Topological Data Analysis — Exercise Session

April 07, 2016

1 Triangulation and (persistent) homology

The dunce hat is a classical example of a space that is contractible (homotopy equivalent to a point) but not collapsible (does not deformation retracts onto a point). It is obtained by indentifying the three edges of a triangle as shown in Figure 1. Our goal here is to show by a calculation that it is at least homologically equivalent to a point.

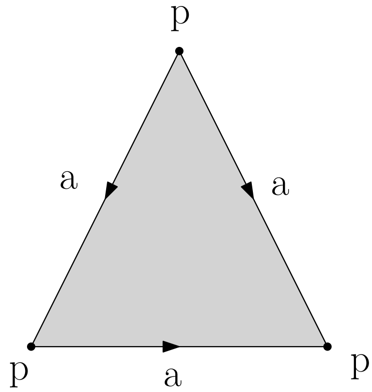

Figure 1: The Dunce Hat.

Question 1. Build a triangulation of the Dunce Hat (you may draw a picture to represent it). Beware that your triangulation must be a simplicial complex, not a general cell complex.

Question 2. Use your simplicial complex to compute the homology of the Dunce Hat.
Hint: to avoid tedious calculations, you can proceed as follows: pick a filtration of your complex then apply the persistence algorithm; for each simplex σ_j inserted, use the following property to predict its effect at the homology level (identify the created d_j-cycle or the killed (d_j-1)-cycle):

Lemma 1. *At each step j, the insertion of simplex σ_j either creates an independent d_j-dimensional cycle (i.e. increases the dimension of $\mathrm{H}_{d_j}(X_{j-1};\mathbf{k})$ by 1) or kills a (d_j-1)-dimensional cycle (i.e. decreases the dimension of $\mathrm{H}_{d_j-1}(X_{j-1};\mathbf{k})$ by 1), where d_j is the dimension of σ_j. The criterion to decide which scenario we are in is whether $\partial\sigma_j$ is already a boundary in $\mathrm{C}_{d_j-1}(X_{j-1};\mathbf{k})$ or not.*

2 Persistence diagrams

Question 3. Compute the persistence diagrams of the following functions:

- the height function on a vertical torus in $\mathbb{R}^3$,
- the Gaussian curvature on a torus in $\mathbb{R}^3$.

3 Reeb graph and Mapper

Consider the function f depicted in Figure 2.

Question 4. Compute the Reeb graph of f as well as the extended persistence diagram of the quotient map $\tilde{f}$.

Consider now the interval cover of $\operatorname{Im} f$ depicted on the left-hand side of Figure 2.

Question 5. Compute the corresponding Mapper and its signature.

