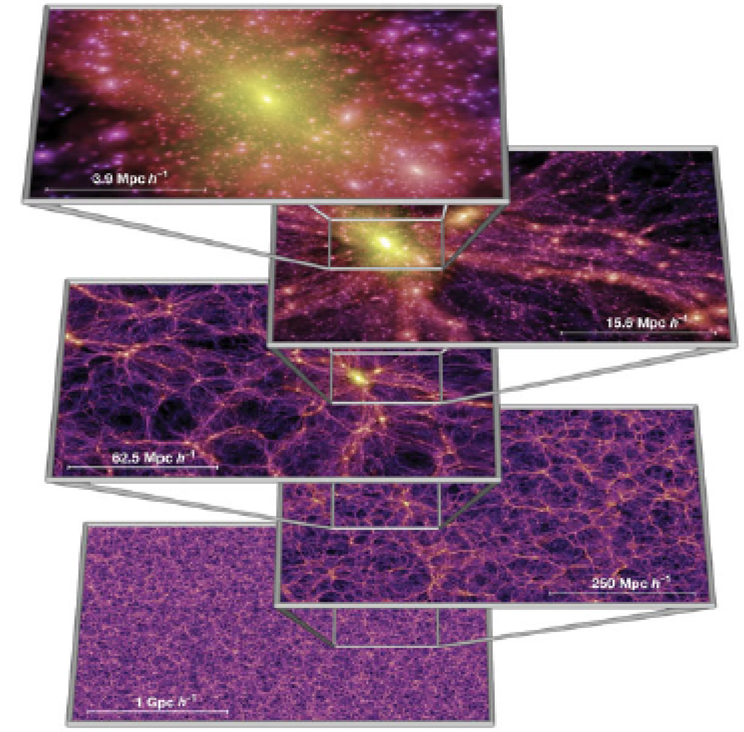
# Topological Inference

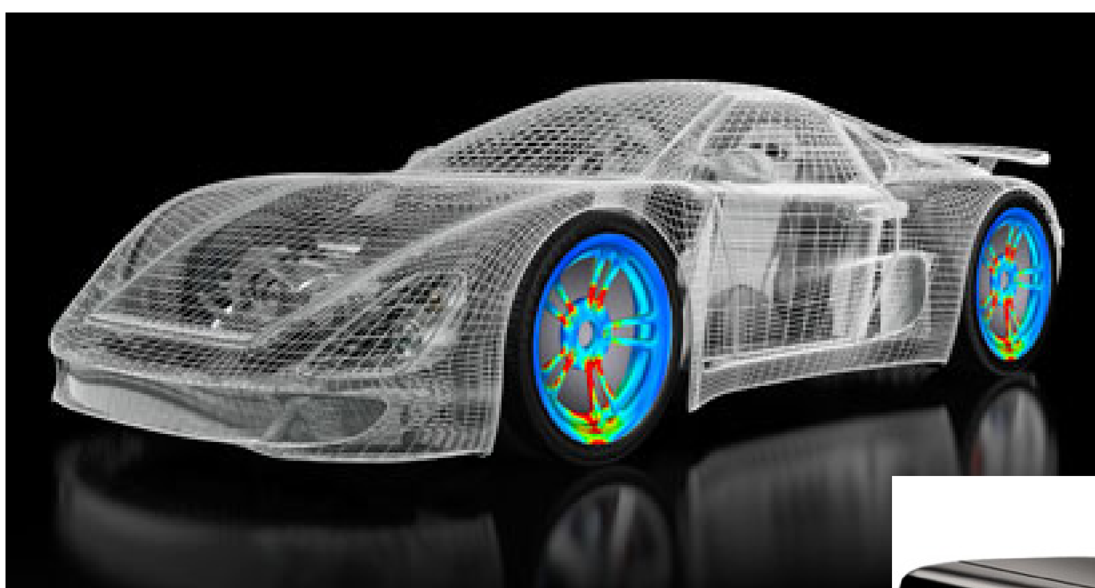
http://geometrica.saclay.inria.fr/team/Steve.Oudot/courses/EMA/

# Context: The data deluge



Modern data sets are ever more massive and complex:

- academia
- industry
- general public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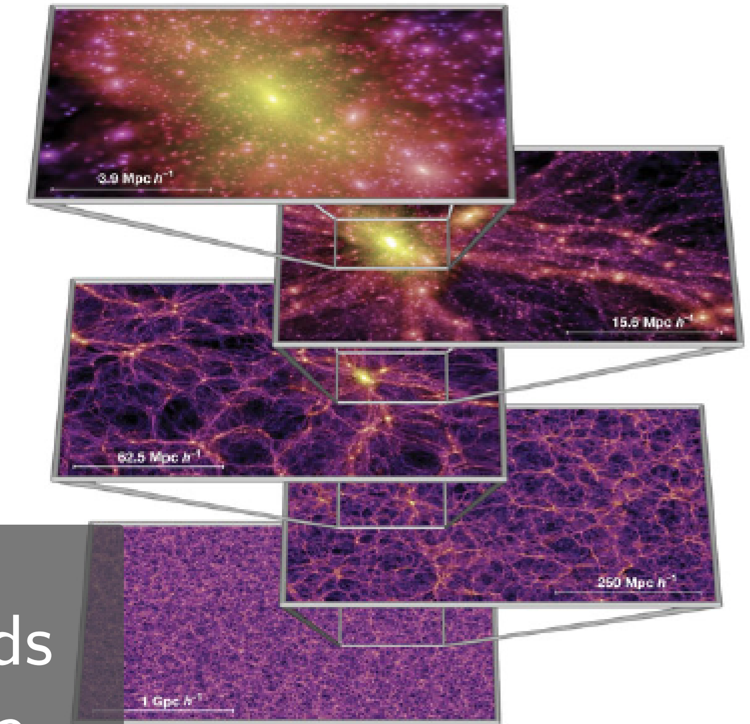
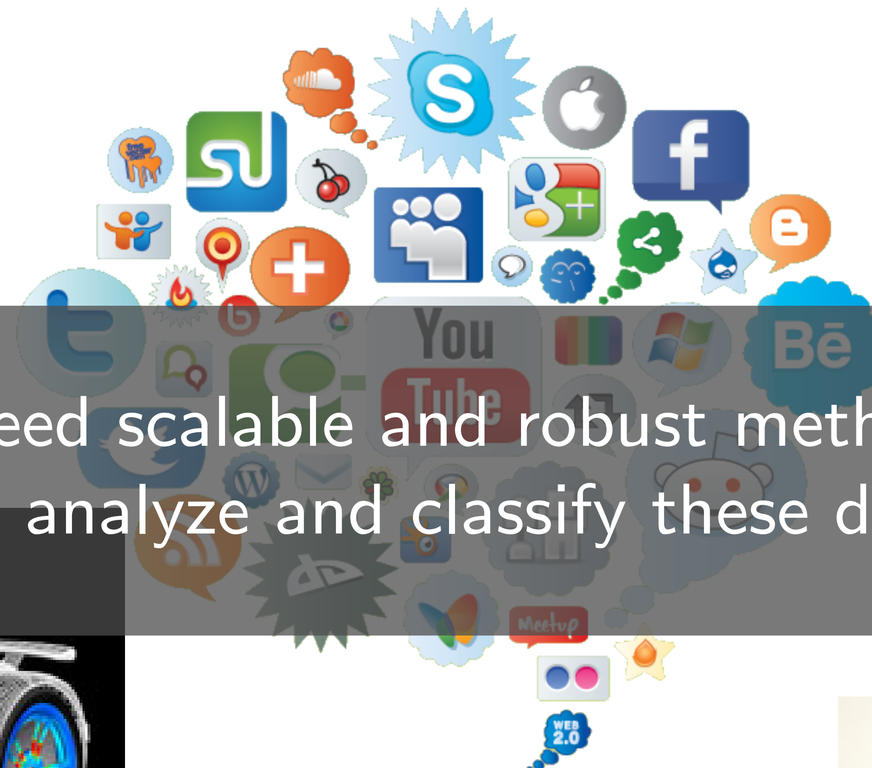


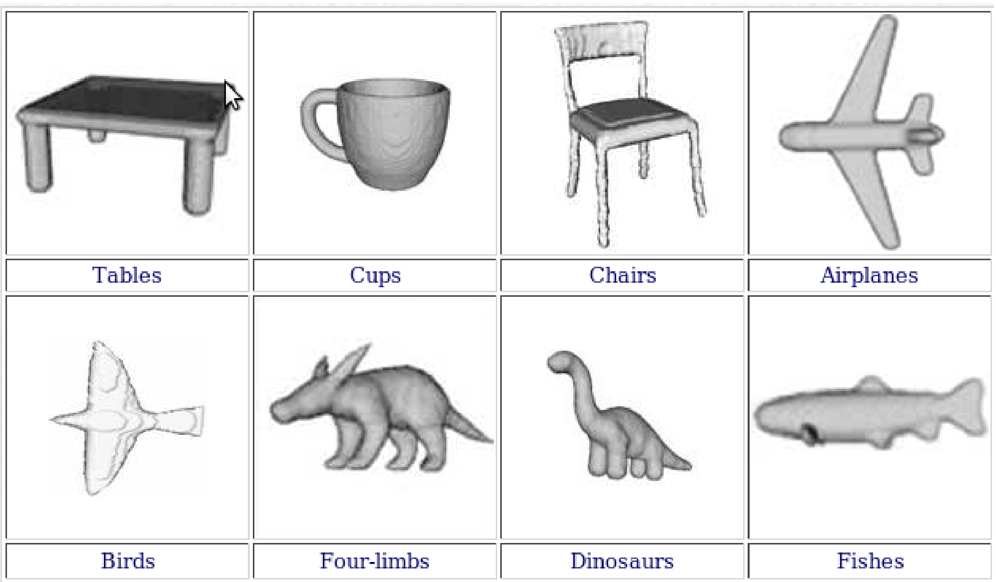


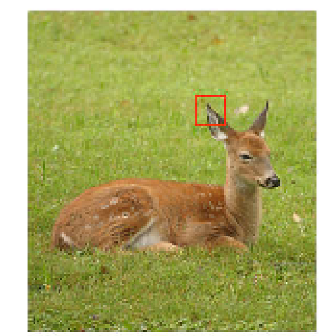
Need scalable and robust methods to analyze and classify these data

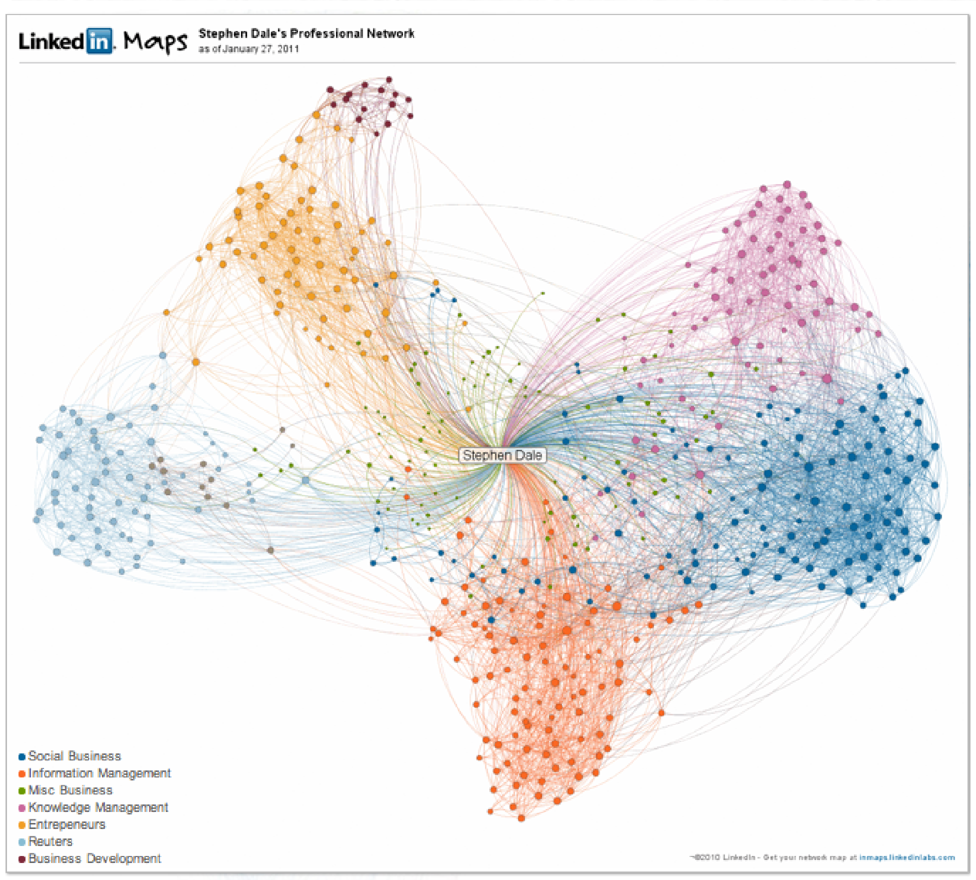

# Exploratory analysis of geometric data

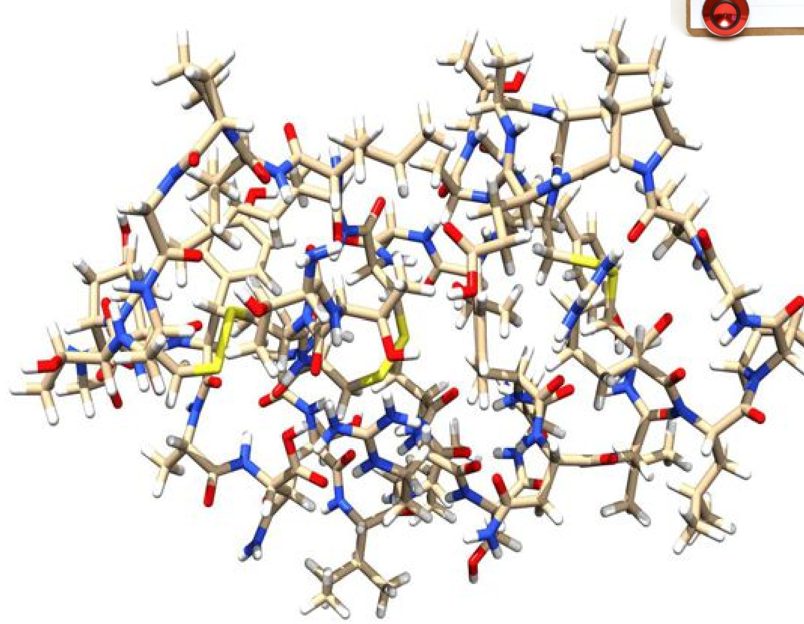
**Input:** point cloud equipped with a metric or (dis-)similarity measure

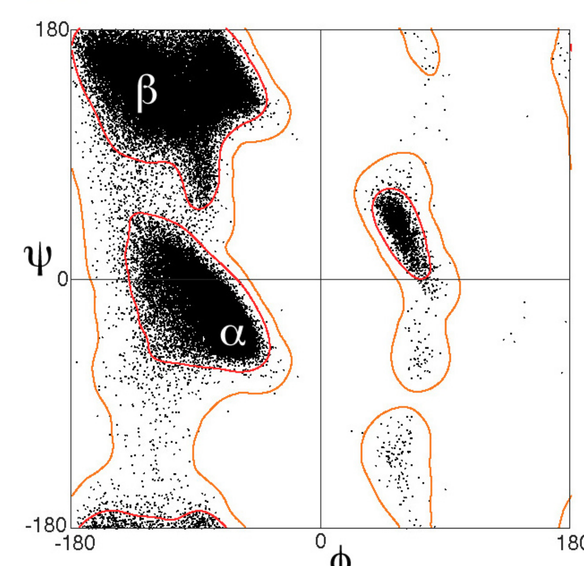
**data point** $\equiv$ image/patch, geometric shape, protein conformation, patient, LinkedIn user...





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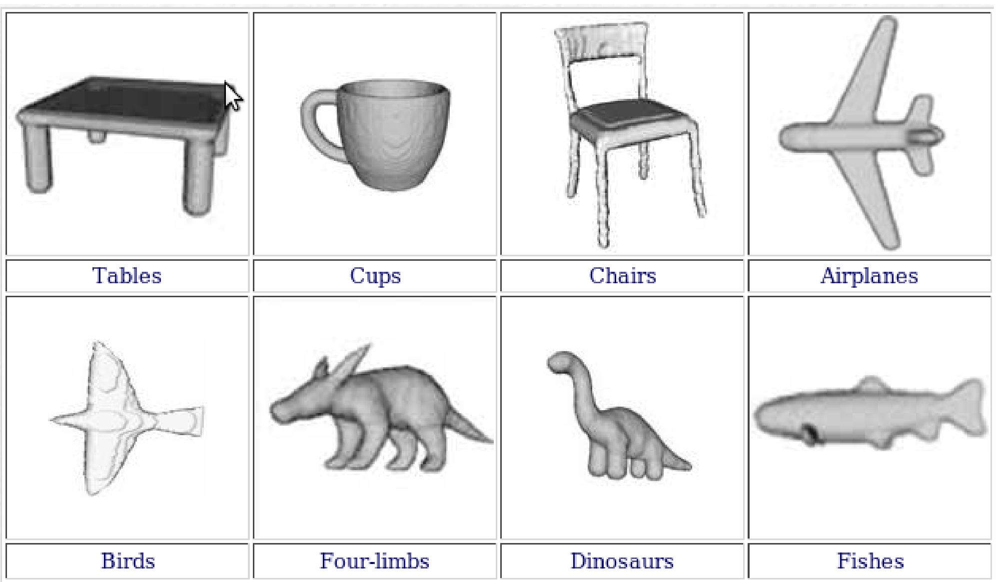
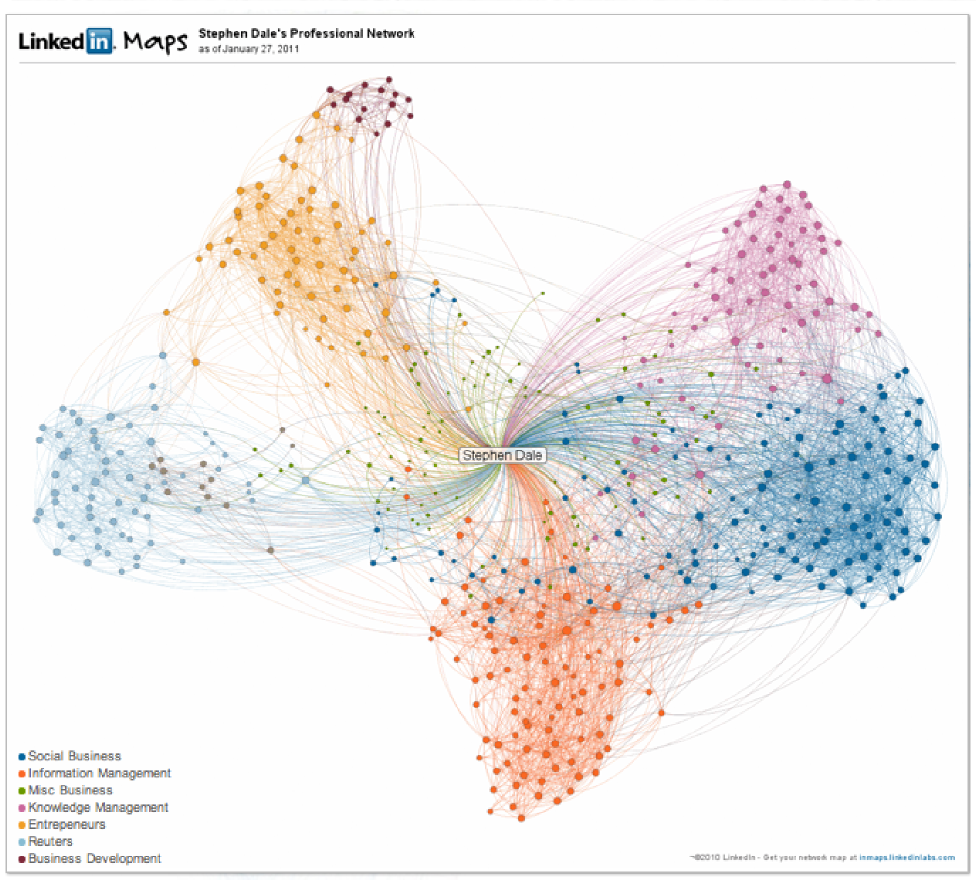
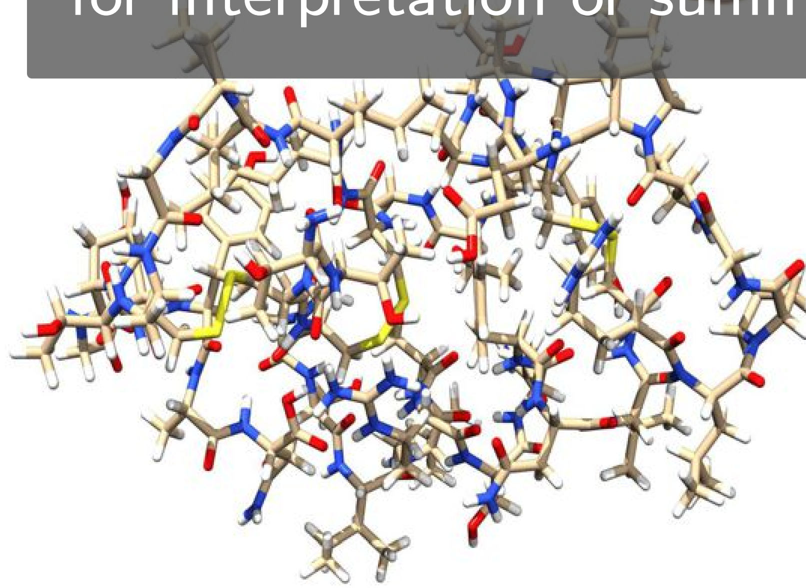
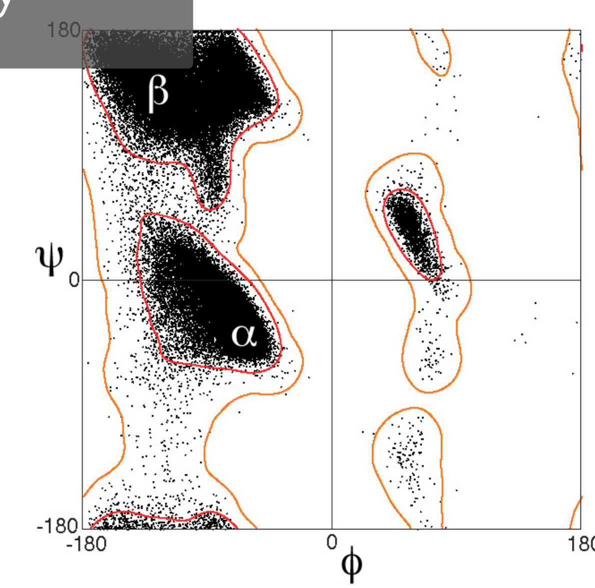


