① Kernel trick:

Def: $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a reproducing kernel if $\forall n \in \mathbb{N}$, $\forall x_1, \dots, x_n \in \mathcal{X}$, the gram matrix $(G_{ij})_{ij}$ with $G_{ij} = k(x_i, x_j)$ is positive semi-definite.

Thm: [Moore, Aronszajn]
k reproducing kernel $\Rightarrow \exists$ (essentially unique) pair $(\mathcal{H}_k, \varphi_k)$ such that $k(\cdot,\cdot) = \langle \varphi_k(\cdot), \varphi_k(\cdot) \rangle_{\mathcal{H}_k}$.

RKHS ↗ ↖ feature map

② Kernels for PDs:

$\mathcal{X}$: space of diagrams
$k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ kernel
φ_k : feature map
$\mathcal{H}_k$: RKHS, $\langle \cdot, \cdot \rangle_{\mathcal{H}_k}$, $\|\cdot\|_{\mathcal{H}_k}$

- stability:

$\forall D, D' \in \mathcal{X}$, $\|\varphi_k(D) - \varphi_k(D')\|_{\mathcal{H}_k} \leq C \cdot d_B^\infty(D, D')$. or $C \cdot W_p(D, D')$.

- discriminativity: $\forall D, D' \in \mathcal{X}$, $\|\varphi_k(D) - \varphi_k(D')\|_{\mathcal{H}_k} \geq C' \cdot d_B^\infty(D, D')$.

- injectivity: $\varphi_k : \mathcal{X} \to \mathcal{H}_k$ is injective

- universality: $\mathcal{H}_k := \text{span}\{k_x : x \in \mathcal{X}\}$ is dense in $C^0(\mathcal{X}, \mathbb{R})$.

- additivity: $k(D \cup D', D'') = k(D, D'') + k(D', D'')$ $\forall D, D', D'' \in \mathcal{X}$.

↳ overkill → characteristic: the mean map $P \mapsto \mu_P = \mathbb{E}_{x \sim P}[k_x]$ is injective
(proba. measure on $\mathcal{X}$)

→ reminder:

$c_\infty(M : A \leftrightarrow B) = \max \left\{ \sup_{(a,b) \in M} \|a - b\|_\infty , \sup_{\substack{s \in A \cup B \\ \text{unmatched}}} \|s - \bar{s}\|_\infty \right\}$ ↑ ortho. proj. onto Δ

→ $c_p(M : A \leftrightarrow B) = \left(\sum_{(a,b) \in M} \|a - b\|_\infty^p + \sum_{\substack{s \in A \cup B \\ \text{unmatched}}} \|s - \bar{s}\|_\infty^p \right)^{1/p}$.

→ $W_p(A, B) = \inf_{M : A \leftrightarrow B} c_p(M)$.

③ Landscapes:

Note: pts at infinity are removed.

$D \longmapsto \lambda_D : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$

$\lambda_D \in L^2(\mathbb{N} \times \mathbb{R})$

↳ equipped with the product of μ
- → counting measure on $\mathbb{N}$
- → Lebesgue measure on $\mathbb{R}$

$$\int_{\mathbb{N}\times\mathbb{R}} \lambda_D \, d\mu := \sum_{k=1}^{\infty} \int_{\mathbb{R}} \lambda_D(k,t)\, dt$$

Note: $\sum_k \int_{\mathbb{R}} \lambda_D(k,t)^2 dt < +\infty$ ~~therefore~~ when $|D|$ is finite.

ie: $\lambda_D \in L^2(\mathbb{N}\times\mathbb{R})$ when $|D|$ is finite.

④ Discrete measures:

$D = \sum_{p\in D} \delta_p$ (no ~~renormalization~~ normalization)

Heat equation: $u : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta_x u(x,t) \\ u|_{x\in\Delta} = 0 \quad \forall x \in \Delta \text{ (diagonal) and } t \in \mathbb{R}_{\geq 0} \text{ (Dirichlet cond.)} \\ u|_{t=0} = \sum_{p\in D} \delta_p - \delta_{\bar{p}} . \end{cases}$$

↳ $\phi_\lambda :=$ solution $u \Big|_{t=\sigma}$.

↳ $k := \langle \phi_\lambda(\cdot), \phi_\lambda(\cdot) \rangle_{L^2(\mathbb{R}^2)}$.

(5) Bootstrap:

- View $\hat{X}_n$ itself as an observation of a random variable $\sim \mu^{\otimes n}$ (n pts sampled iid according to μ).

- Study the statistics of $\underbrace{d_B\left(\mathrm{Dg}\,\mathcal{F}(\hat{X}_n),\ \mathrm{Dg}\,\mathcal{F}(X)\right)}$.

  measures the amplitude of variation of $\mathrm{Dg}\,\mathcal{F}(\hat{X}_n)$ around $\mathrm{Dgm}\,\mathcal{F}(X)$ in the space of diagrams.

- Main idea:

  the above statistics is similar to the one of

$$d_B\left(\mathrm{Dg}\,\mathcal{F}(\hat{X}^*),\ \mathrm{Dg}\,\mathcal{F}(\hat{X}_n)\right)$$

$$\hat{X}^* \sim (\underbrace{\mu_{\hat{X}_n}}_{\text{empirical measure on } \hat{X}_n})^{\otimes n}$$

  (n pts drawn from $\hat{X}_n$ with unif. proba. and with replacement)

- $(1-\alpha)$ quantile:

$$q_\alpha := \inf\left\{ t : \frac{1}{N}\sum_{i=1}^{N} \underbrace{I(\sqrt{n}\, d^*_i \geq t)}_{=\begin{cases} 1 & \text{if } \sqrt{n}\, d^*_i \geq t \\ 0 & \text{otherwise} \end{cases}} \leq \alpha \right\}.$$