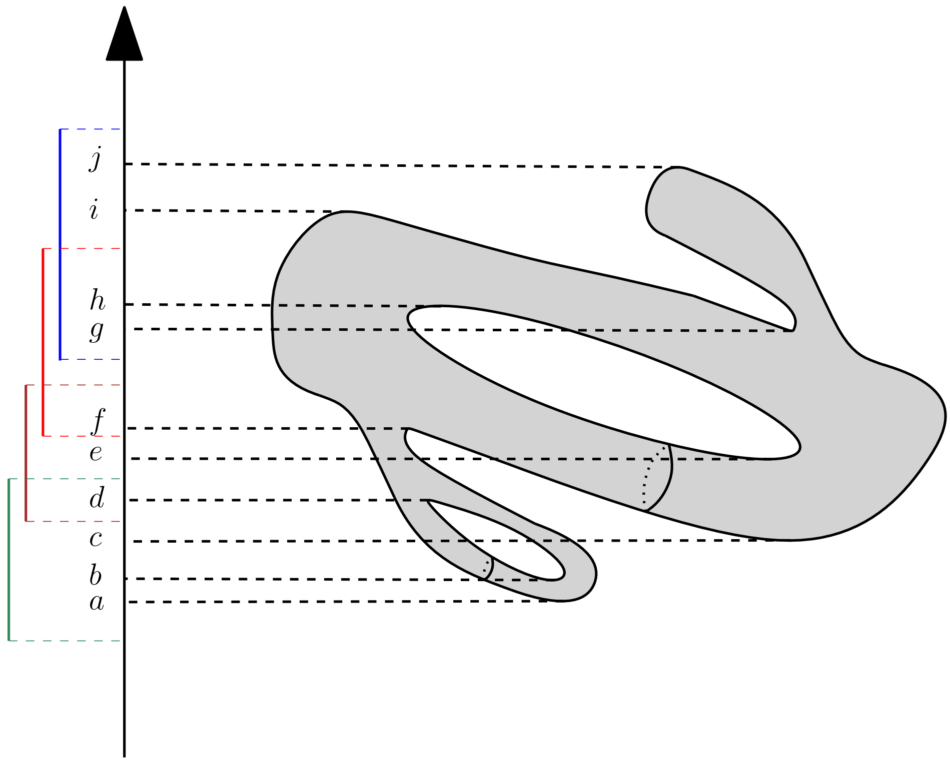

Figure 2: The height function f on a double torus.

4 [Problem] Eccentricity-based signatures

Let (X, d_X) be a finite metric space. Define the eccentricity as follows:

$$\forall x \in X,\ \operatorname{ecc}(x) = \frac{1}{2} \max\{\mathrm{d}_X(x, x') \mid x' \in X\}.$$

This function takes its values in $\mathbb{R}^+$. For any $t \in \mathbb{R}^+$, let X_t denote the t-sublevel set of ecc, that is:

$$X_t = \operatorname{ecc}^{-1}([0, t]) = \{x \in X \mid \operatorname{ecc}(x) \leq t\}.$$

Consider the filtration $\mathcal{E}(X, \mathrm{d}_X)$ defined by:

$$\forall t \in \mathbb{R}^+,\ E_t = R_t(X_t, \mathrm{d}_X).$$

where $R_t(X_t, \mathrm{d}_X)$ denotes the Rips complex of X_t of parameter t. Our goal here is to show that this filtration defines a stable signature, that is:

Theorem 2. *For any finite metric spaces* (X, d_X) *and* (Y, d_Y)*, we have*

$$\mathrm{d}_{\mathrm{b}}^{\infty}(\mathrm{Dgm}\,\mathcal{E}(X, \mathrm{d}_X),\ \mathrm{Dgm}\,\mathcal{E}(Y, \mathrm{d}_Y)) \leq 2\,\mathrm{d}_{\mathrm{GH}}(X, Y).$$

We will use the following embedding result:

Lemma 3. *Any finite metric space* (Z, d_Z) *embeds isometrically into* $(\mathbb{R}^n, \ell^\infty)$*, where* n *denotes the cardinality of* Z*.*

Question 6. Prove Lemma 3.
Hint: letting $Z = \{z_1, \cdots, z_n\}$, for each point z_i consider the vector $(\mathrm{d}_Z(z_i, z_1), \mathrm{d}_Z(z_i, z_2), \cdots, \mathrm{d}_Z(z_i, z_n)) \in \mathbb{R}^n$.

Let (X, d_X) and (Y, d_Y) be two finite metric spaces, and let $\varepsilon > \mathrm{d}_{\mathrm{GH}}(X, Y)$.

Question 7. Show that (X, d_X) and (Y, d_Y) can be jointly embedded isometrically into $(\mathbb{R}^d, \ell^\infty)$, for some $d > 0$, such that the Hausdorff distance between their images is at most ε.
Hint: look at the embedding outline in Figure 3.