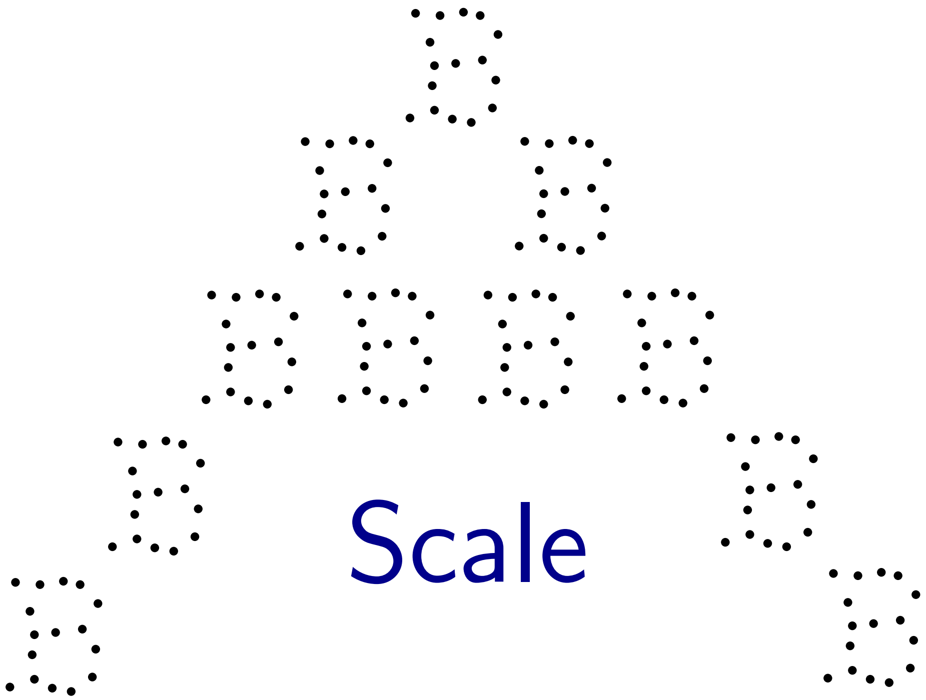



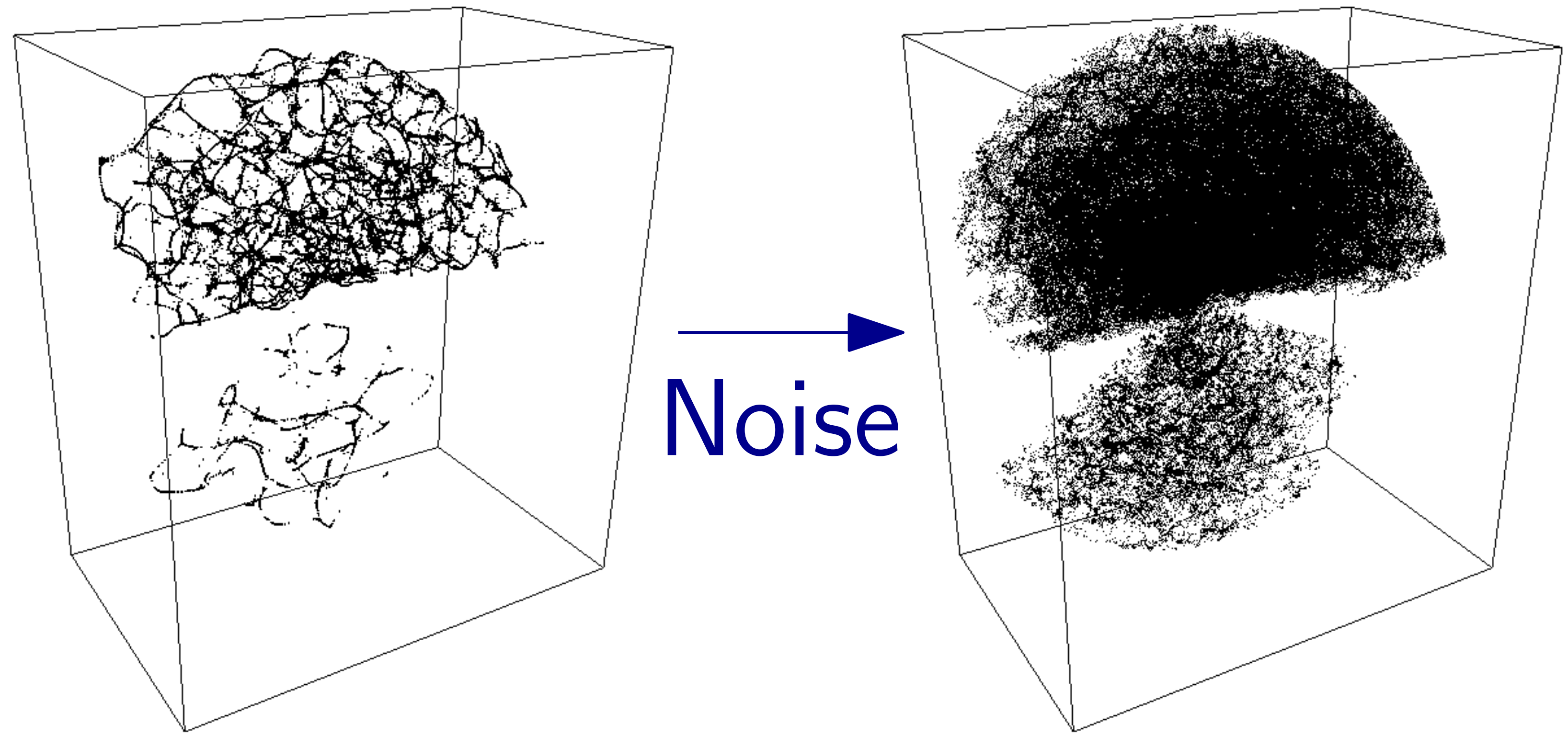

**Goal:** describe the structure of the geometry underlying the data, for interpretation or summary

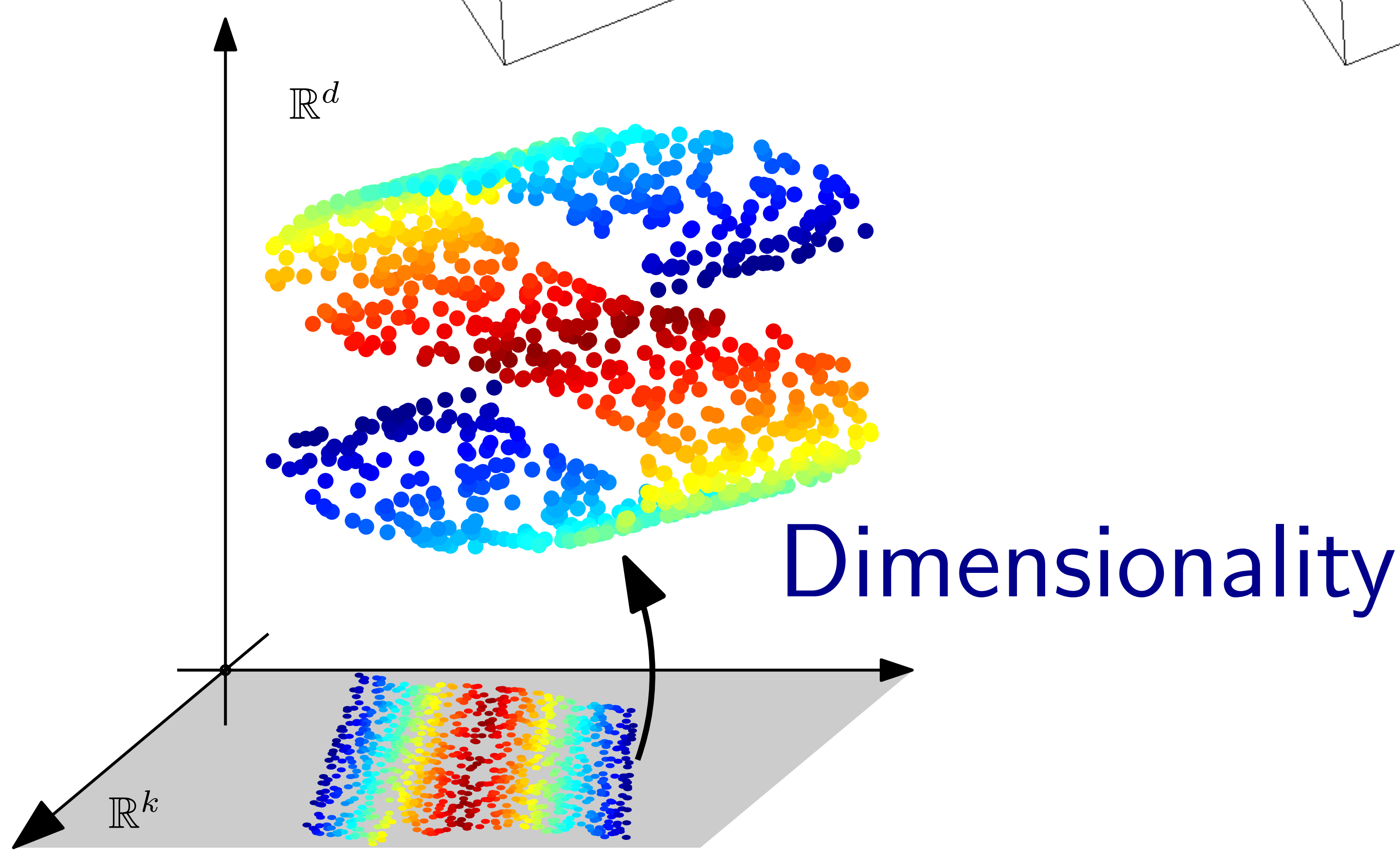




# Challenges



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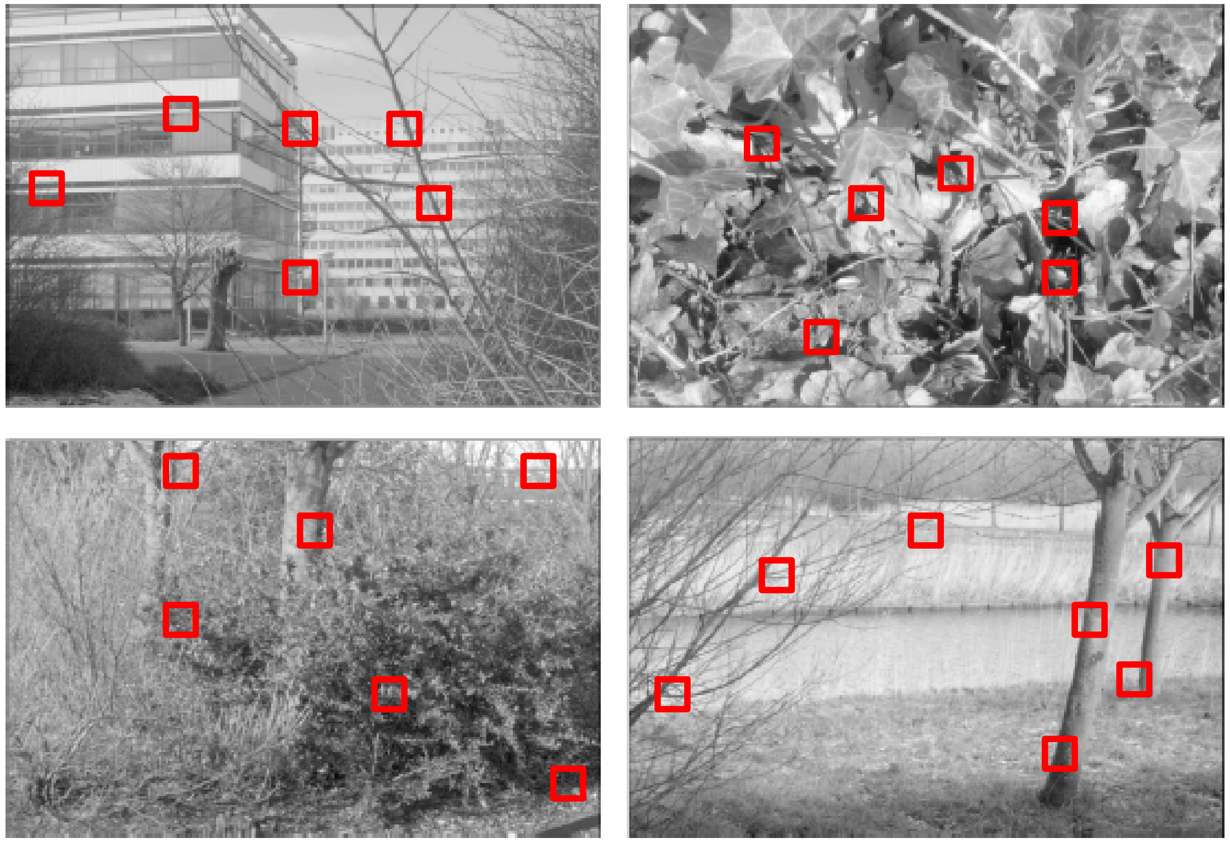
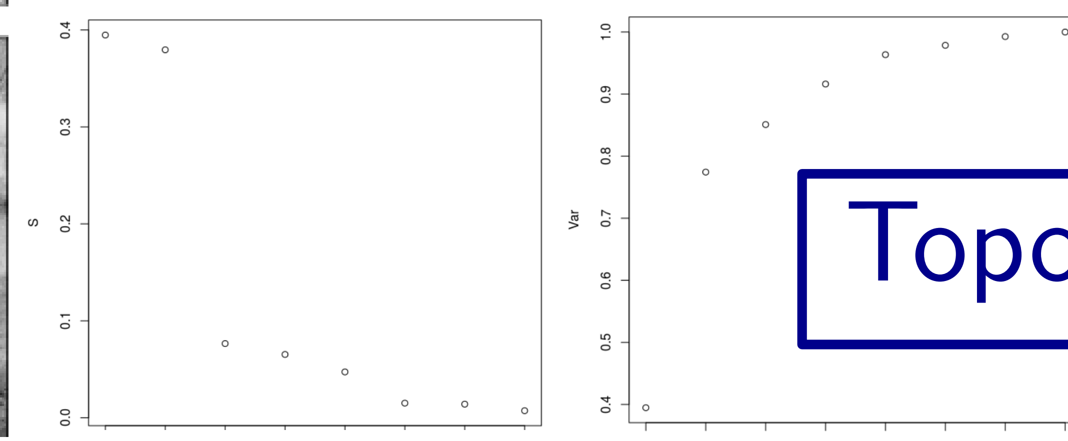


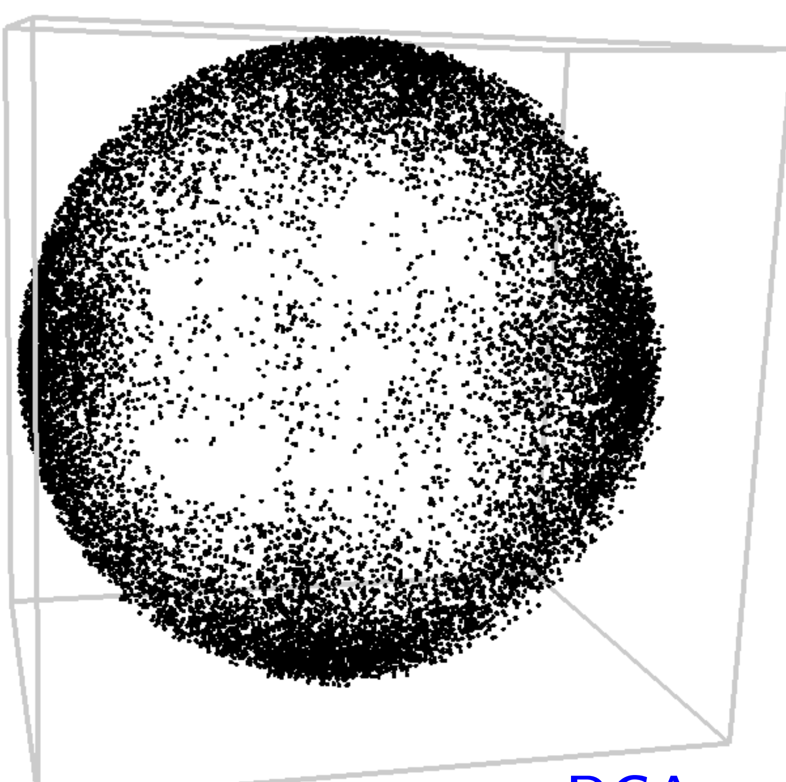
$4$ million data points in $\mathbb{R}^9$

(source: [Lee, Pederson, Mumford 2003])

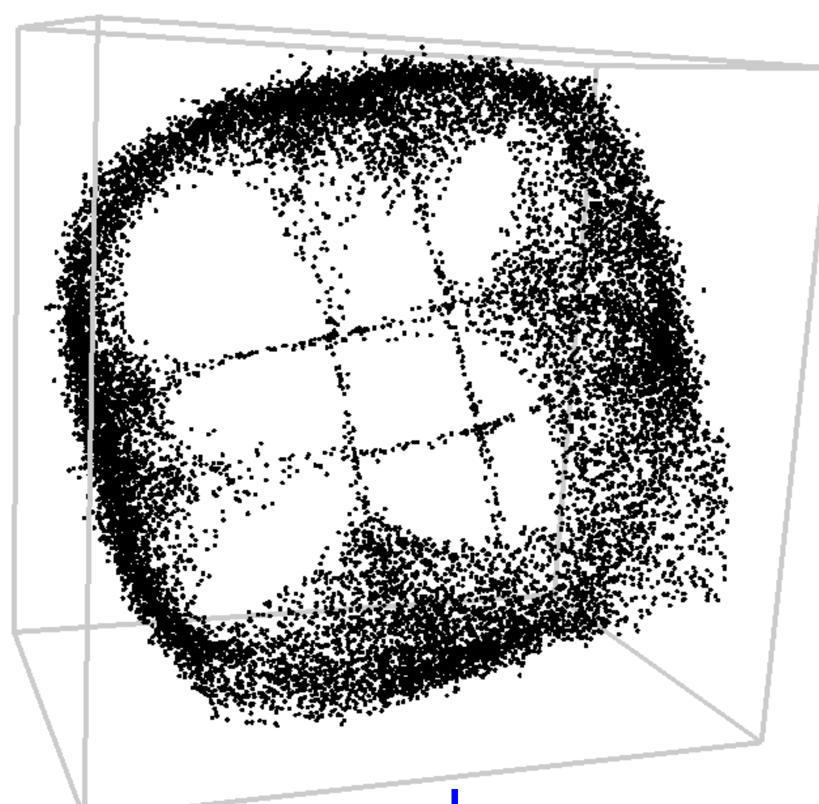
Motivation: study cognitive representation of space of images



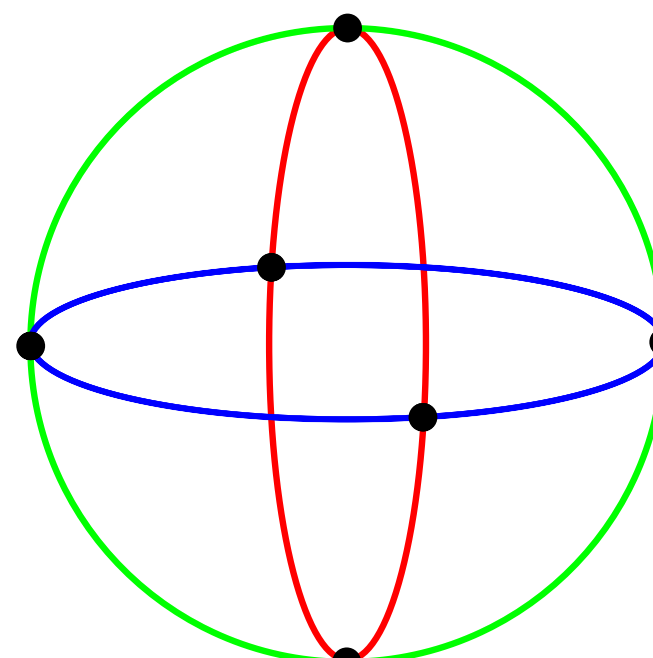

Topology



PCA



Isomap



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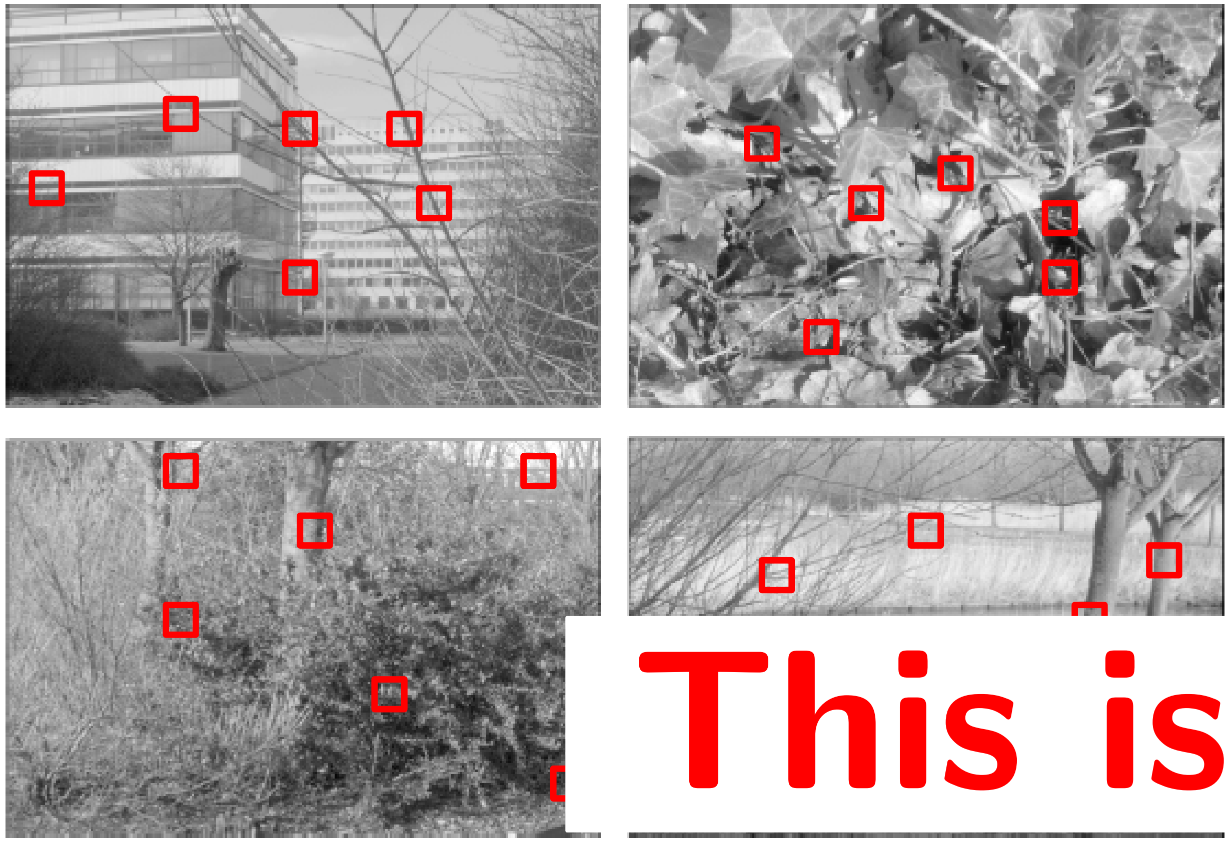

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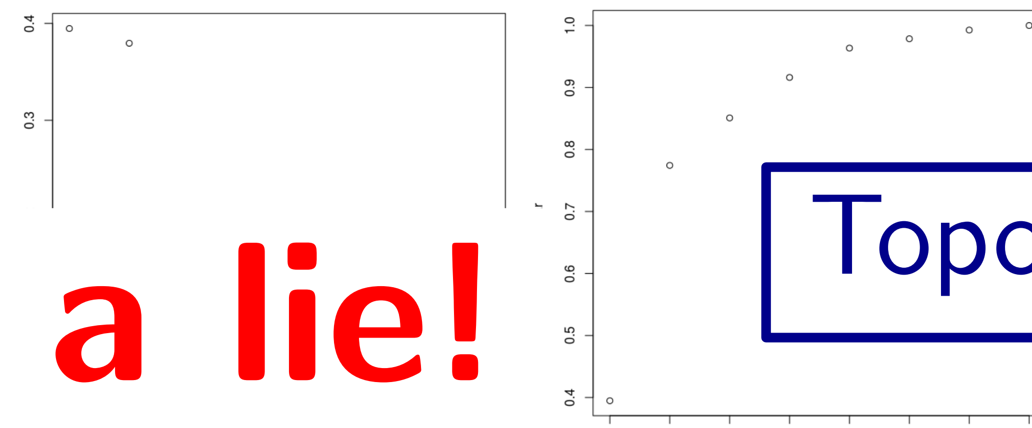
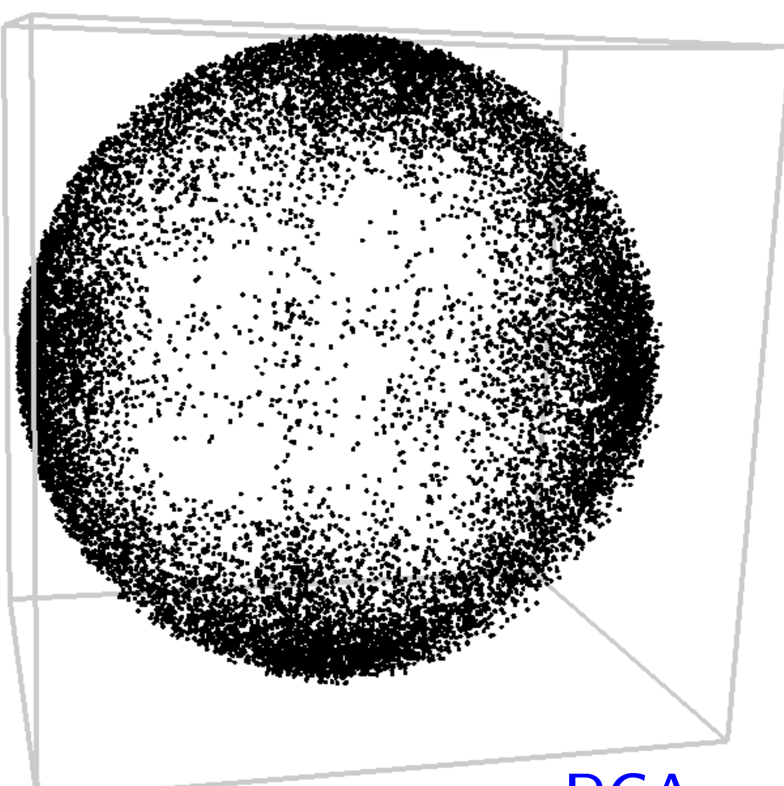


**This is a lie!**

Topology



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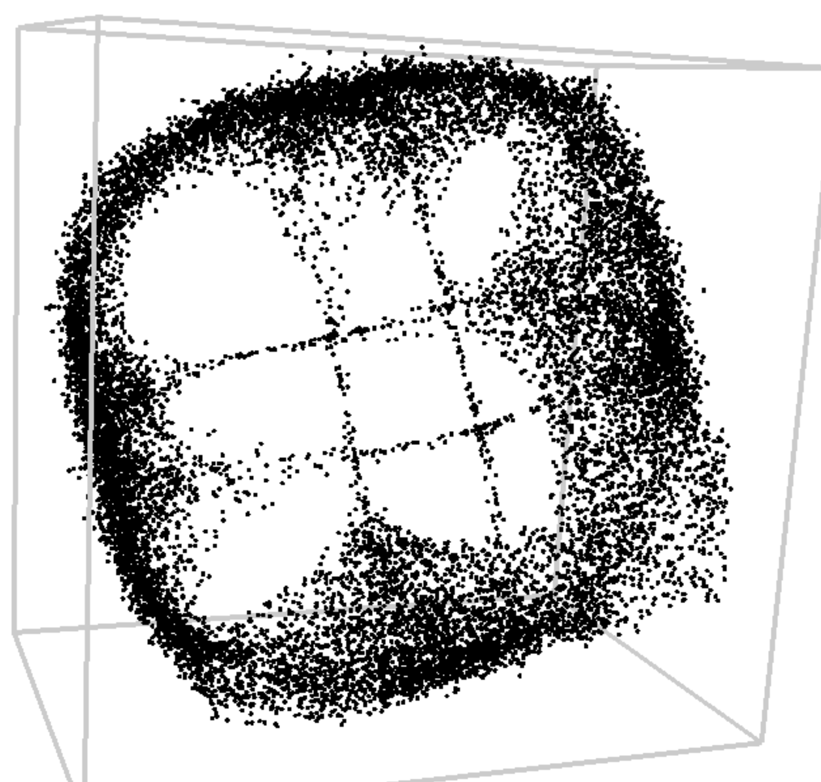

