# Heat Kernel Signature: A Concise Signature Based on Heat Diffusion

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STANFORD UNIVERSITY

# This talk is based on:

Jian Sun, Maks Ovsjanikov, Leonidas Guibas[1]
*A Concise and Provably Informative Multi-scale Signature Based on Heat Diffusion*
Proceedings of SGP 2009, to appear

Maks Ovsjanikov, Jian Sun, Leonidas Guibas[1]
*Global Intrinsic Symmetries of Shapes*
Computer Graphics Forum (Proc. of SGP) 2008



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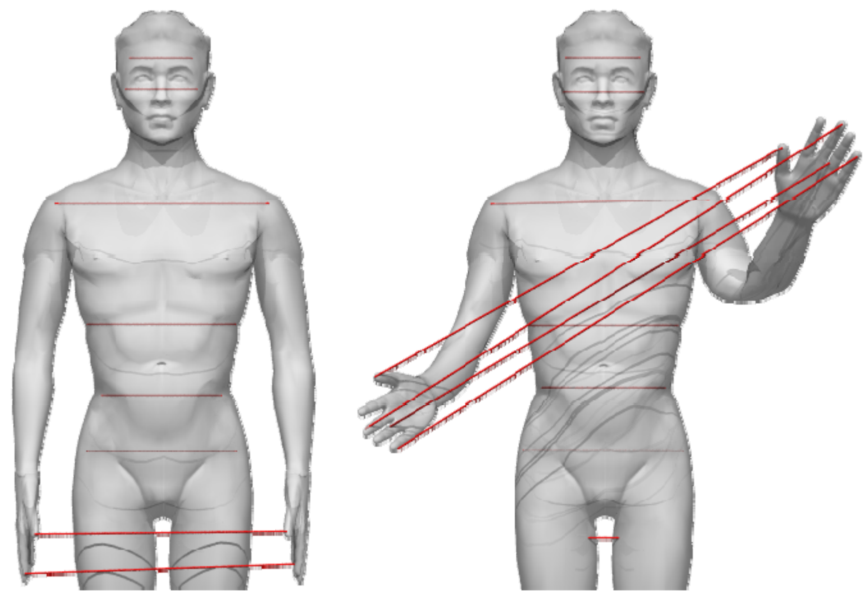
# Goal

Intrinsic Comparison of Points



Isometry Invariant Symmetries

$$T : M \rightarrow M, \text{ s.t. } d_M(x,y) = d_M(T(x),T(y)) \; \forall \, x,y$$

# Goal

Intrinsic Comparison of Points

Isometry Invariant Symmetries

$$T : M \to M, \text{ s.t. } d_M(x,y) = d_M(T(x),T(y)) \; \forall \, x,y$$

# Goal

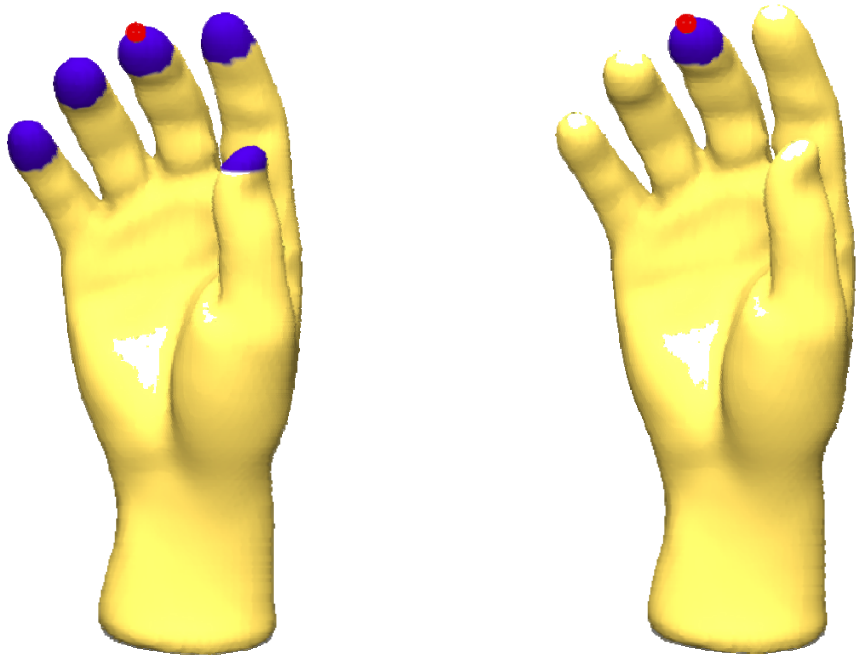
Multi-Scale Intrinsic Comparison of Points

Isometry Invariant Symmetries

$$T: M \to M, \text{ s.t. } d_M(x, y) = d_M(T(x), T(y)) \; \forall \, y \in \mathcal{N}(x)$$

# Goal

Multi-Scale Intrinsic Comparison of Points



Isometry Invariant Symmetries

$$T : M \to M, \text{ s.t. } d_M(x, y) = d_M(T(x), T(y)) \; \forall \; y \in \mathcal{N}(x)$$

# Goal

Find "similar" points at multiple scales.