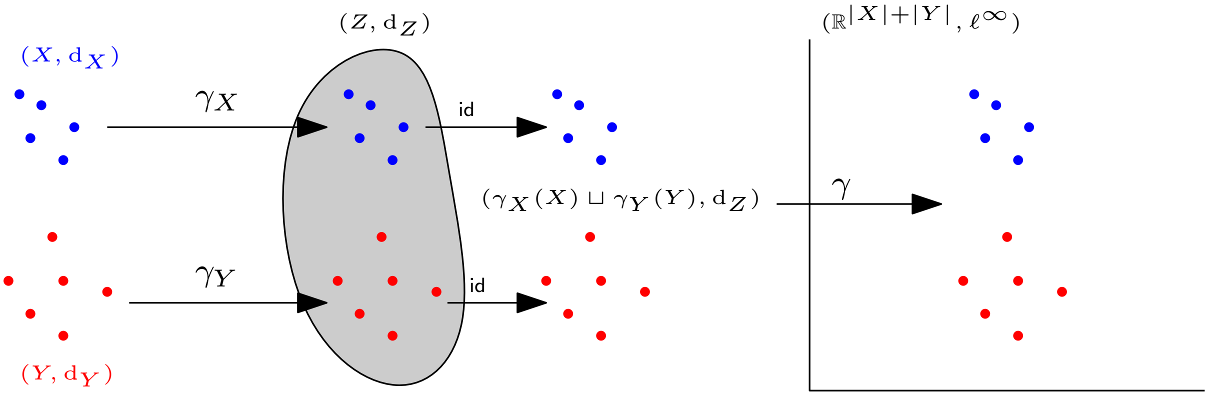

Figure 3: Outline of the embedding for the proof of Theorem 2.

We call respectively X' and Y' the images of X and Y through the joint isometric embedding.

Question 8. Show that $\mathcal{E}(X', \ell^\infty)$ is isomorphic to $\mathcal{E}(X, \mathrm{d}_X)$ as a simplicial filtration.
Hint: this means that there is a bijection $X \to X'$ that induces a bijection between the simplices of the two filtrations, such that the times of appearance of the simplices are preserved.

Similarly, $\mathcal{E}(Y', \ell^\infty)$ is isomorphic to $\mathcal{E}(Y, \mathrm{d}_Y)$. Thus, we have:

$$\mathrm{d}_{\mathrm{b}}^{\infty}(\mathrm{Dgm}\,\mathcal{E}(X, \mathrm{d}_X), \mathrm{Dgm}\,\mathcal{E}(Y, \mathrm{d}_Y)) = \mathrm{d}_{\mathrm{b}}^{\infty}(\mathrm{Dgm}\,\mathcal{E}(X', \ell^\infty), \mathrm{Dgm}\,\mathcal{E}(Y', \ell^\infty)).$$

For any finite set $S \subset \mathbb{R}^d$ and any $t \geq 0$, let S^t denote the t-offset of S in the ℓ^∞-norm, that is:

$$S^t = \{x \in \mathbb{R}^d \mid \min_{s \in S} \|x - s\|_\infty \leq t\}.$$

Question 9. Show that ${X'}_t^t \subseteq {Y'}_{t+\varepsilon}^{t+\varepsilon}$ and ${Y'}_t^t \subseteq {X'}_{t+\varepsilon}^{t+\varepsilon}$ for any $t \geq 0$.

Question 10. Define a function $f_{X'} : \mathbb{R}^d \to \mathbb{R}$ whose t-sublevel set is ${X'}_t^t$ for every $t \in \mathbb{R}^+$. Similarly, define a function $f_{Y'} : \mathbb{R}^d \to \mathbb{R}$ whose t-sublevel set is ${Y'}_t^t$ for every $t \in \mathbb{R}^+$.

Question 11. Deduce that $\|f_{X'} - f_{Y'}\|_\infty \leq \varepsilon$.

Question 12. Deduce now that $\mathrm{d}_{\mathrm{b}}^{\infty}(\mathrm{Dgm}\ f_{X'}, \mathrm{Dgm}\ f_{Y'}) \leq \varepsilon$, where Dgm h denotes the persistence diagram of the filtration of the sublevel sets of h.

Question 13. Deduce now that $\mathrm{d}_{\mathrm{b}}^{\infty}(\mathrm{Dgm}\ \mathcal{EC}(X', \ell^{\infty}), \mathrm{Dgm}\ \mathcal{EC}(Y', \ell^{\infty})) \leq \varepsilon$, where the filtration $\mathcal{EC}(Z', \ell^{\infty})$ has the space $C_t(Z'_t, \ell^{\infty})$ for every $t \in \mathbb{R}^+$ — here C_t stands for the Čech complex of parameter t.
Hint: relate the sublevel sets of $f_{X'}$ to the unions of ℓ^{∞}-balls centered at the points of X'_t, then apply the Nerve Theorem. Same for Y'.

Question 14. Deduce finally that $\mathrm{d}_{\mathrm{b}}^{\infty}(\mathrm{Dgm}\ \mathcal{E}(X', \ell^{\infty}), \mathrm{Dgm}\ \mathcal{E}(Y', \ell^{\infty})) \leq 2\varepsilon$.
Hint: relate the Čech and Rips filtrations to each other in $(\mathbb{R}^d, \ell^{\infty})$.

Question 15. Conclude.