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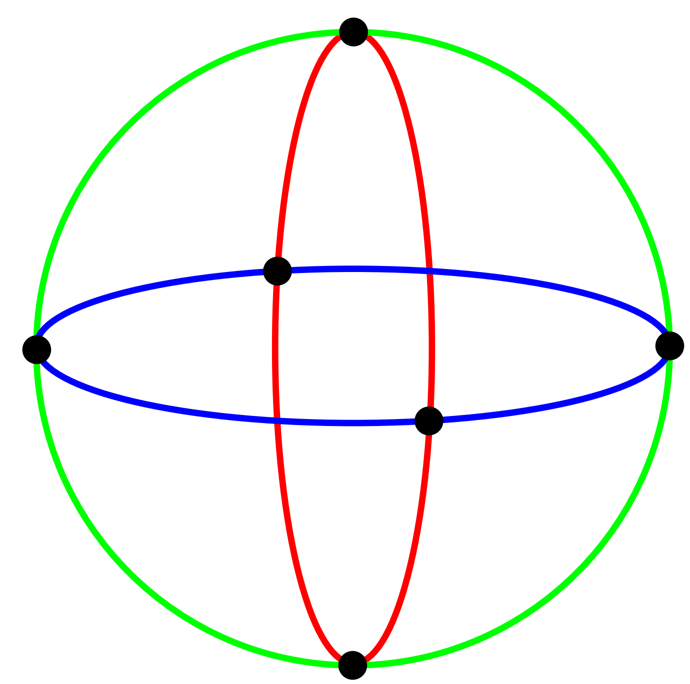
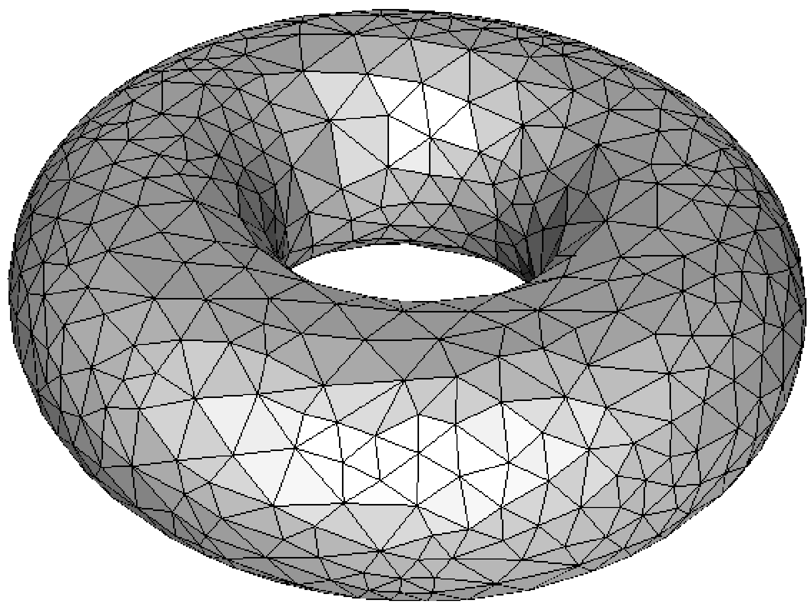


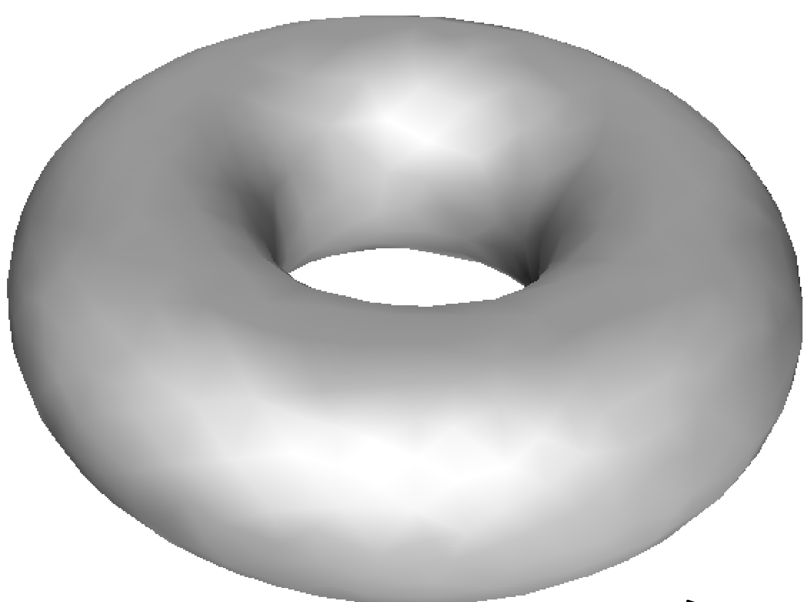
# The topology of data (TDA)

topological invariants for classification

$$\beta_0 = \beta_2 = 1$$
$$\beta_1 = 2$$



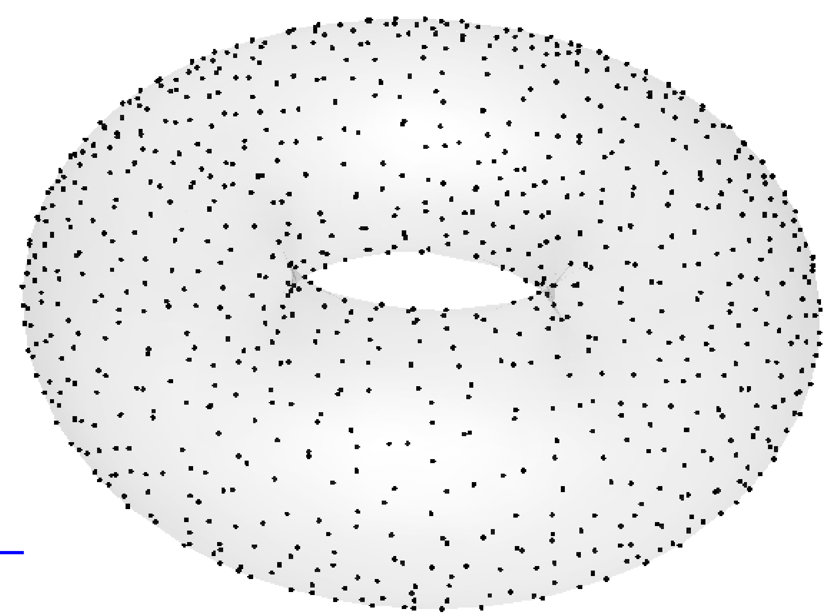
triangulation



compact set

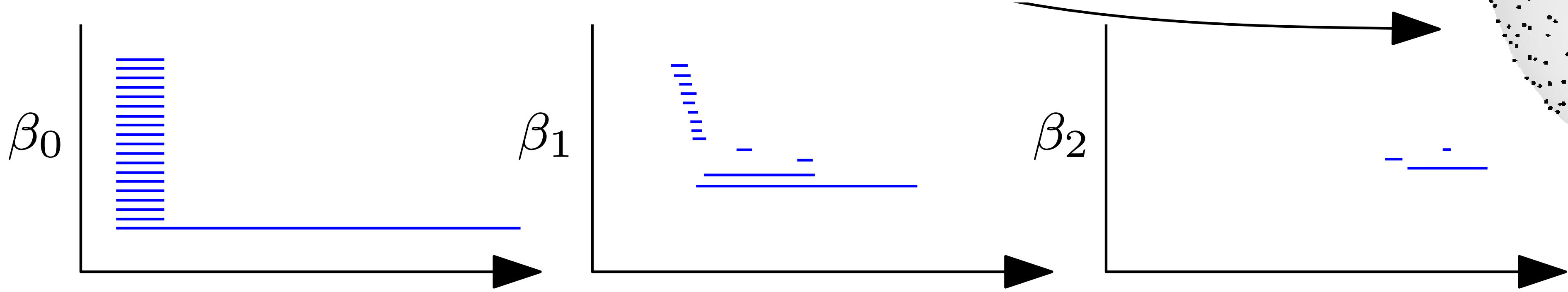
Algebraic topology in the 20th century

Algebraic topology in the 21st century



point cloud

topological descriptors for inference and comparison

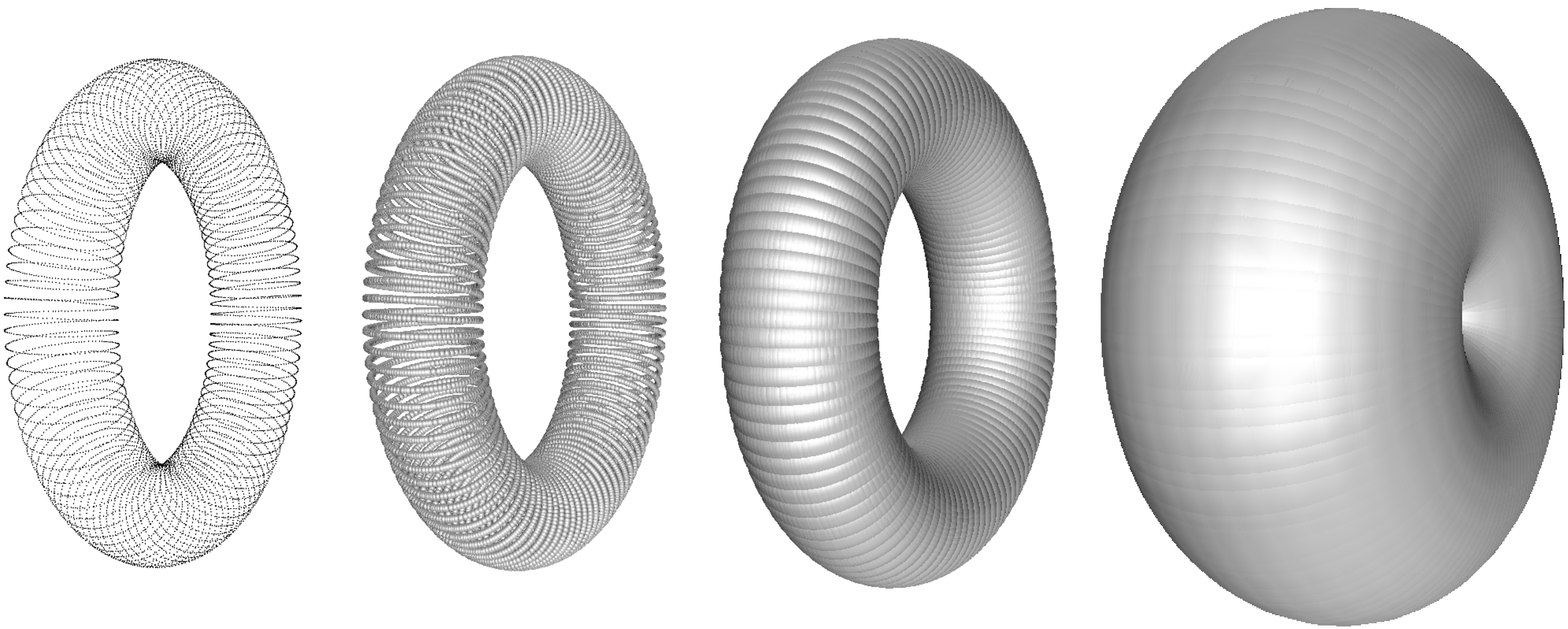


# Topology from Data

Input: point cloud $P \subset \mathbb{R}^d$

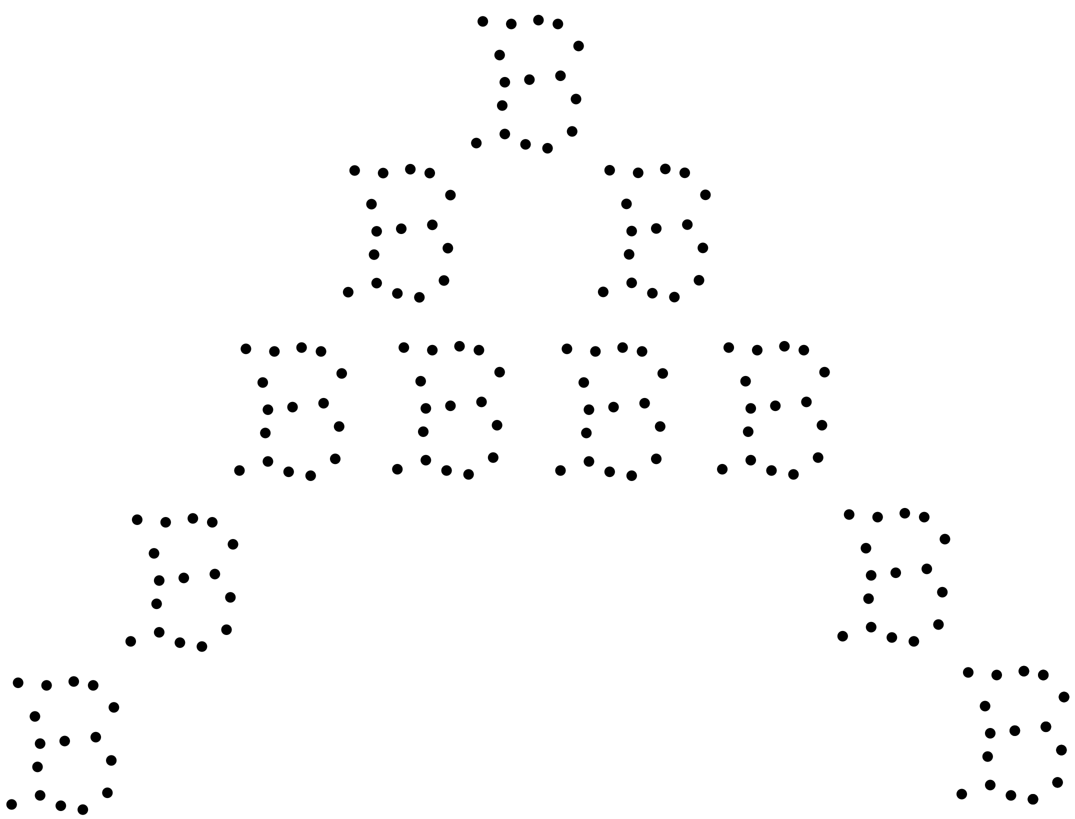
$\rightarrow$ uncover the topological structure of the space(s) underlying the data

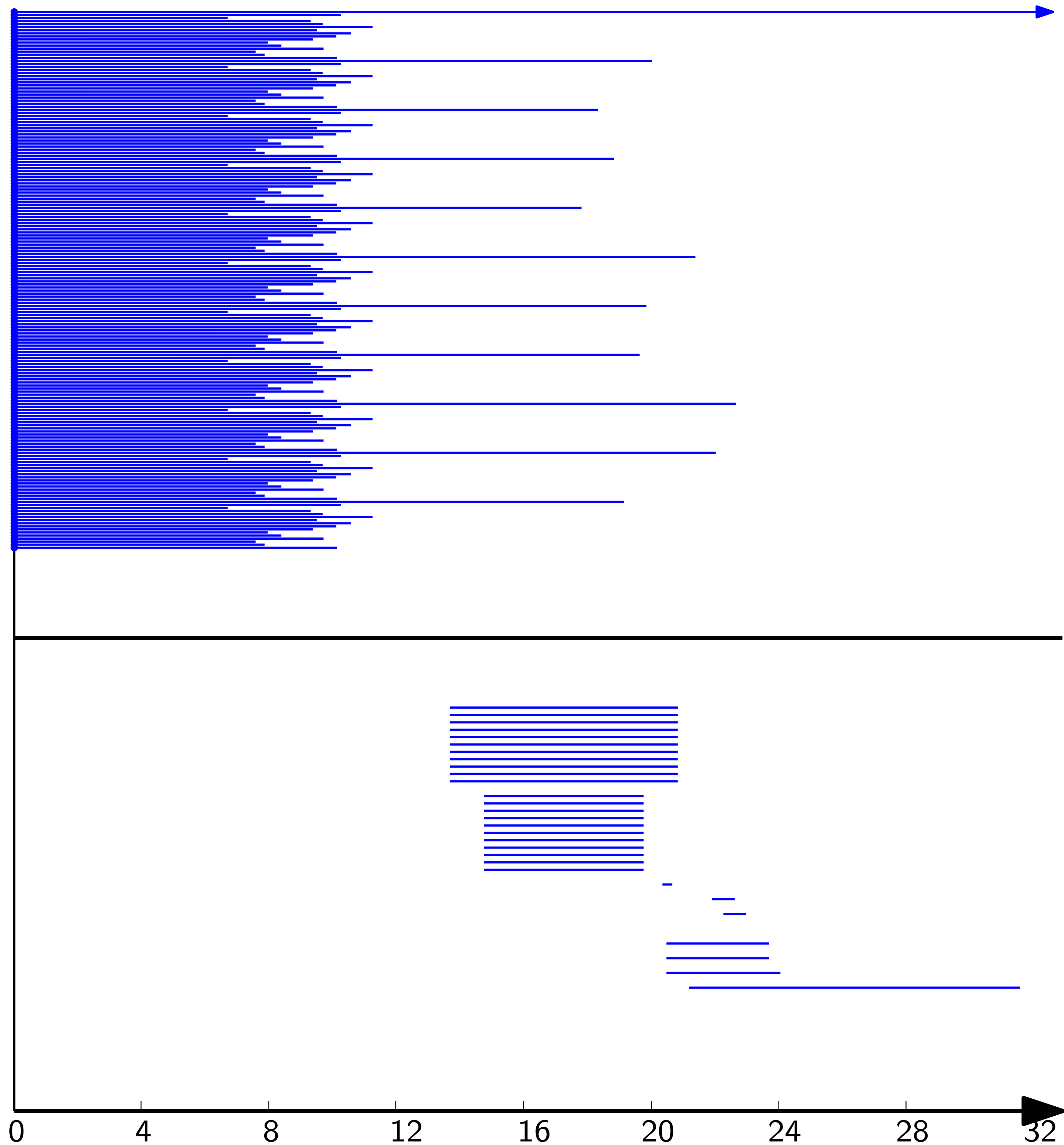
$\rightarrow$ inspect data at all scales and see what 'persists'



# Approach: Compute persistence of distance function

$$\mathrm{d}_P: \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in P} \|x - p\|_2$$





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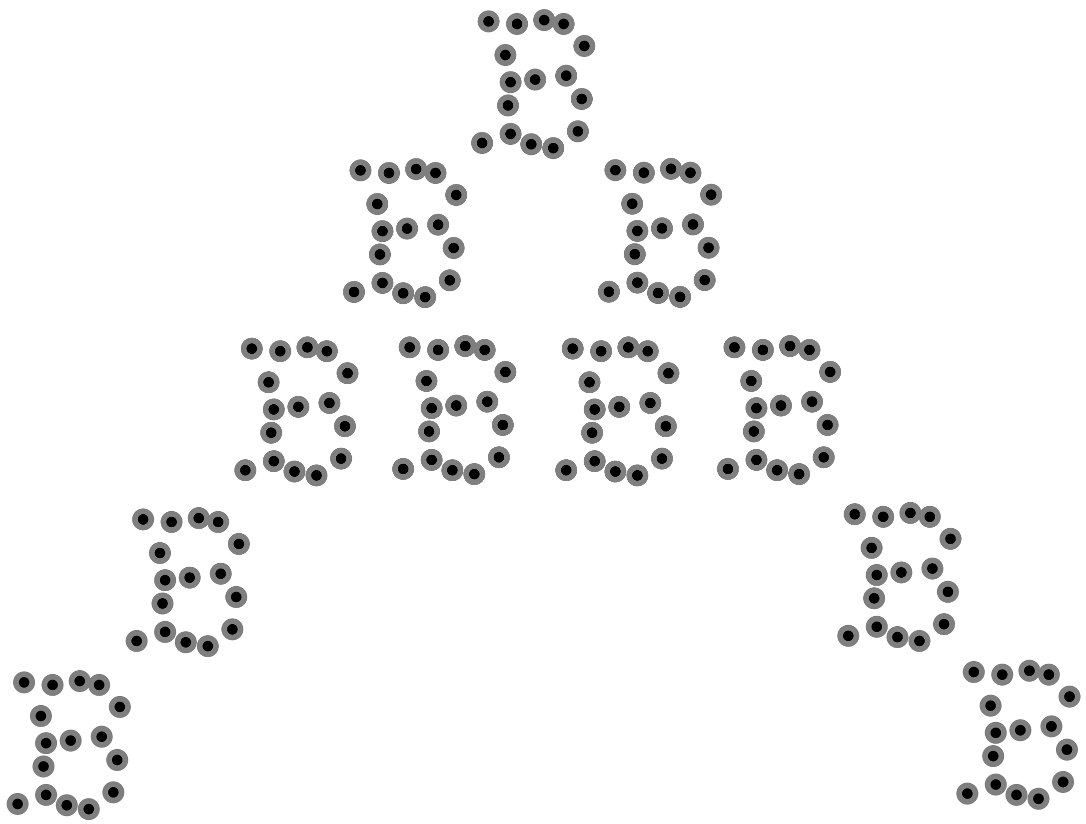

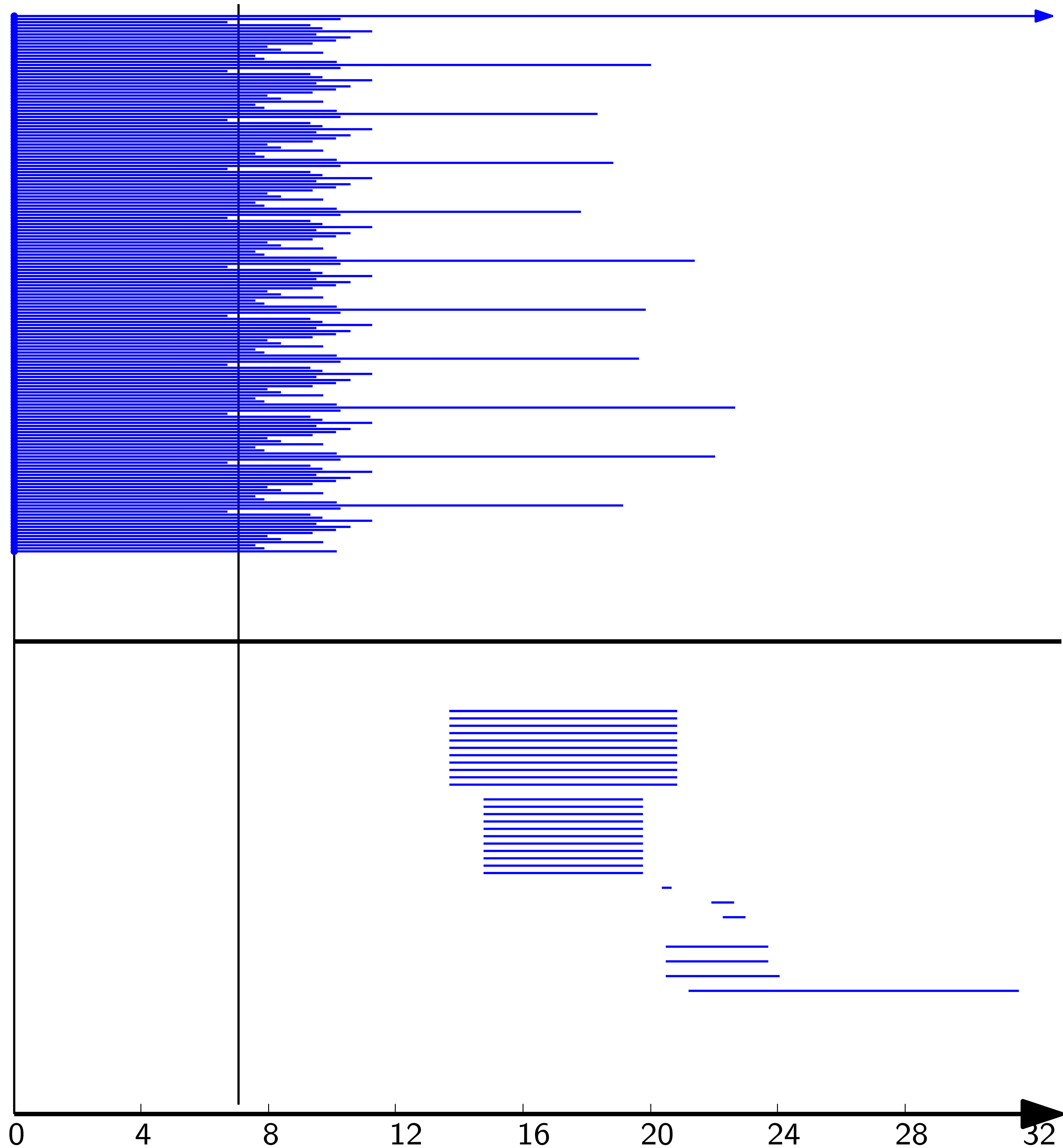

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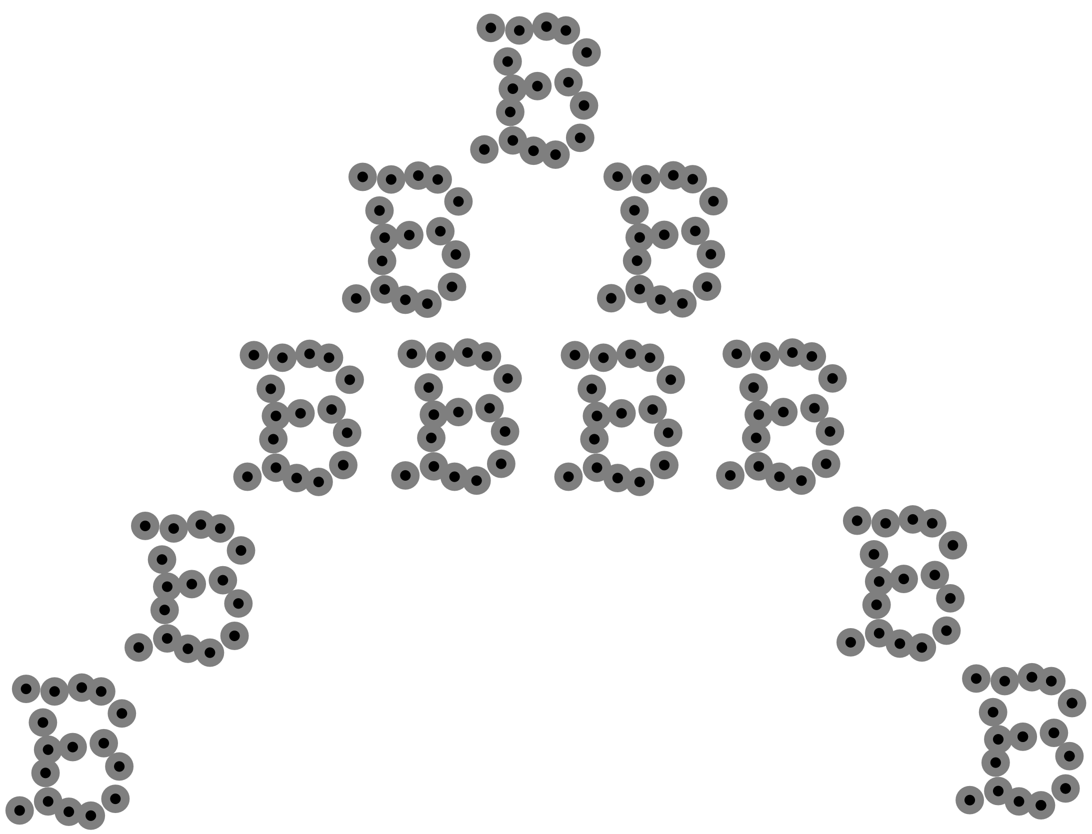