## intrinsic

Invariant to isometric deformations

## robust

Not sensitive to perturbations of the shape

## efficient

Easily computable across many scales

# Old Idea

Define a multiscale signature for every point
Compare points by comparing their signatures

informative

Capture information around the point

commensurable

Easy to compare across points

# Talk Overview

- Background
  heat kernel on a Riemannian manifold
- Heat Kernel Signature
  definition and basic properties
  informative theorem
  computation
- Applications
  multi-scale matching
  shared structure discovery

Conclusions

# Heat Kernel

# Heat Equation on a Manifold

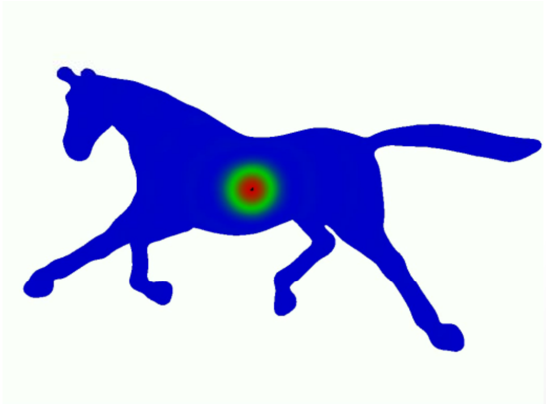
If $u(x,t)$ is the amount of heat at point $x$ at time $t$ .

$$\frac{\partial u}{\partial t} = \Delta u$$

$\Delta$ : Laplace-Beltrami operator.

Given an initial distribution $f(x)$. Heat at time $t$ :

$$f(x,t) = e^{-t\Delta} f$$

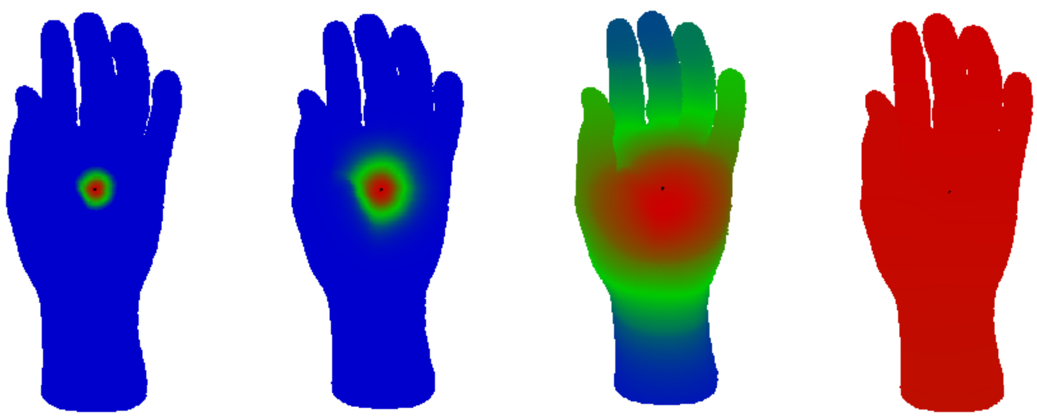


# Heat Equation on a Manifold

Heat kernel $\boldsymbol{k_t(x,y)}$ :

$$f(x,t) = \int_{\mathcal{M}} k_t(x,y) f(y) dy$$

$\boldsymbol{k_t(x,y)}$: amount of heat transferred from $\boldsymbol{x}$ to $\boldsymbol{y}$ in time $\boldsymbol{t}$.



$t = 0.001$ $t = 0.02$ $t = 3$ $t = 40$

# Heat Equation on a Manifold

$k_t(x, y)$ : prob. density function of Brownian motion on $M$.

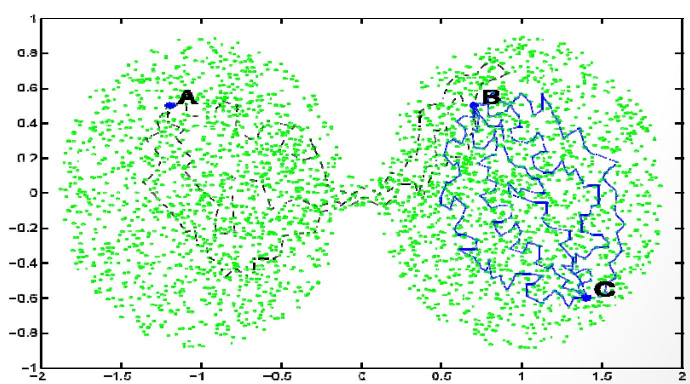
$$\mathbb{P}\left(W_x^t \in C\right) = \int_C k_t(x, y)dy$$

Intuitively: weighted average over all paths possible between $x$ and $y$ in time $t$.