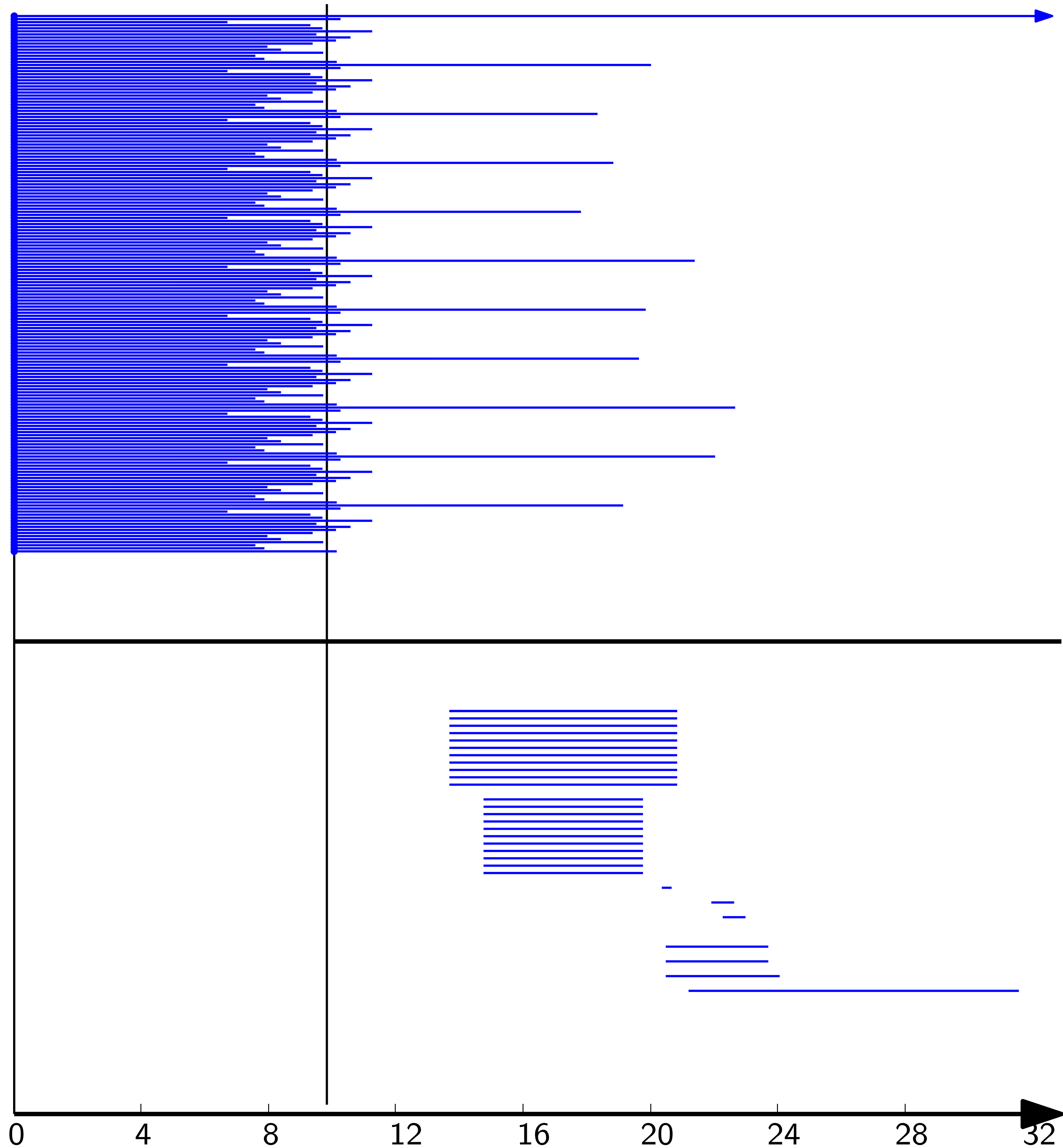

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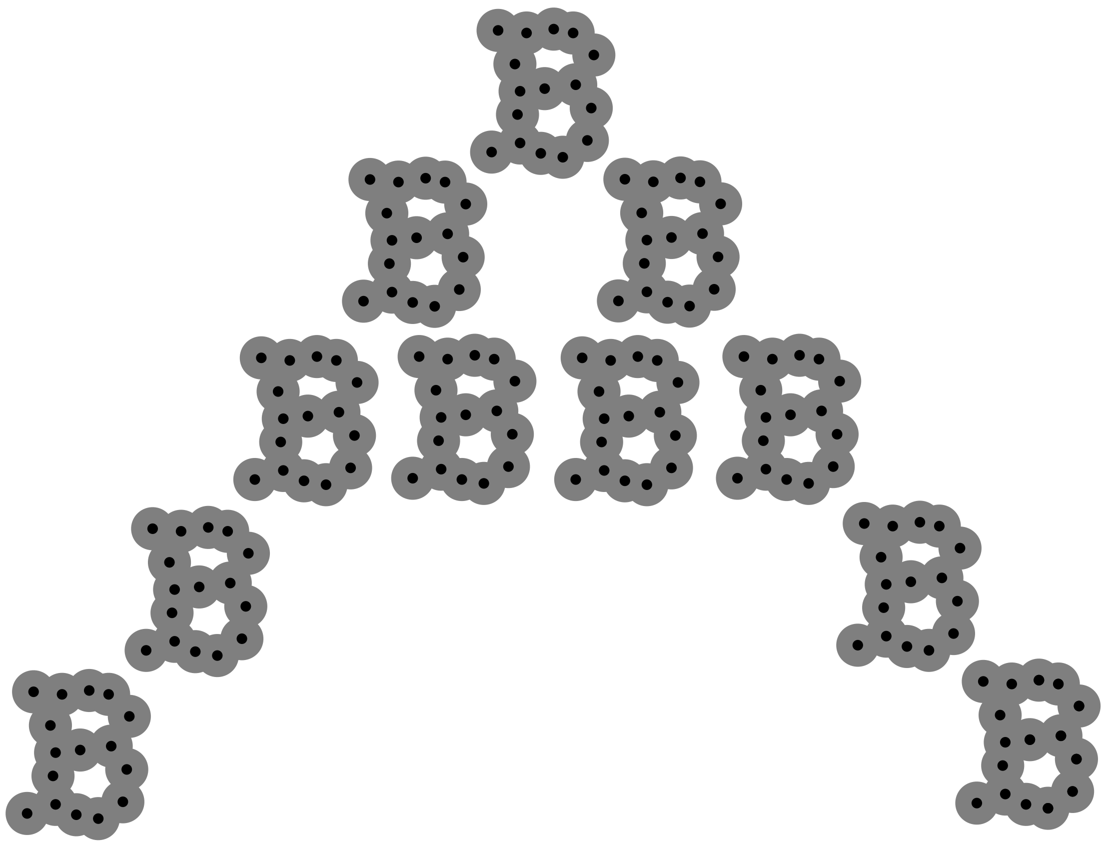

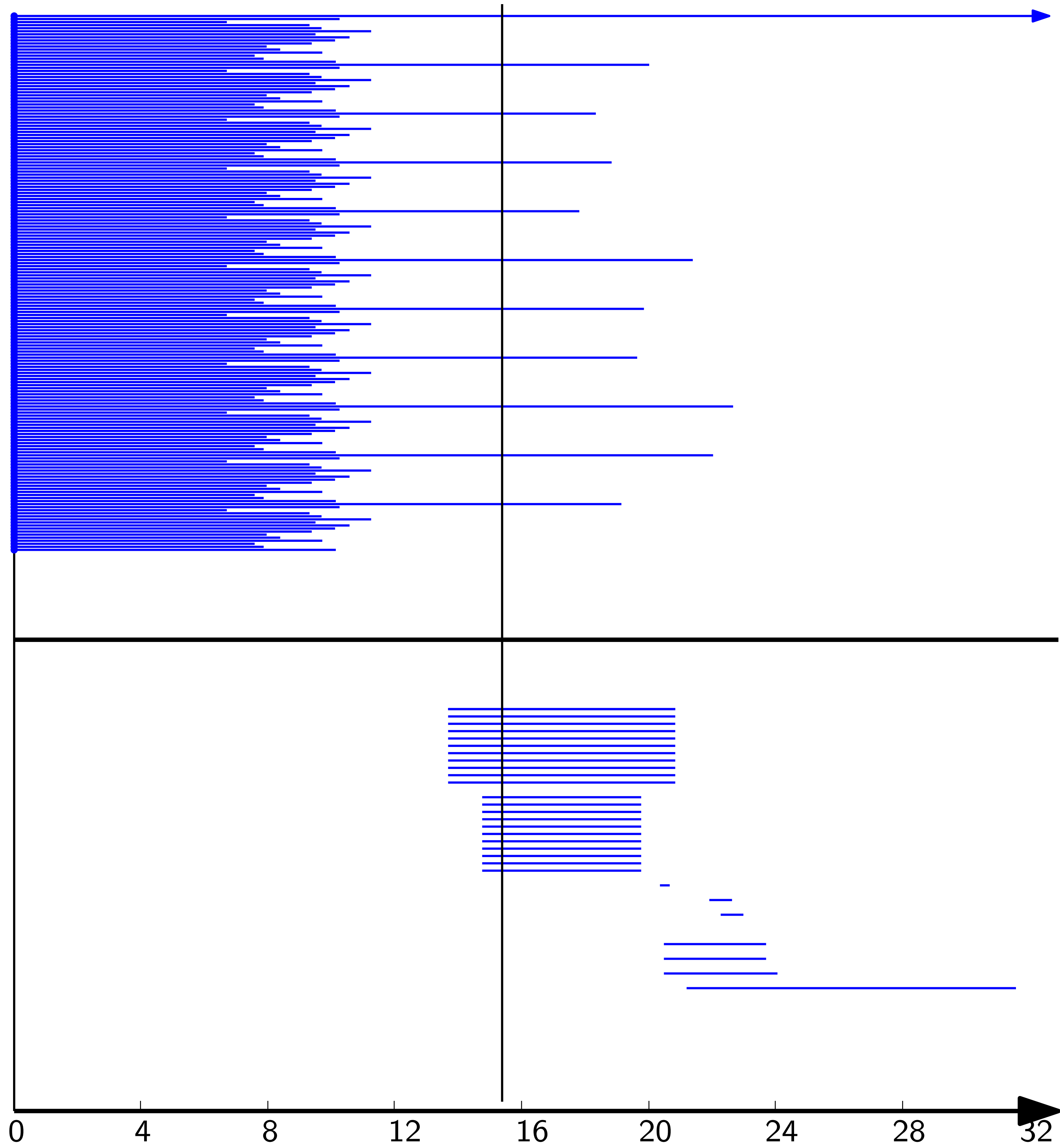

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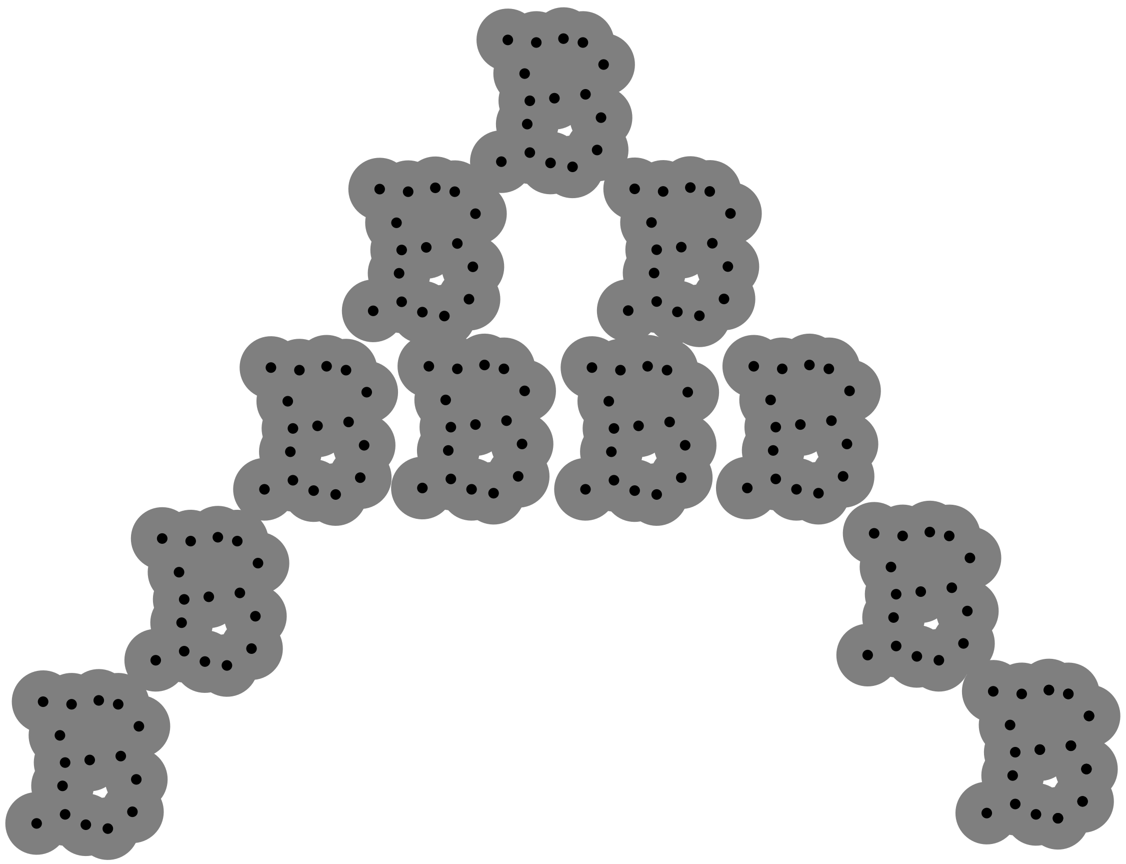

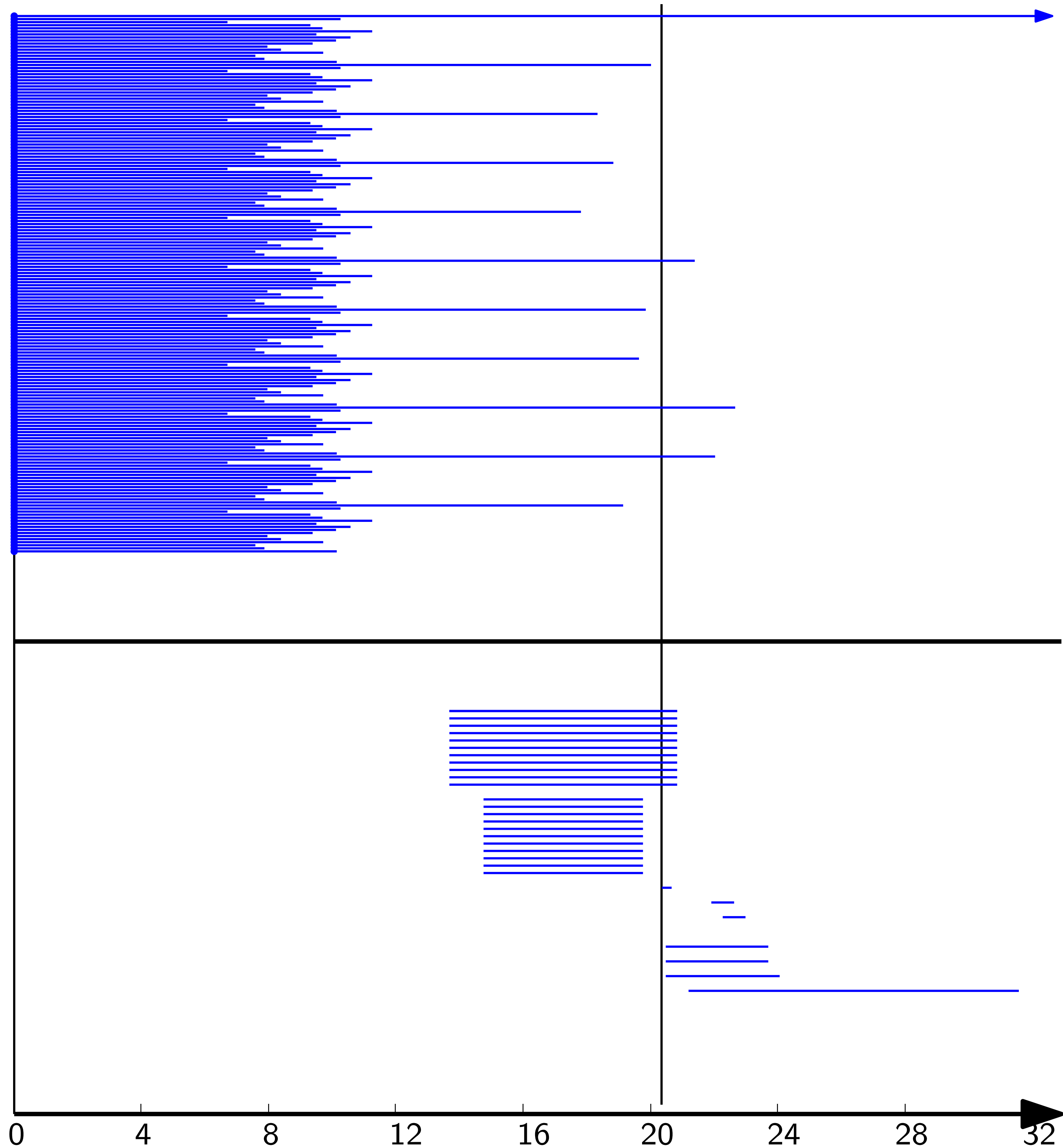

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$$
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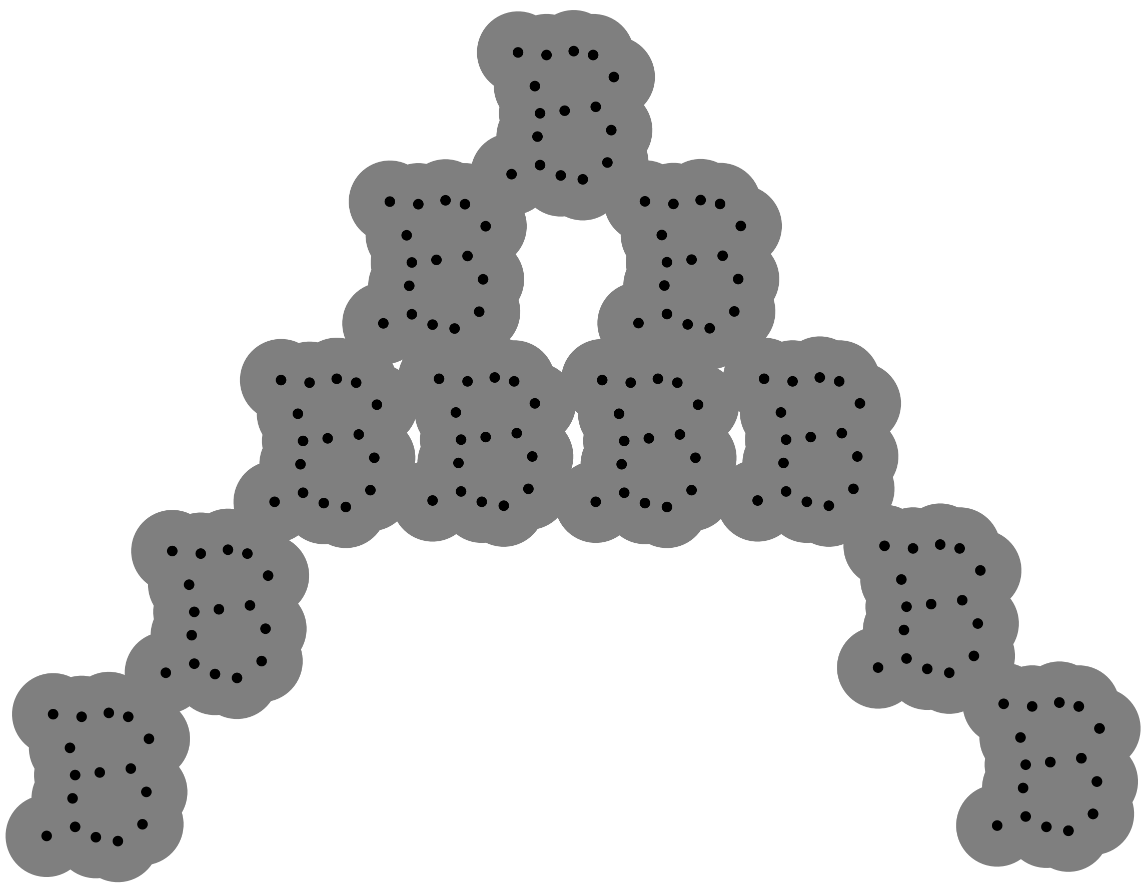

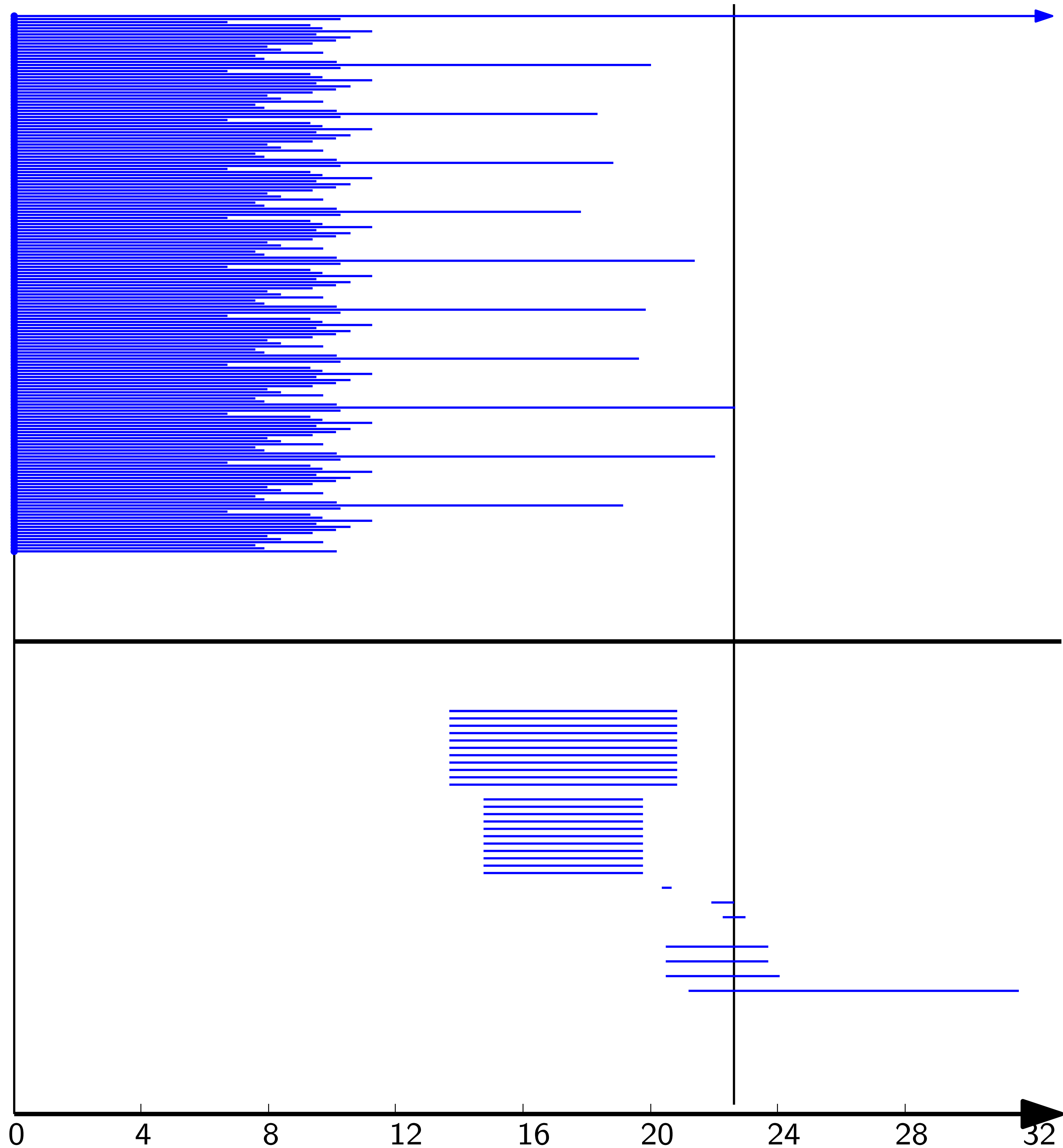

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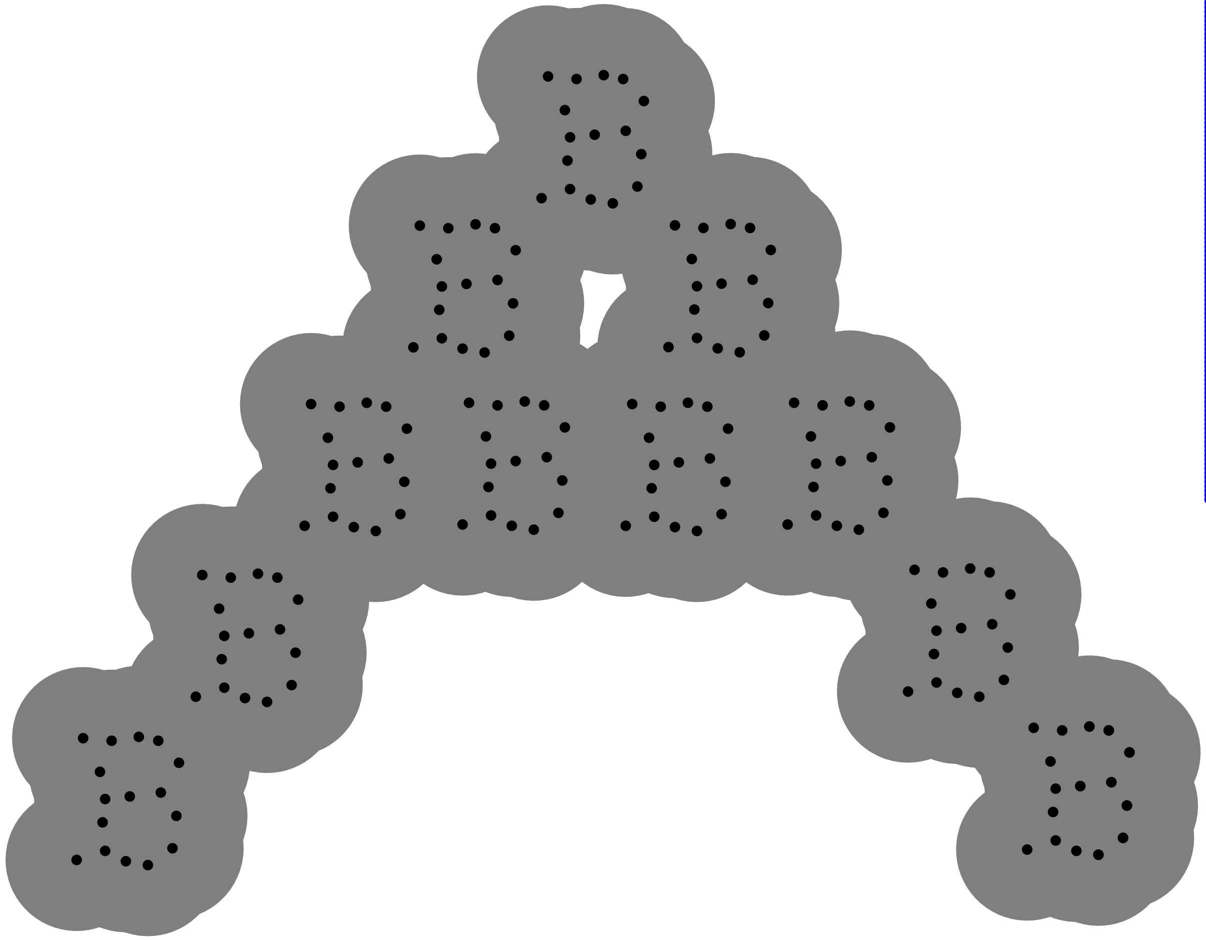

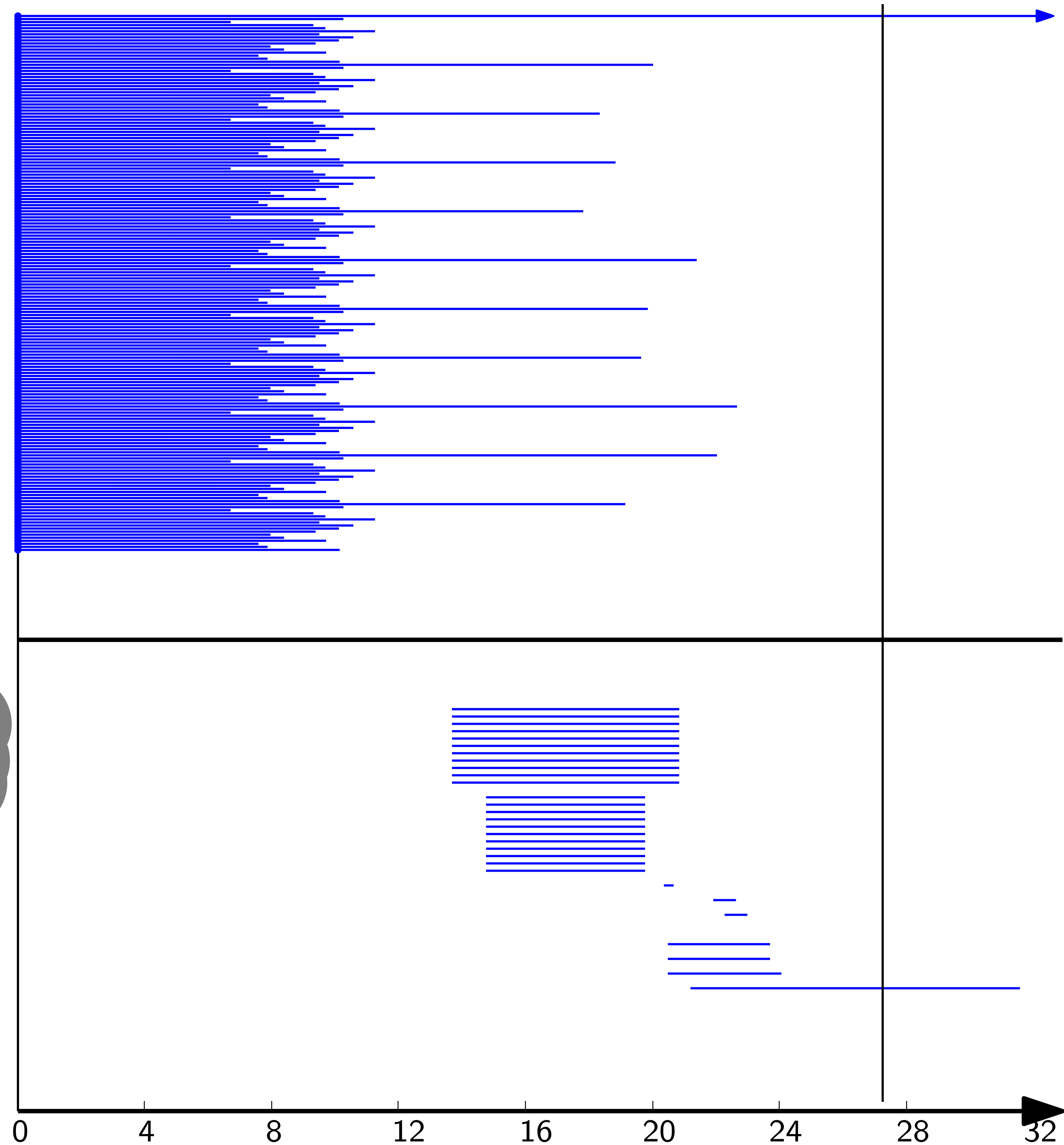

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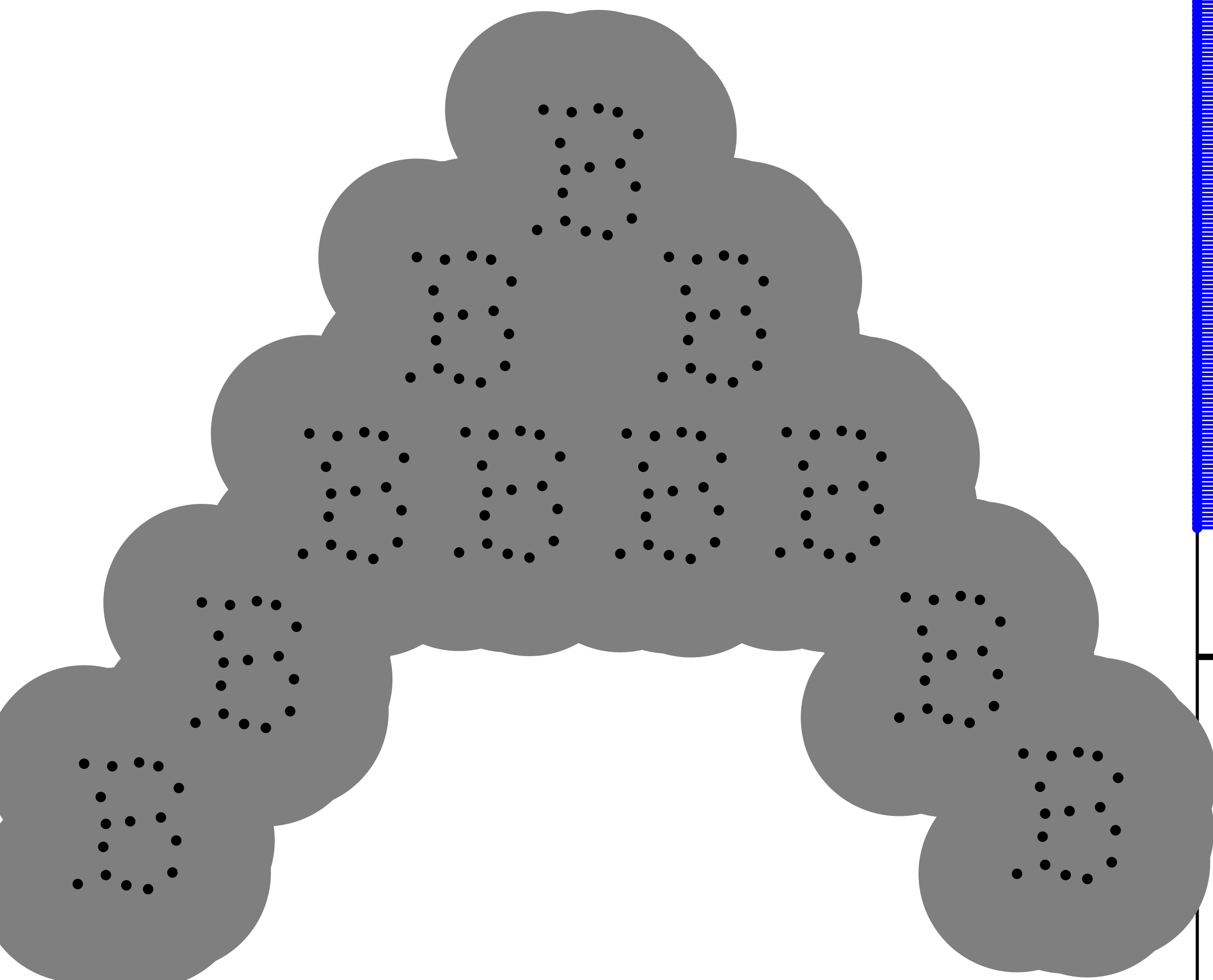

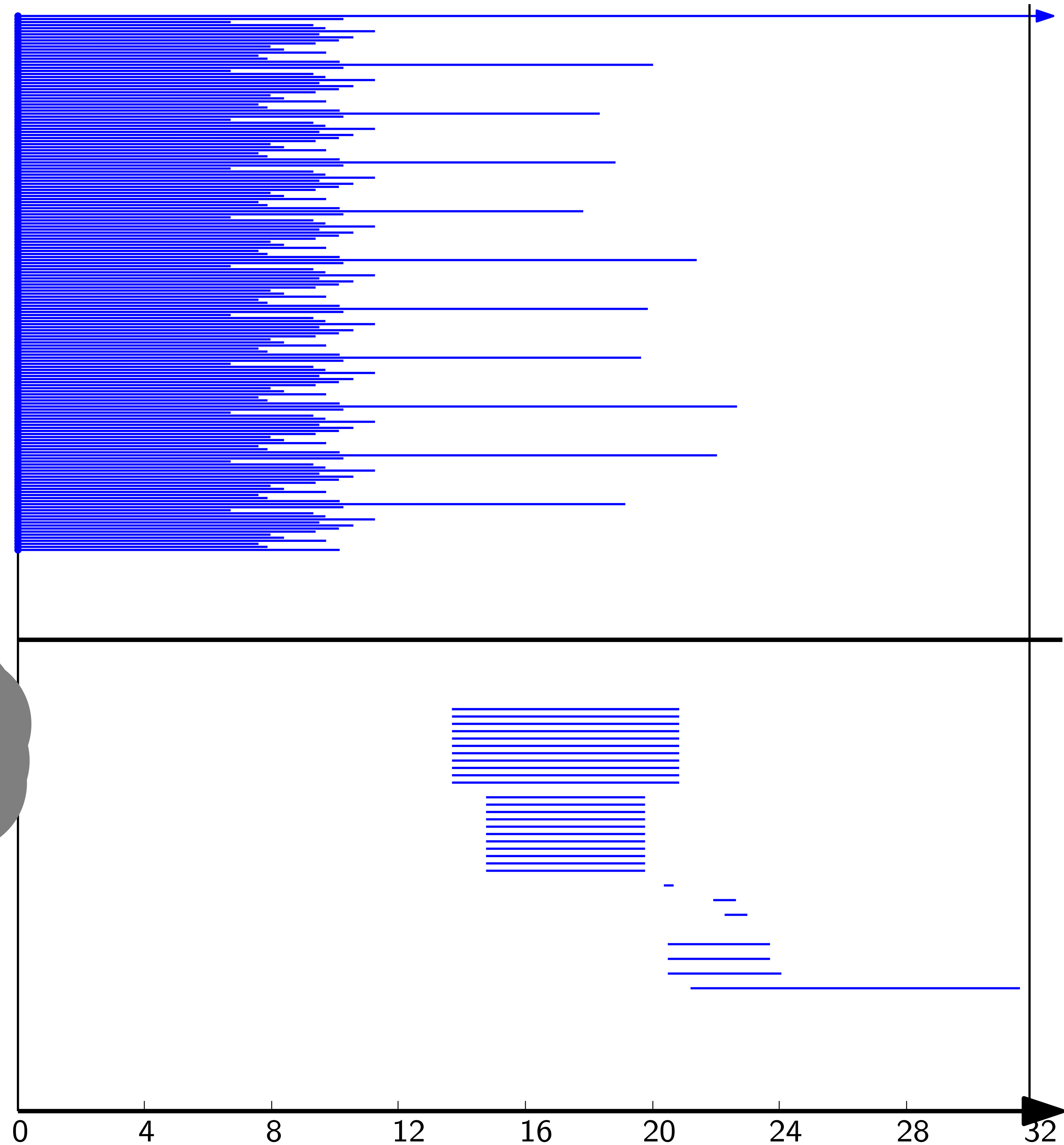

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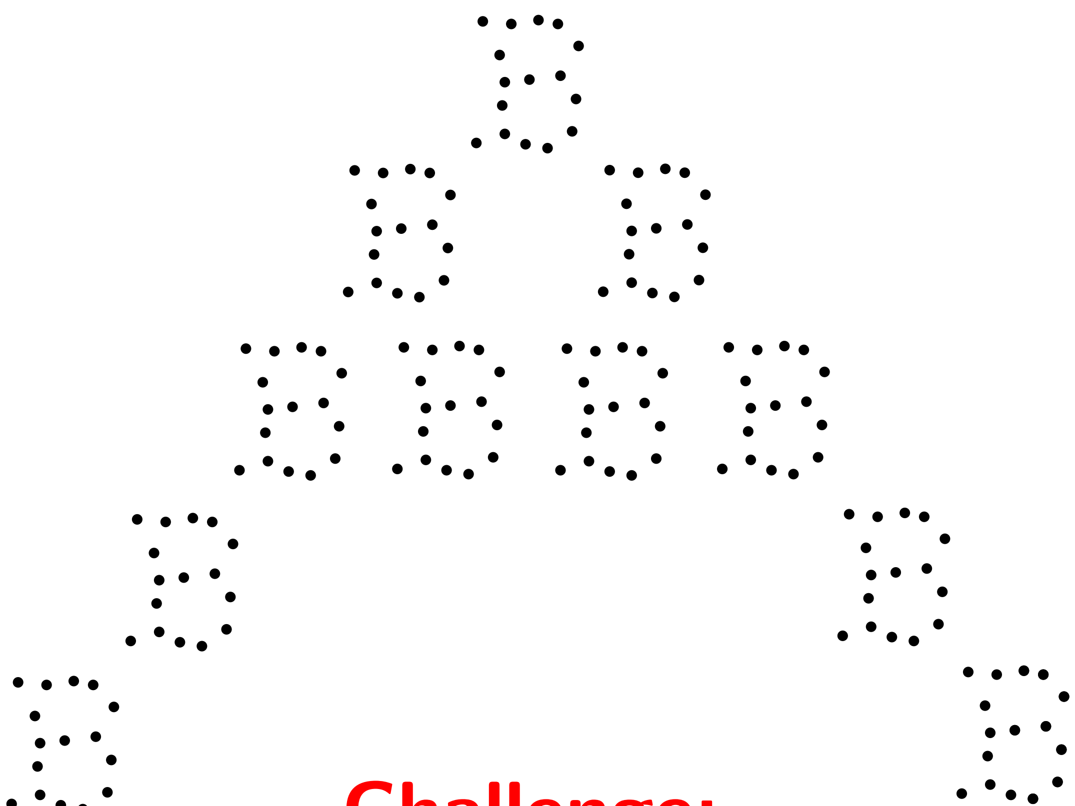
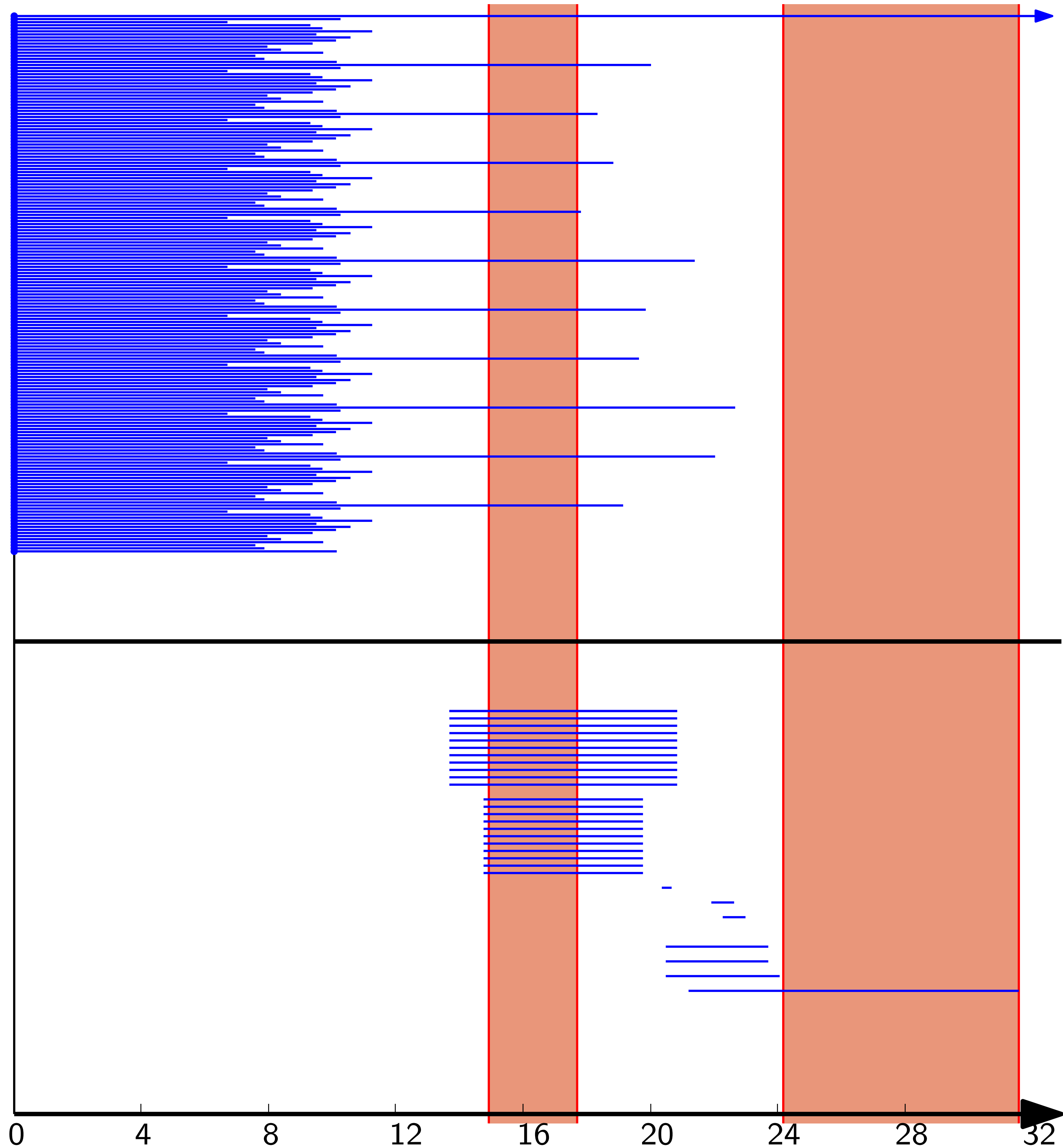
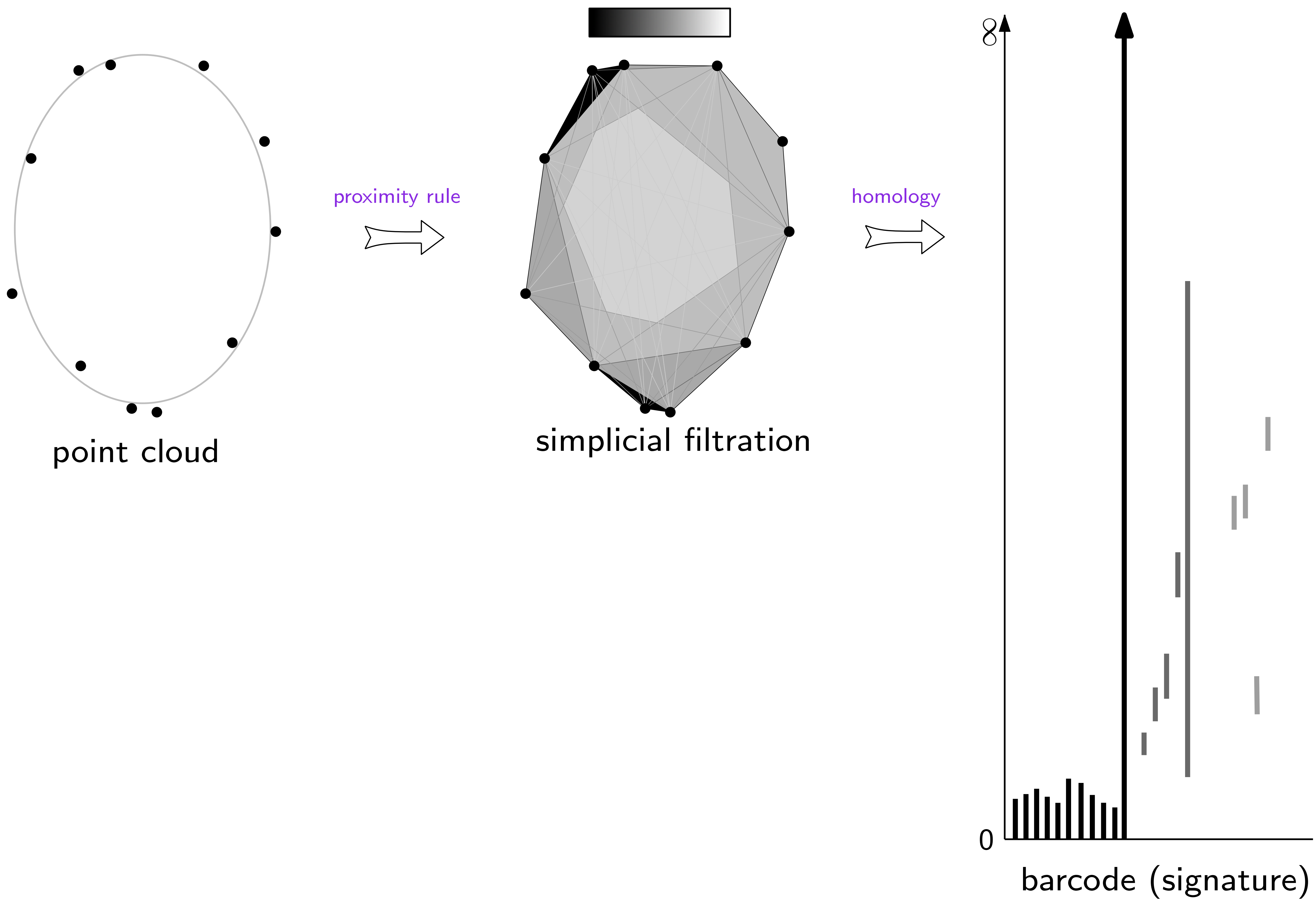





**Challenge:**
provide theoretical guarantees

(sufficient sampling conditions under which the barcode of $\mathrm{d}_P$ reveals the homology of the underlying space)

# In practice: The inference pipeline



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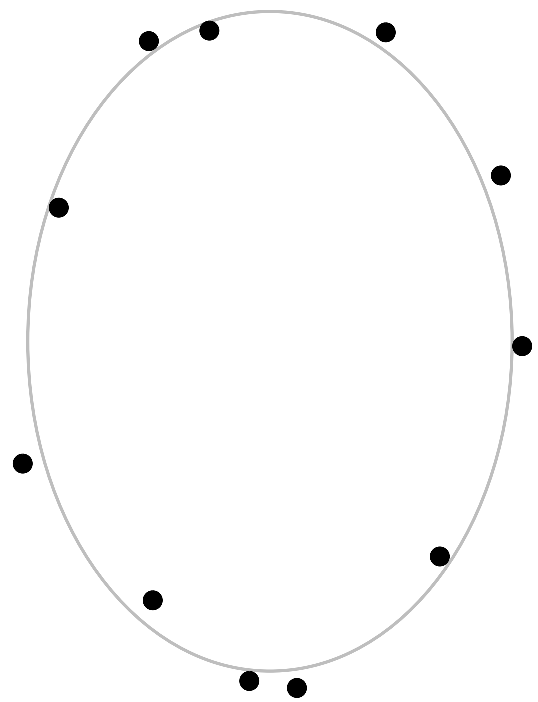
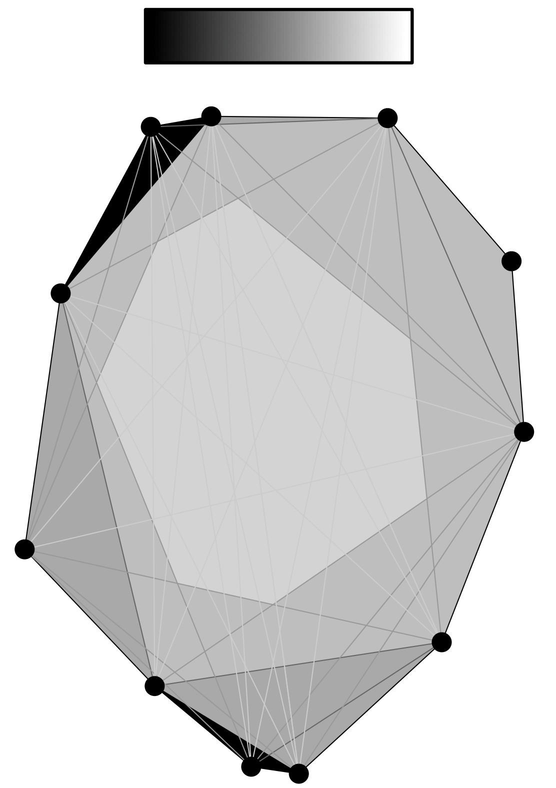


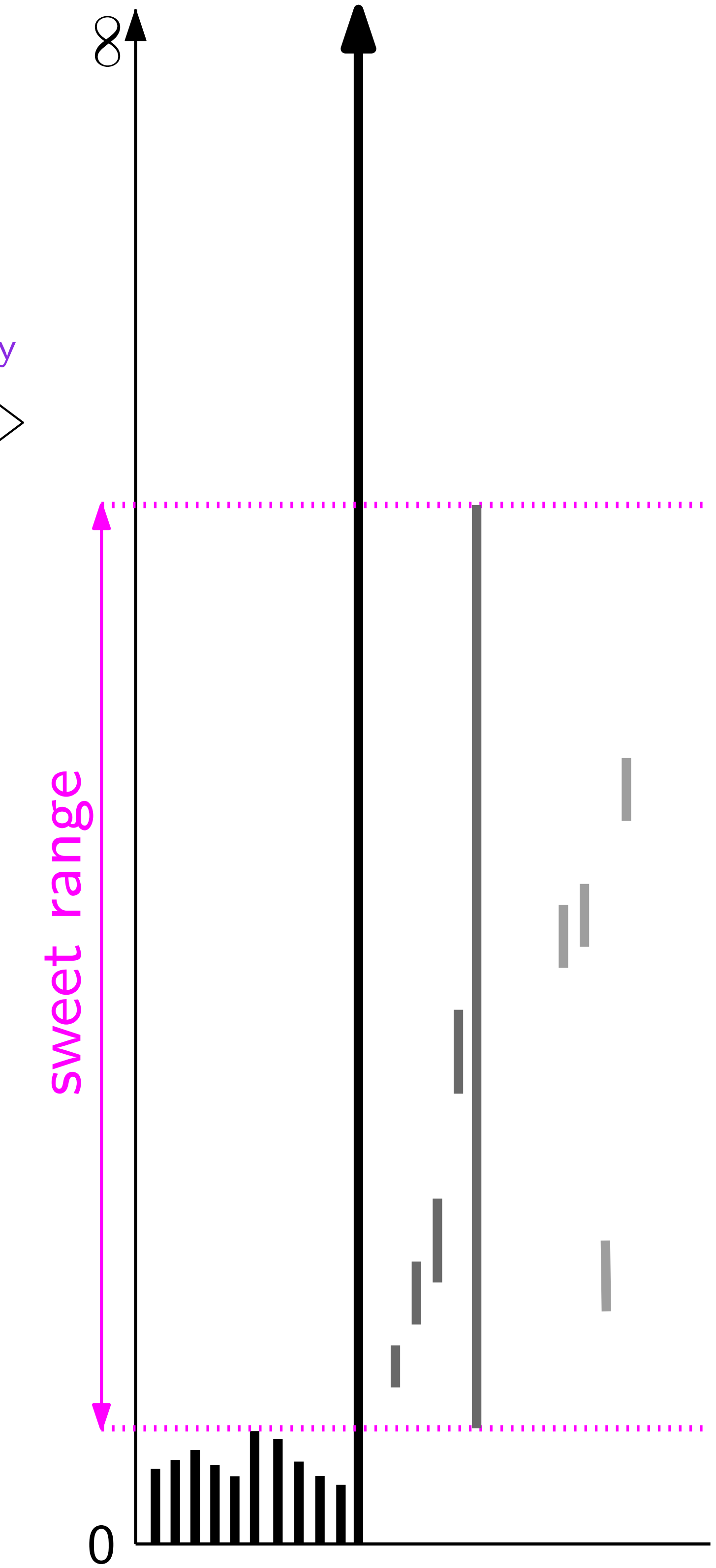
point cloud

proximity rule

homology



simplicial filtration




barcode (signature)

**Challenges:**

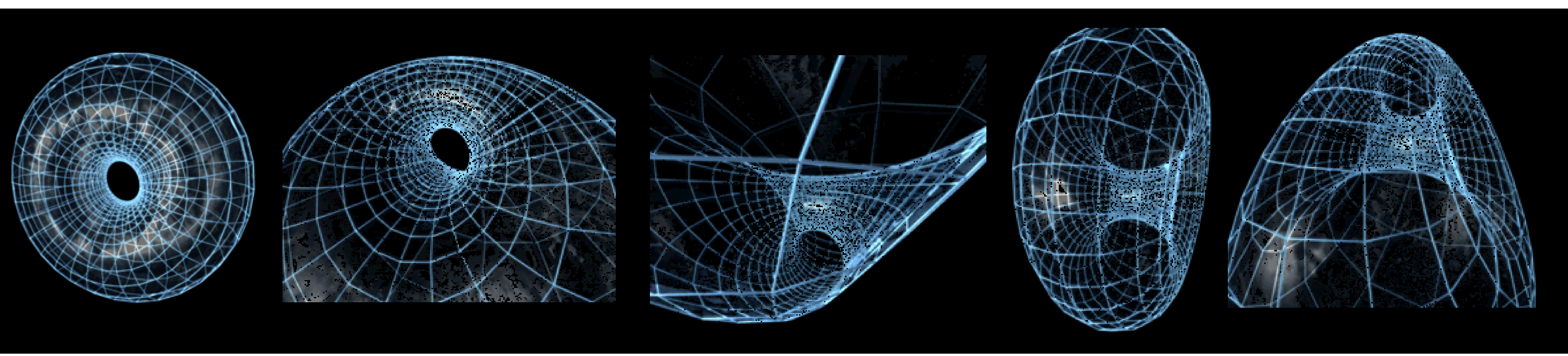
- scaling with input size and dimensionality
- theoretical guarantees (*sweet range*, SNR)

# Motivating example (manufactured data)

$(\mathbb{R} \mod \mathbb{Z})^2$

$$(u, v) \mapsto \frac{1}{\sqrt{2}}(\cos(2\pi u), \sin(2\pi u), \cos(2\pi v), \sin(2\pi v))$$

$\subset \mathbb{S}^3 \subset \mathbb{R}^4$



source: `http://http://en.wikipedia.org/wiki/Clifford_torus`

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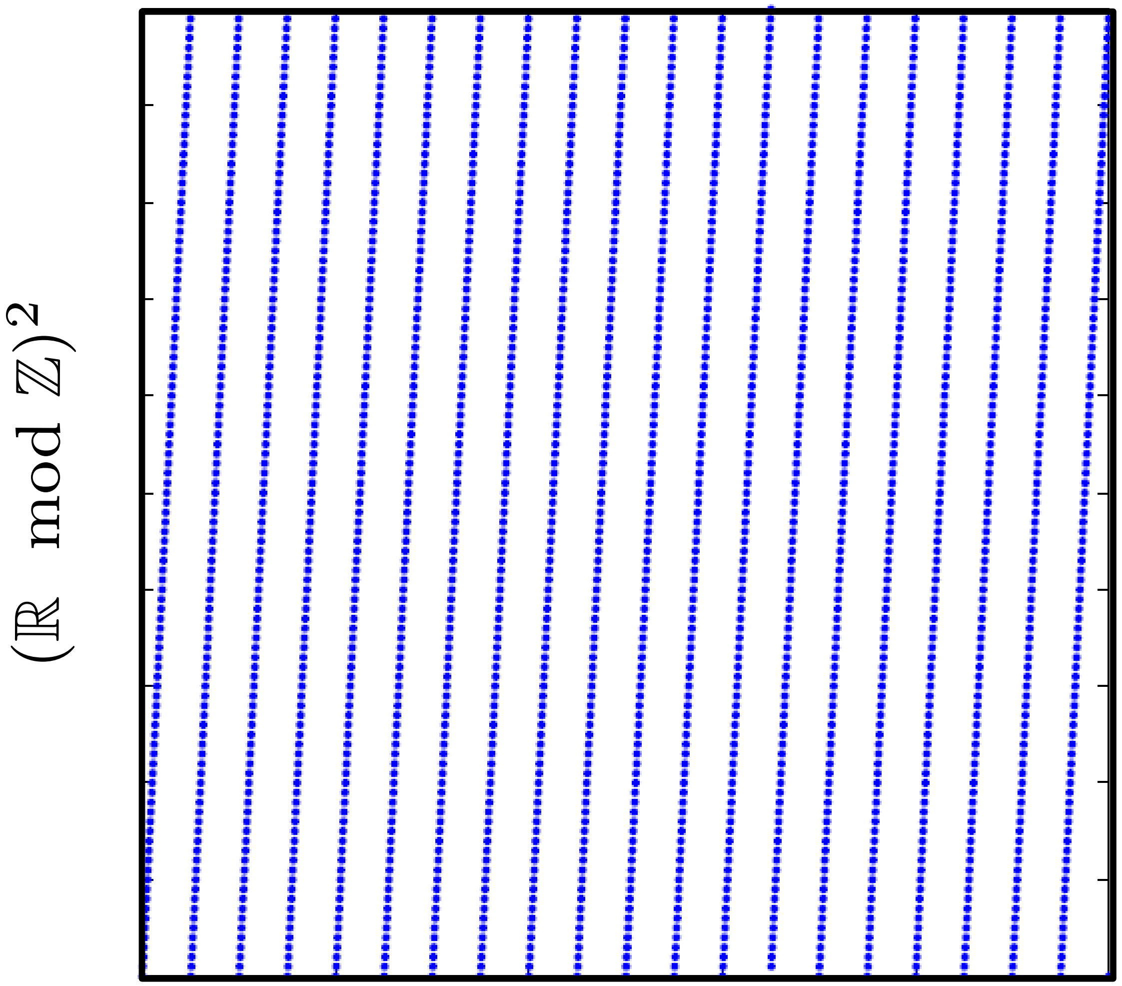


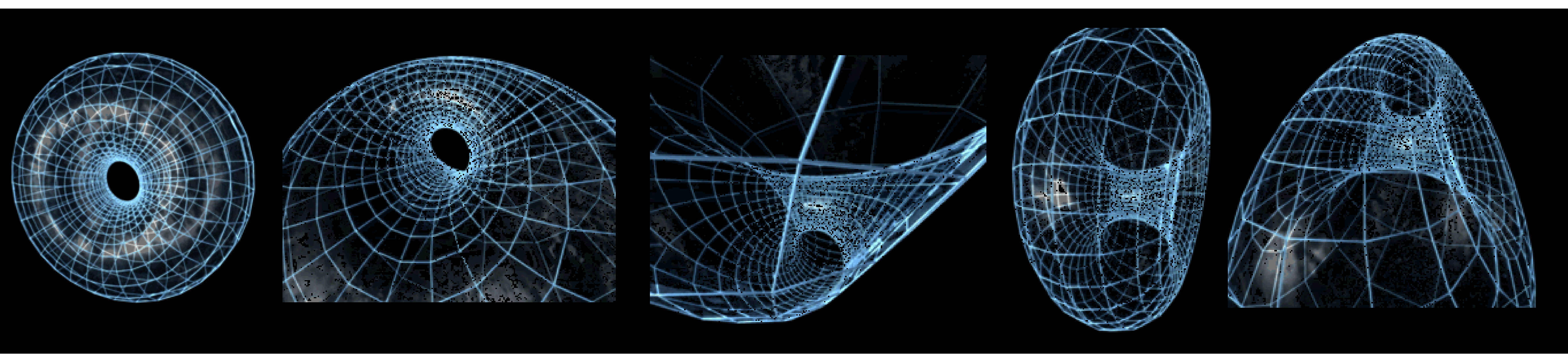

$(u, v) \mapsto \frac{1}{\sqrt{2}}(\cos(2\pi u), \sin(2\pi u), \cos(2\pi v), \sin(2\pi v))$

$n = 2000$ data points

ambient dimension $d = 4$

intrinsic dimension $k = 1, 2, 3$

$\subset \mathbb{S}^3 \subset \mathbb{R}^4$



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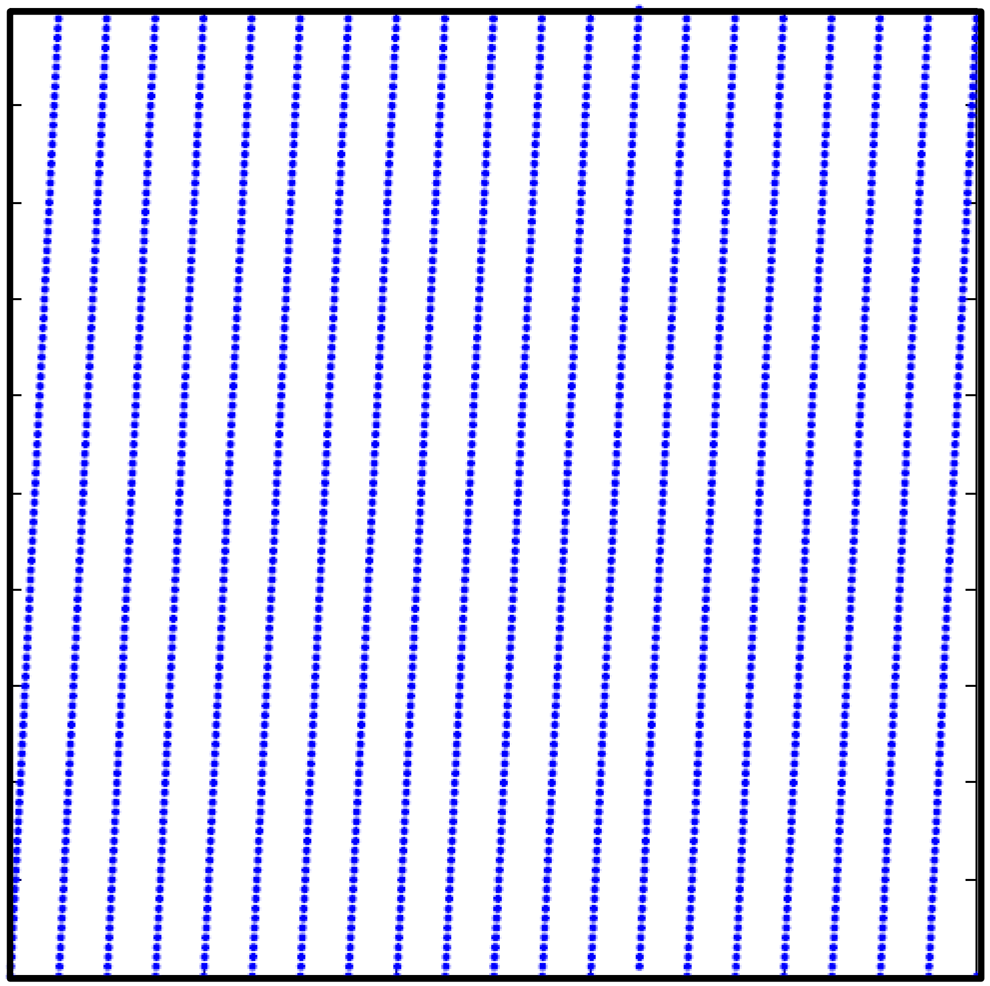

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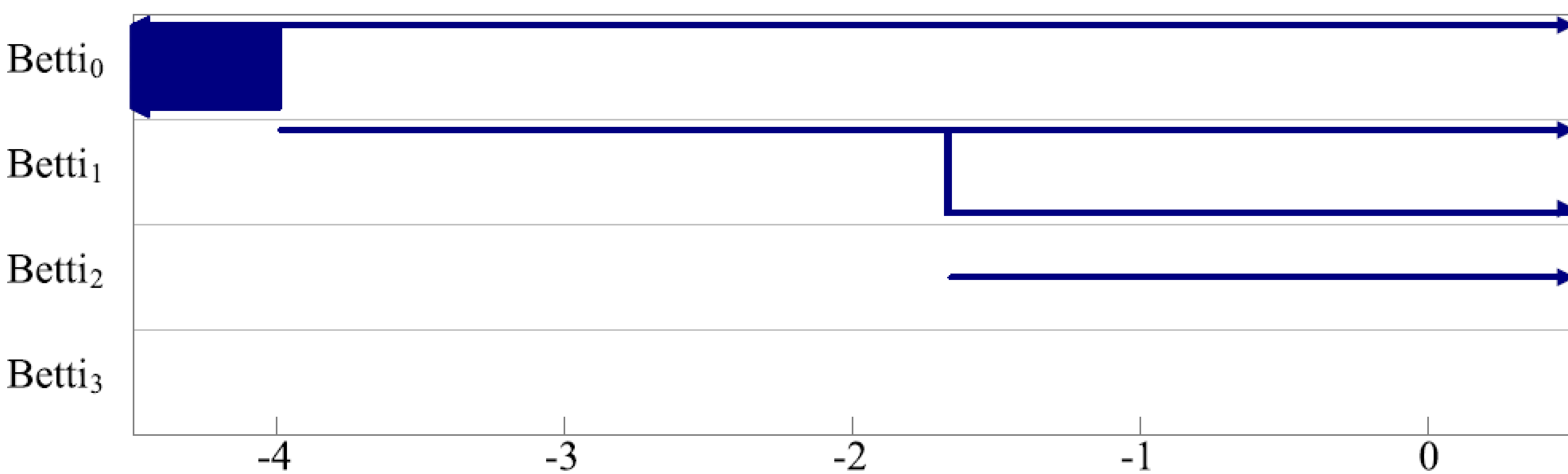


Vietoris-Rips filtration

size $\sim 2^n \mid n^{d+1}$

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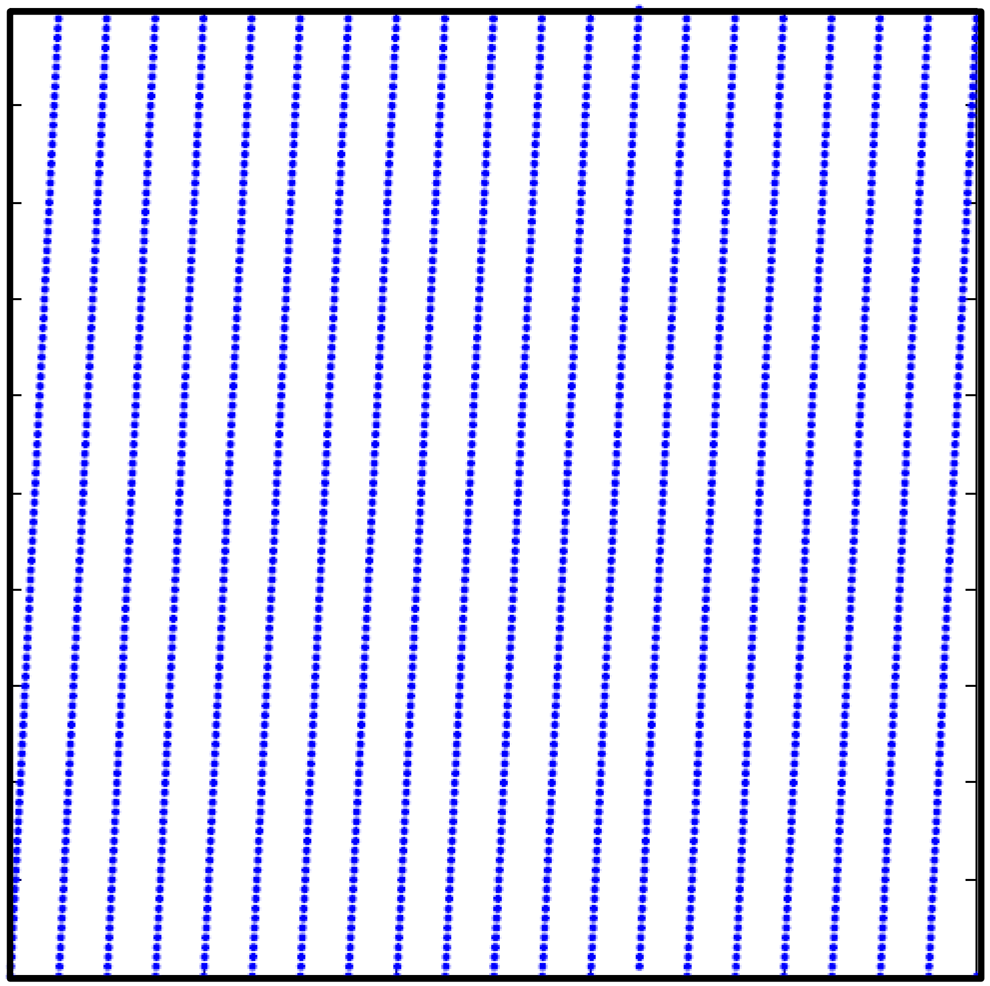
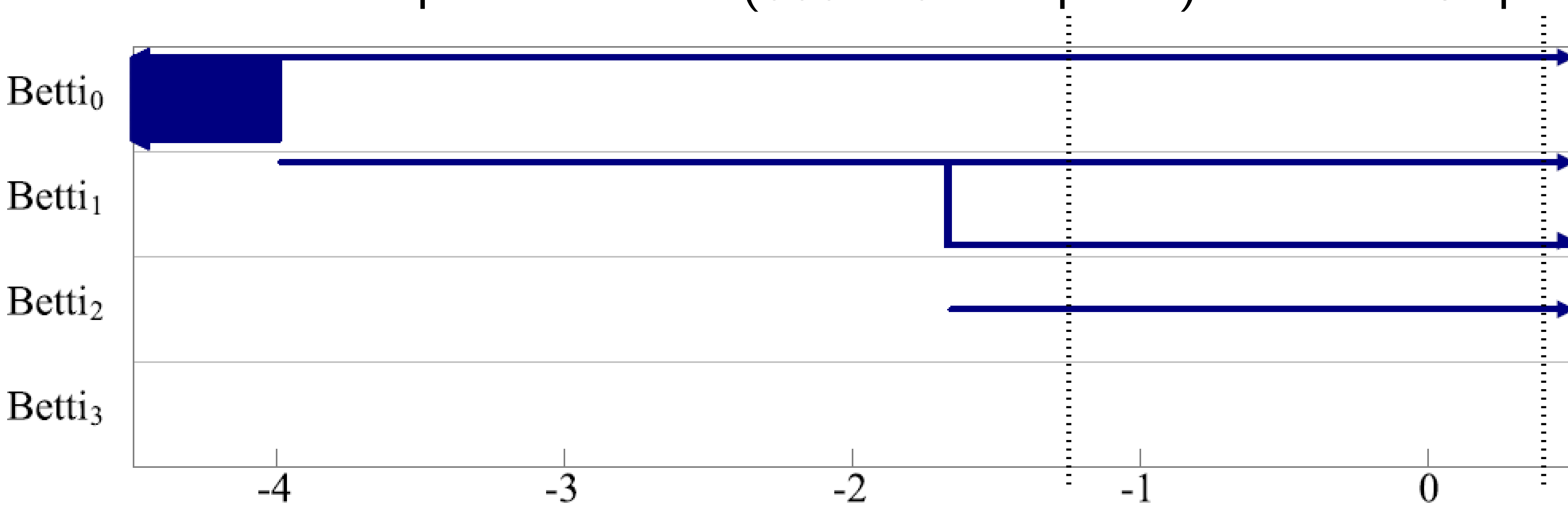


$n = 2000$ data points
ambient dimension $d = 4$
intrinsic dimension $k = 1, 2, 3$

computation limit $(500 \cdot 10^6$ simplices)

3-sphere $(37 \cdot 10^9$ simplices)

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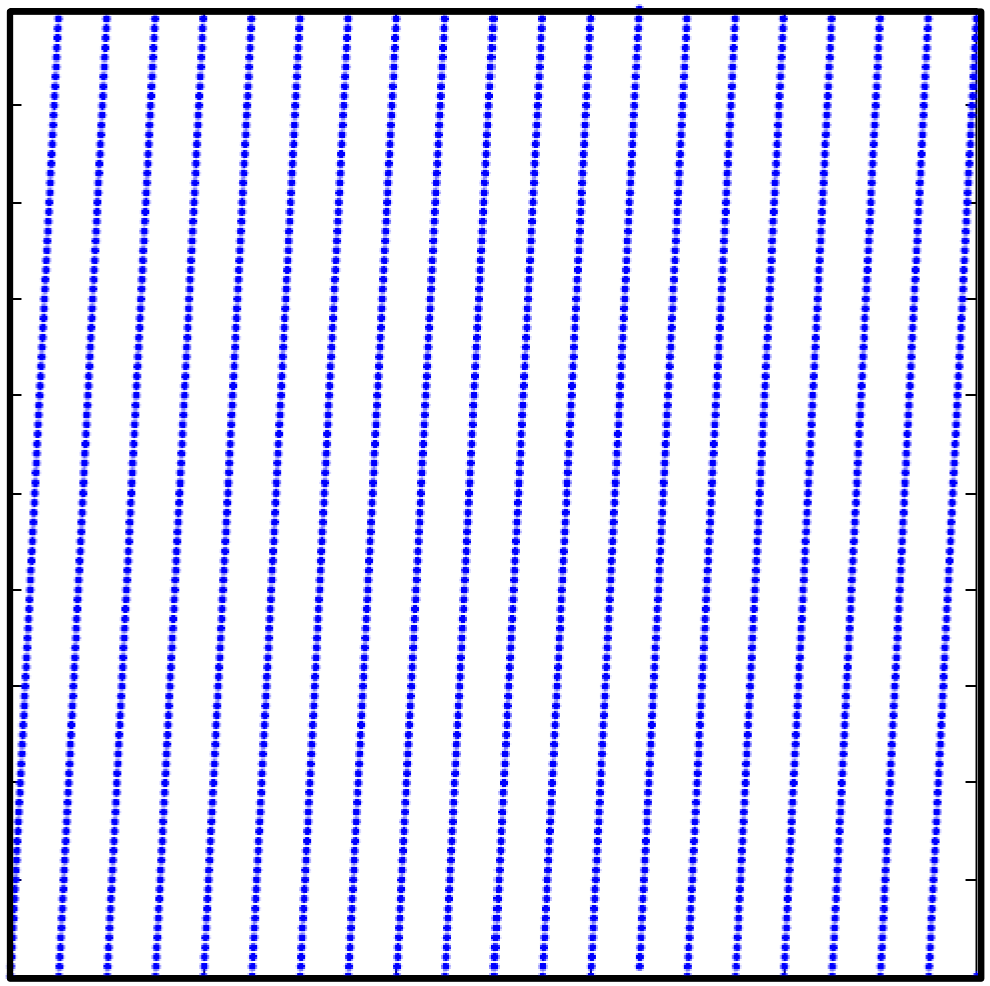
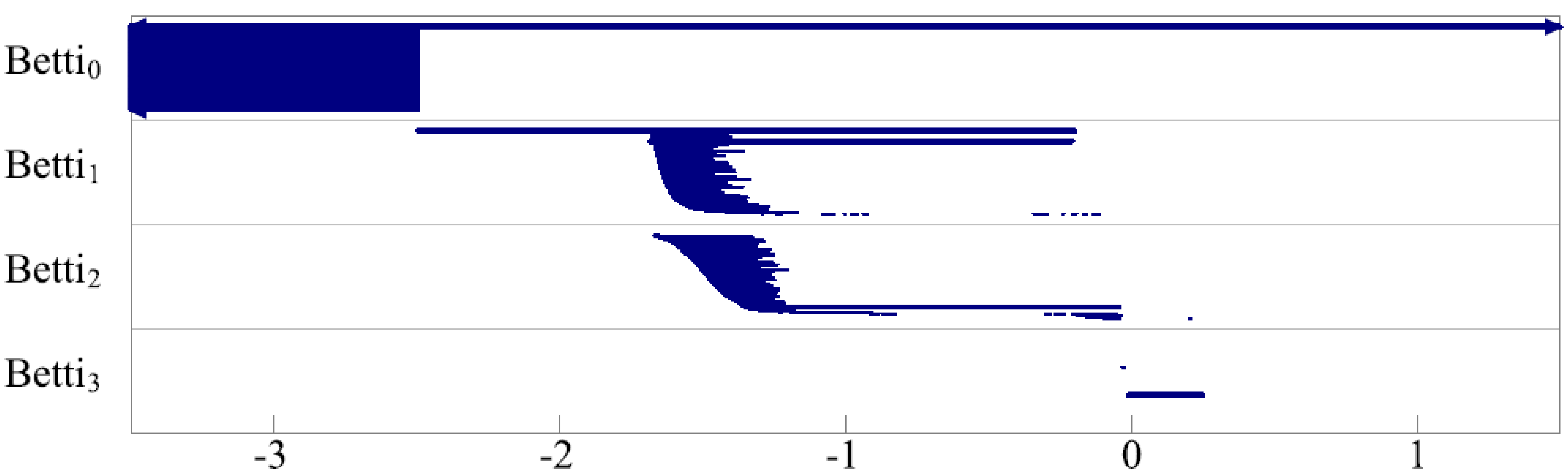


$n = 2000$ data points
ambient dimension $d = 4$
intrinsic dimension $k = 1, 2, 3$

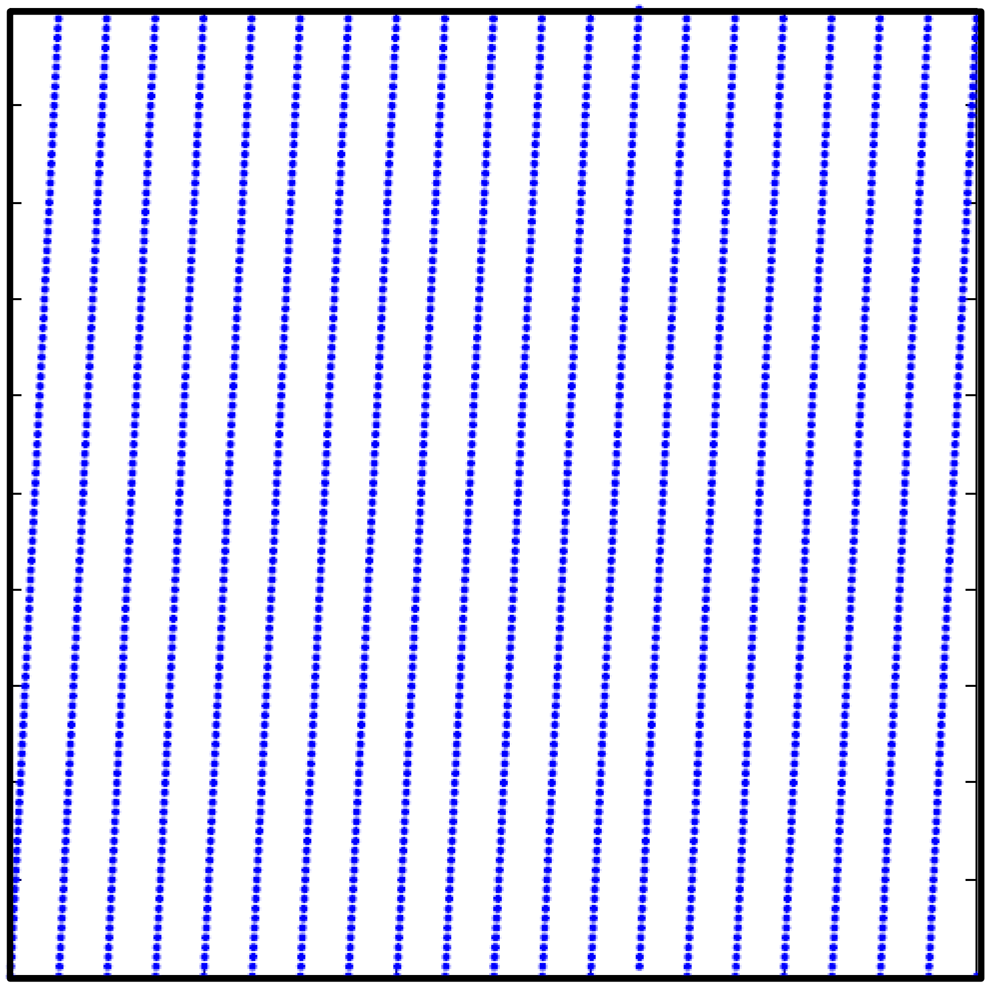
($12 \cdot 10^6$ simplices)




mesh-based filtration

size $\sim 2^{d^2} n$

# Motivating example (manufactured data)

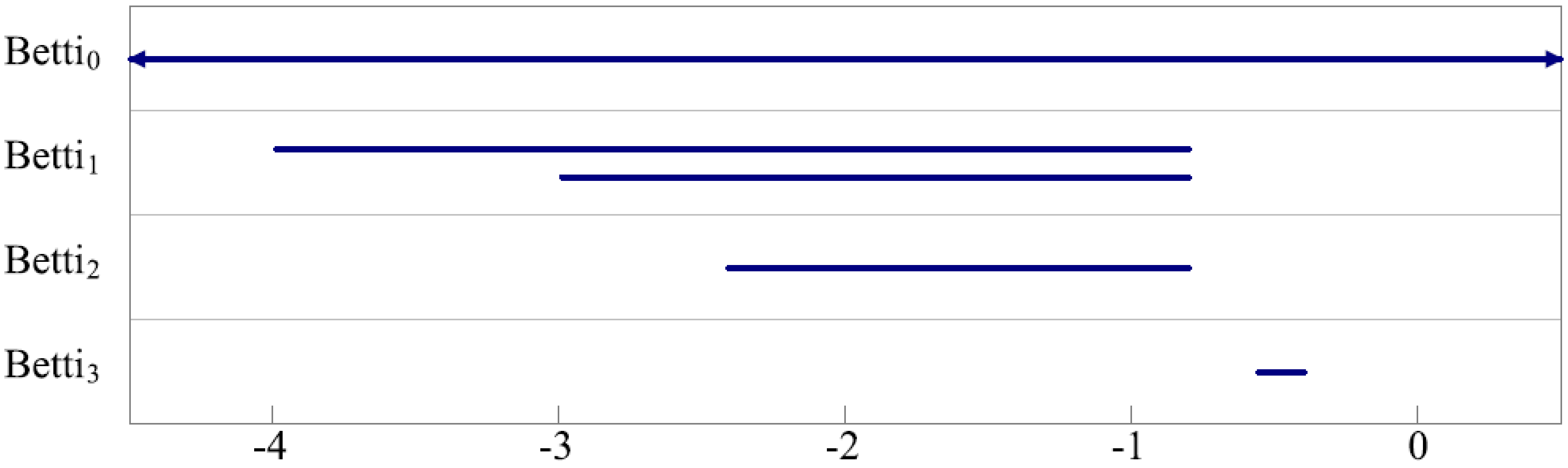


$n = 2000$ data points

ambient dimension $d = 4$

intrinsic dimension $k = 1, 2, 3$

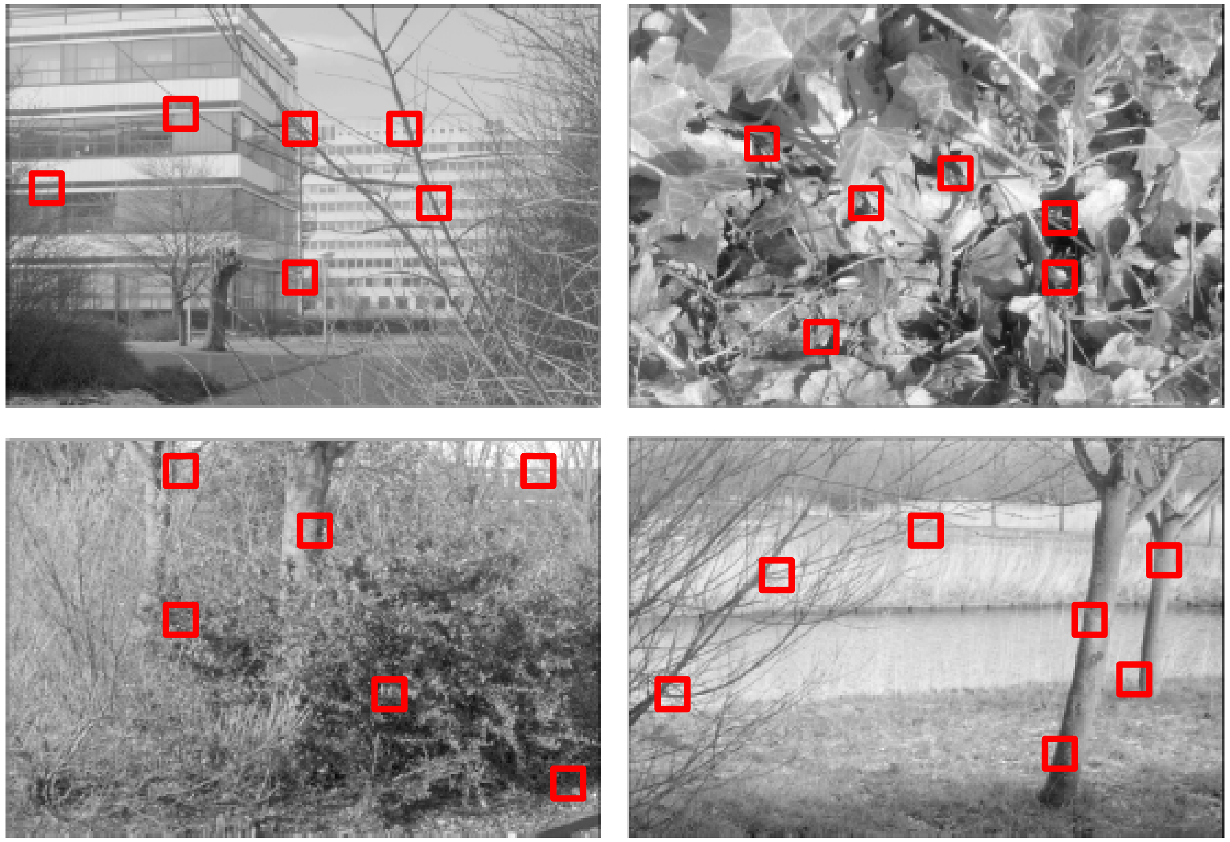
($200 \cdot 10^3$ simplices)




Rips zigzags

size $\sim 2^{k^2} n$

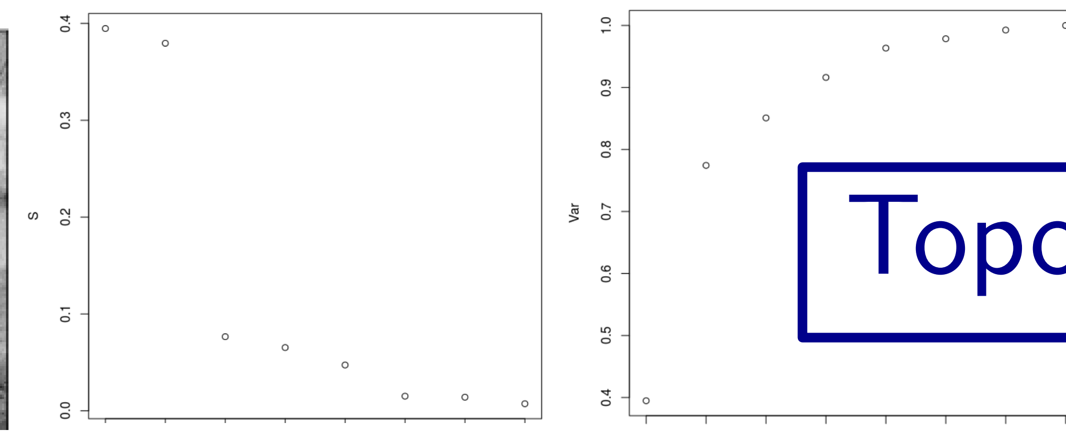
# Natural images data



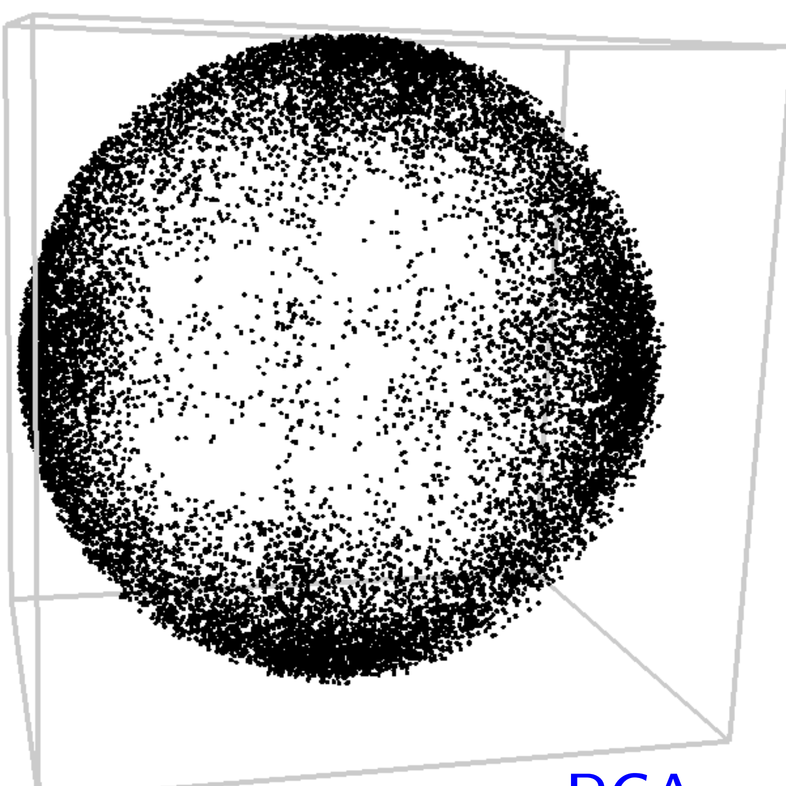
$4$ million data points in $\mathbb{R}^9$

(source: [Lee, Pederson, Mumford 2003])

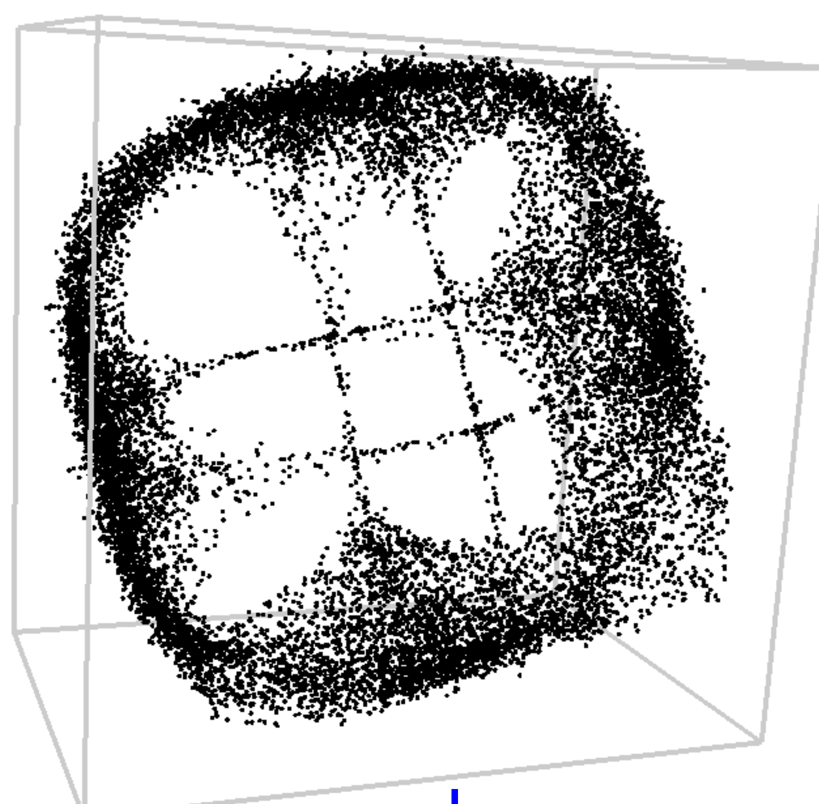
Motivation: study cognitive representation of space of images



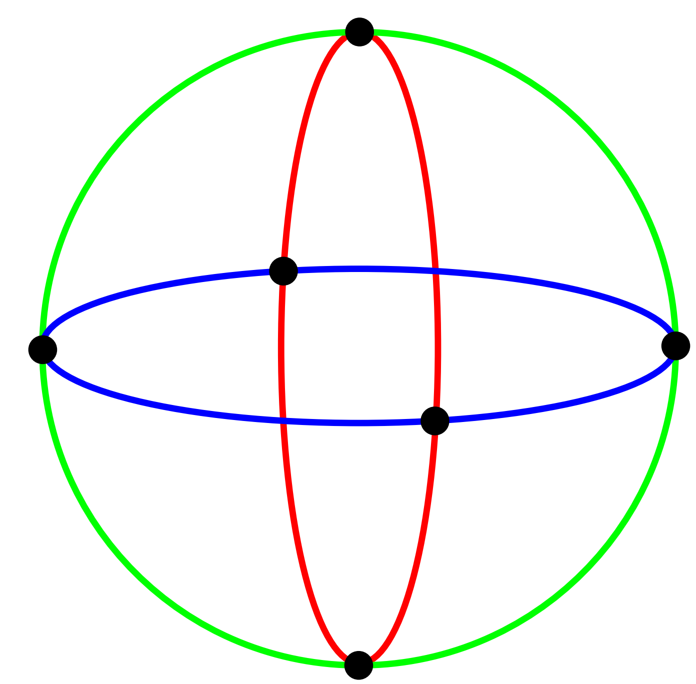
Topology



PCA

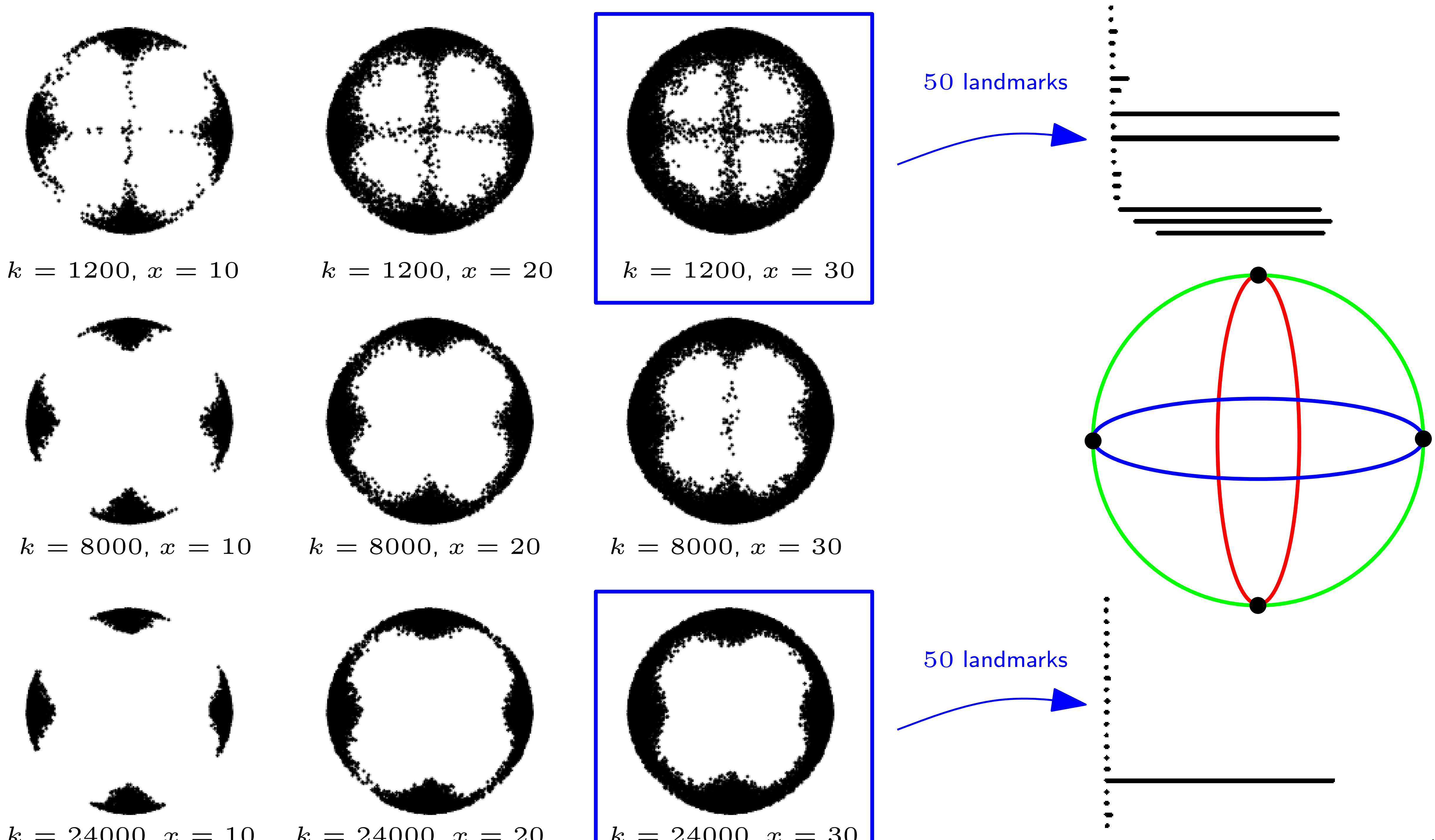


Isomap



# Natural Images Data

**Preprocessing:** - select bottom $x\%$ of data points according to $k$-NN distance
- sample 5000 points uniformly at random from filtered point set



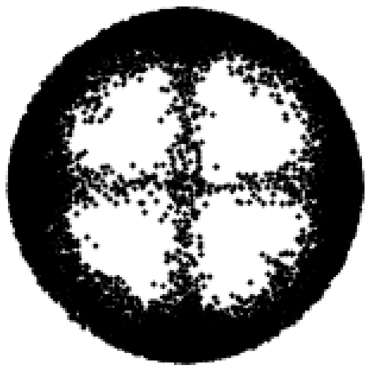

(source: [de Silva, Carlsson 04])

# Natural Images Data

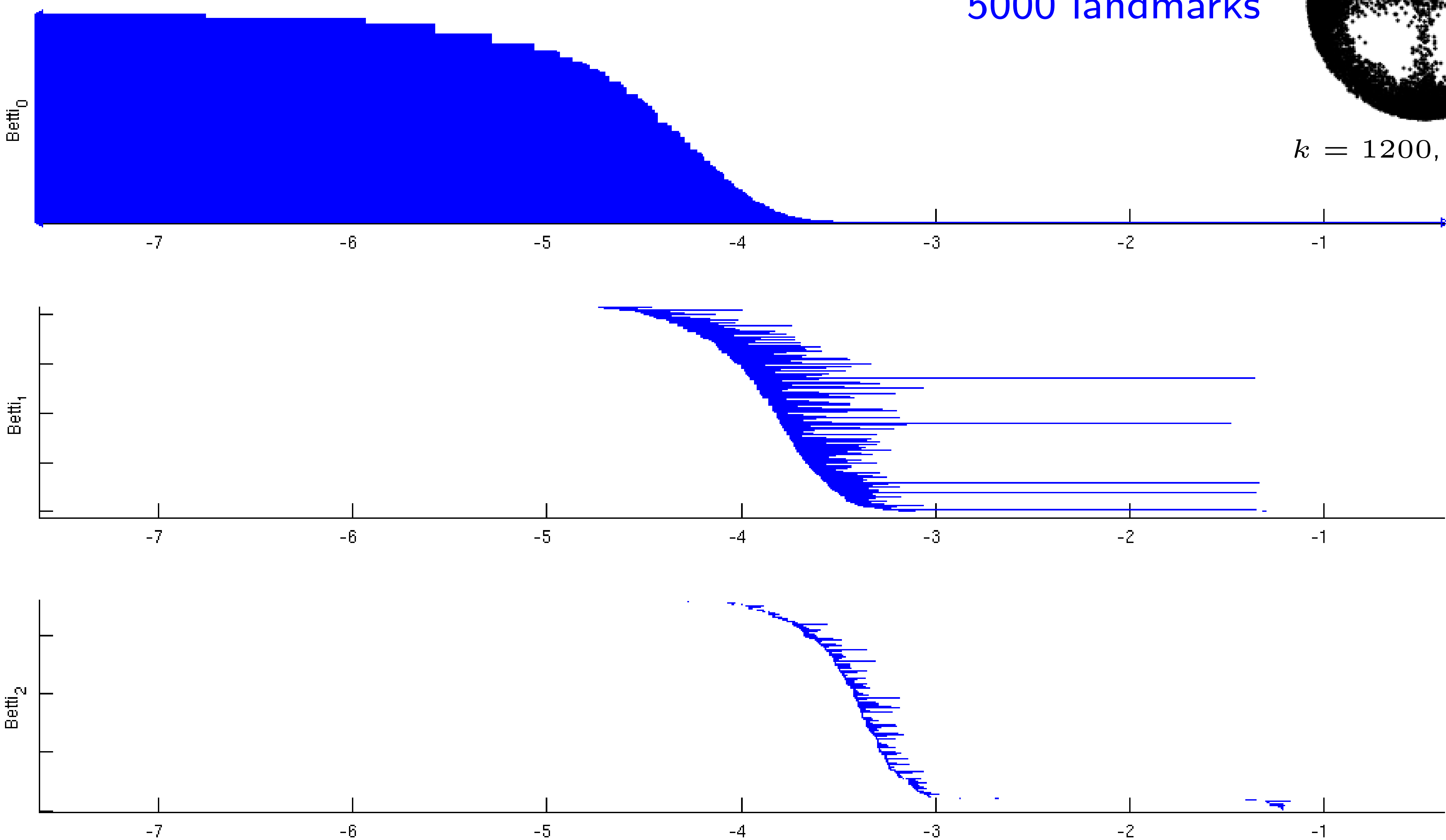
**Preprocessing:** - select bottom $x\%$ of data points according to $k$-NN distance

- sample 5000 points uniformly at random from filtered point set

5000 landmarks



$k = 1200$, $x = 30$

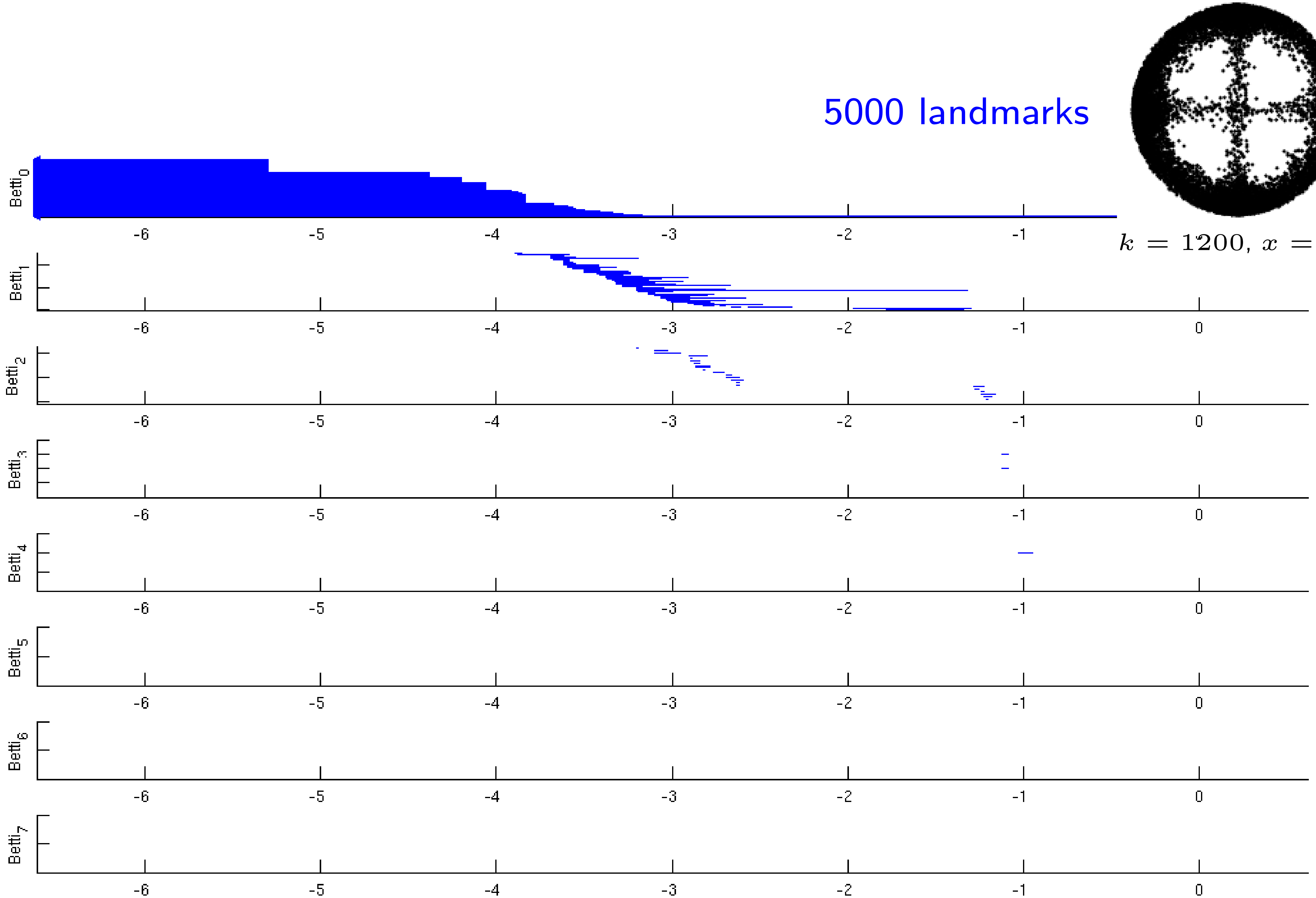



(source: [O., Sheehy 13])

# Natural Images Data

**Preprocessing:** - select bottom $x\%$ of data points according to $k$-NN distance

- sample 5000 points uniformly at random from filtered point set

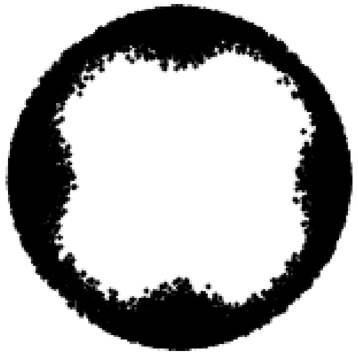



(source: [O., Sheehy 13])

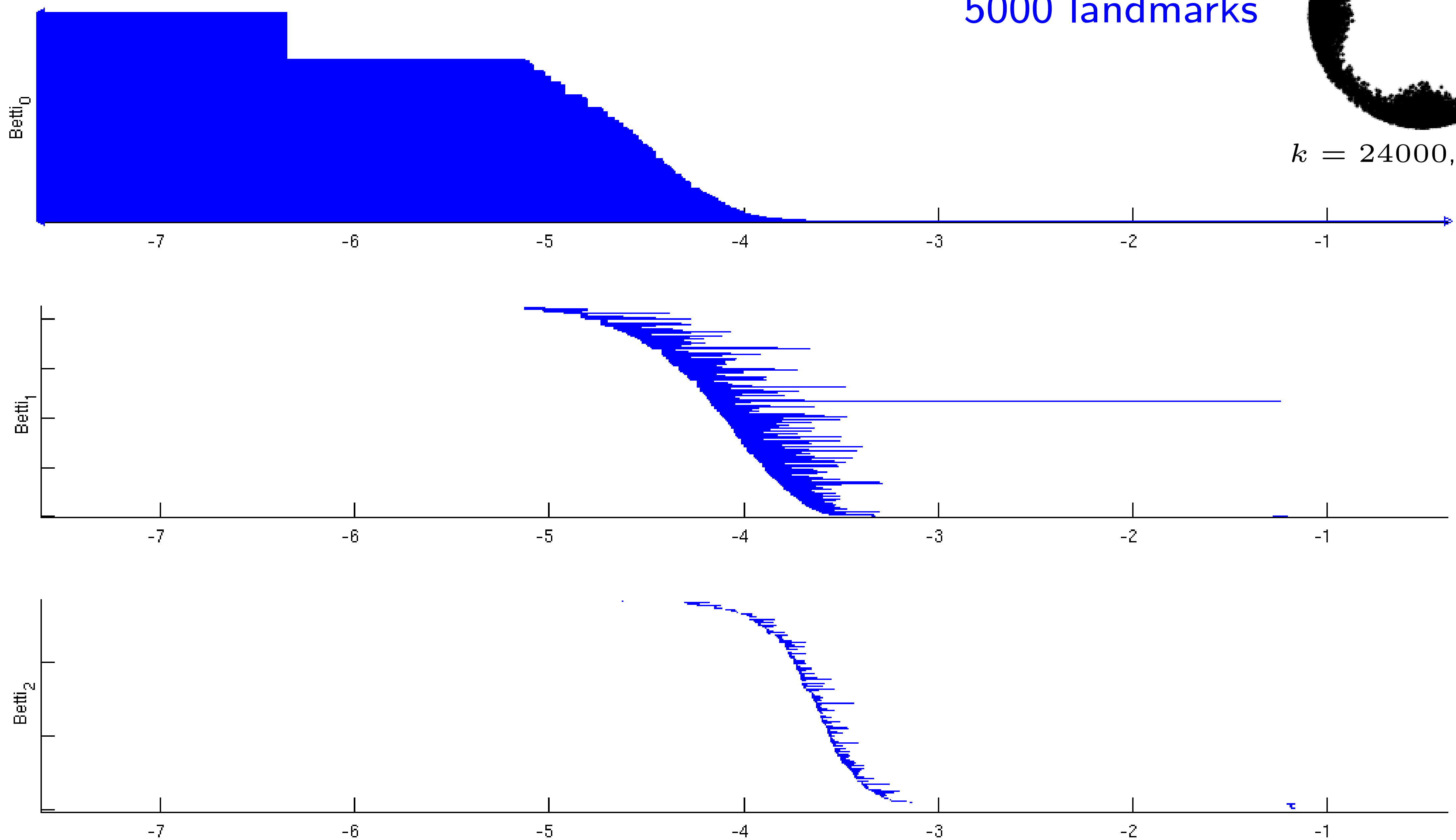
# Natural Images Data

**Preprocessing:** - select bottom $x\%$ of data points according to $k$-NN distance
- sample 5000 points uniformly at random from filtered point set

5000 landmarks



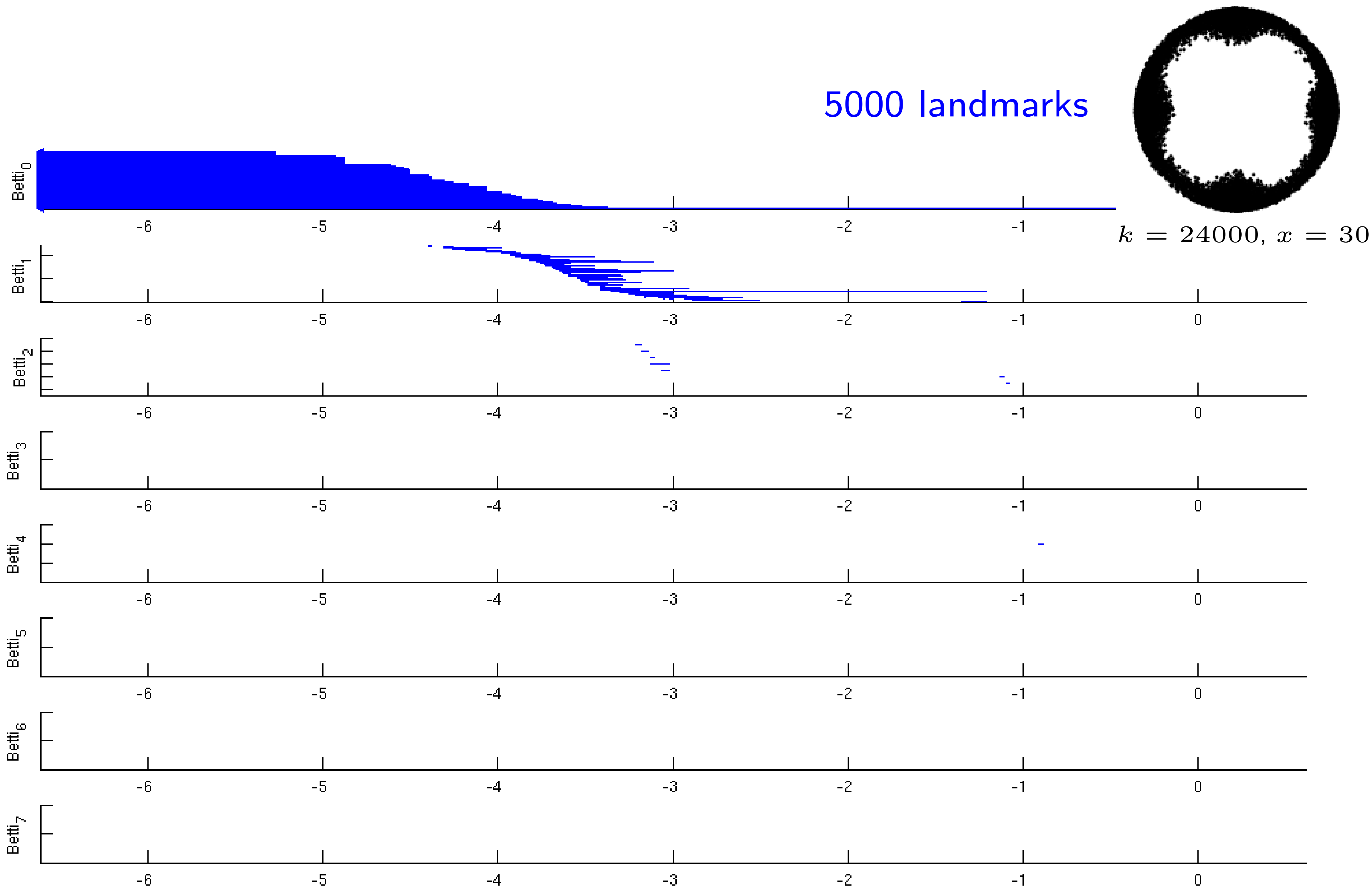
$k = 24000$, $x = 30$




(source: [O., Sheehy 13])

# Natural Images Data

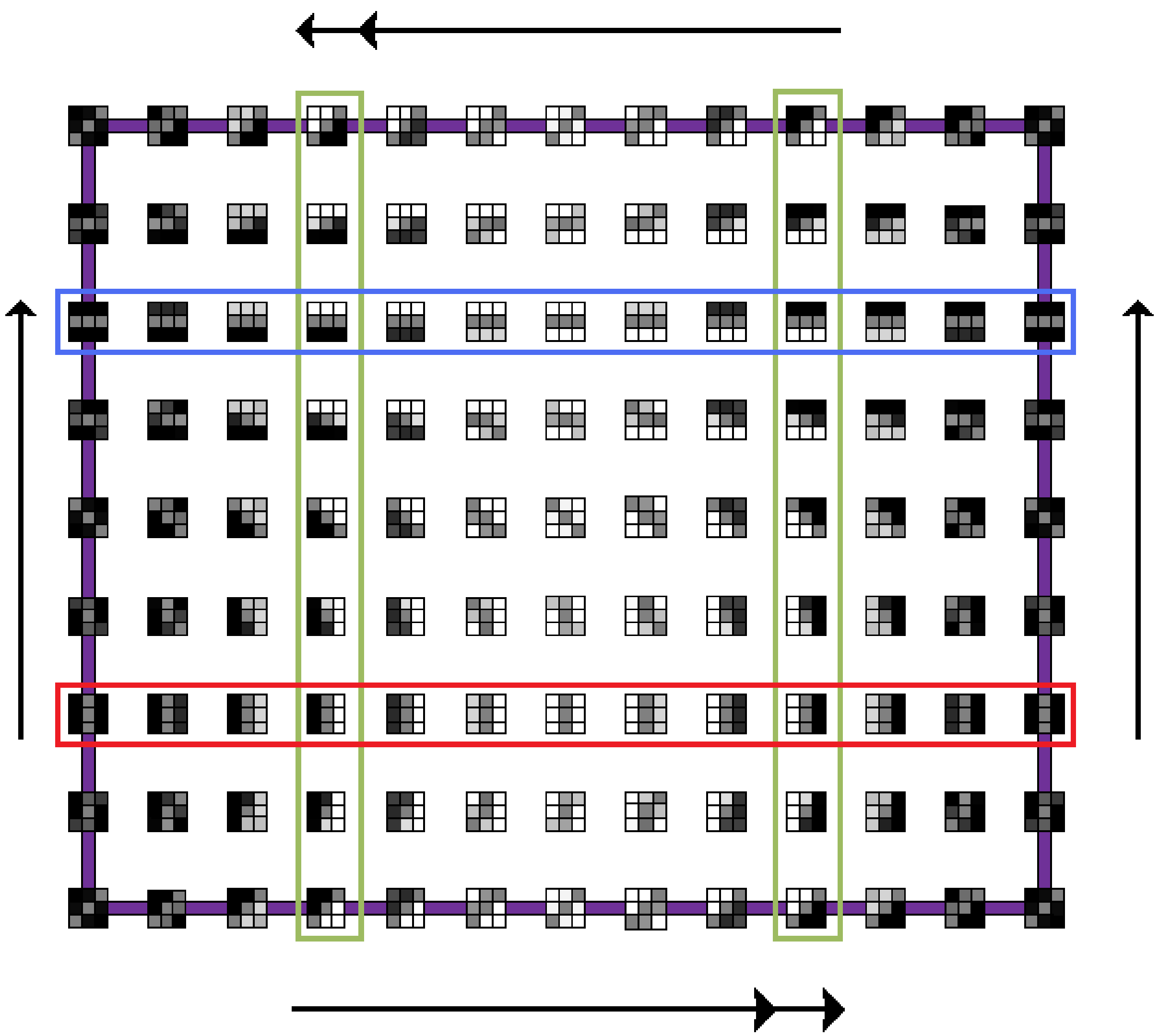
**Preprocessing:** - select bottom $x\%$ of data points according to $k$-NN distance
- sample 5000 points uniformly at random from filtered point set

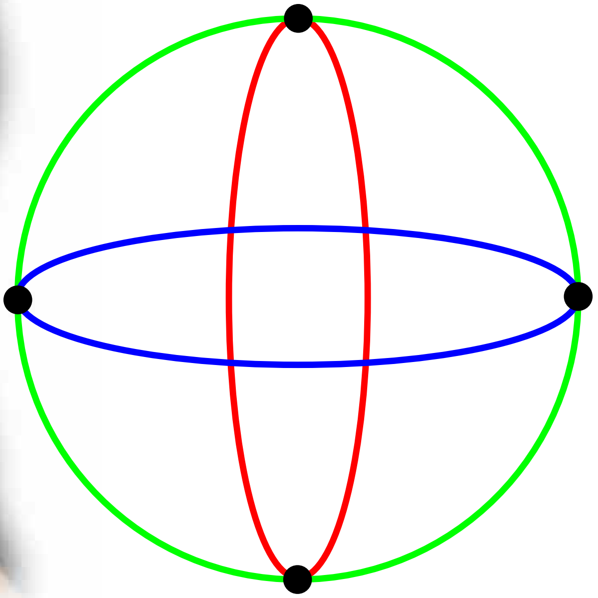



(source: [O., Sheehy 13])

# Natural Images Data

**Preprocessing:** - select bottom $x\%$ of data points according to $k$-NN distance
- sample 5000 points uniformly at random from filtered point set





(source: [Carlsson, Ishkhanov, de Silva, Zomorodian 2008])