Related to Diffusion Distance:

$$D_t(x, y) = k_t(x, x) - 2k_t(x, y) + k_t(y, y)$$

a robust multi-scale measure of proximity.




S. Lafon

# Heat Kernel Properties

Invariant under isometric deformations

If $T : X \to Y$ is an isometry then:

$$k_t(X, Y) = k_t(T(x), T(y))$$

Conversely: characterizes the shape up to isometry.

If $k_t(X, Y) = k_t\,(T(x), T(y))\ \forall\ x, y, t$ then:

$T$ is an isometry.

This is because:

$$\lim_{t \downarrow 0} (t \log k_t(x, y)) = -\frac{1}{4} d^2_{\mathcal{M}}(x, y)\ \forall\ x, y$$

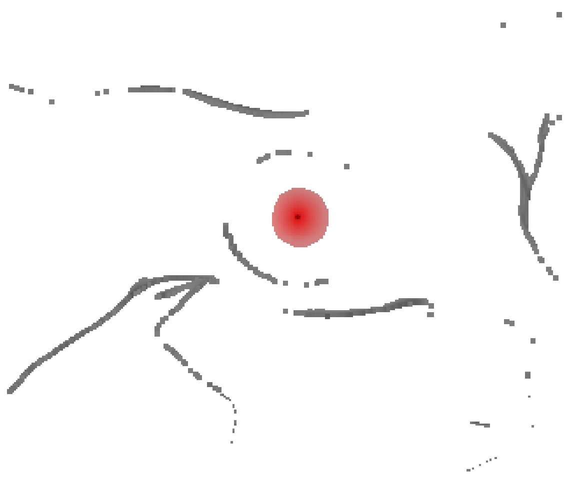
where $d_{\mathcal{M}}(\cdot, \cdot)$ is the geodesic distance.

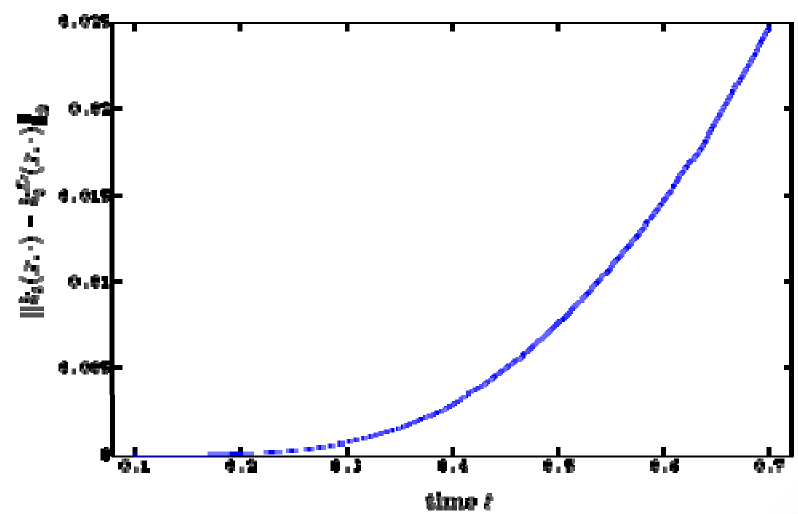
# Heat Kernel Properties

- Multi-Scale:

  For a fixed $x$, as $t$ increases, heat diffuses to larger and larger neighborhoods.

  Therefore, $k_t(x, \cdot)$ is determined (reflects the properties of) a neighborhood that grows with $t$.

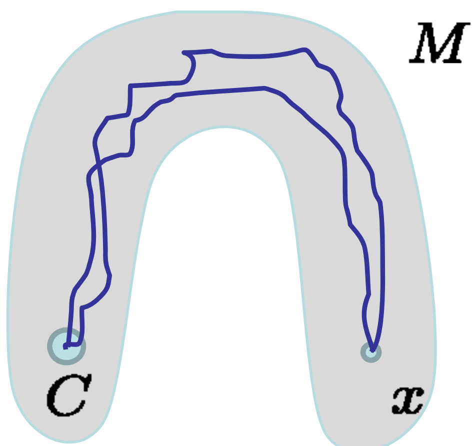




# Heat Kernel Properties

- Robust:

  $k_t(x, \cdot)$ is a probability density function, a weighted average over all paths, which should not be sensitive to local perturbations.

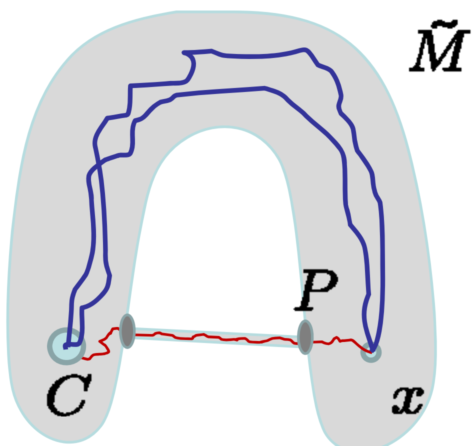



$$k_t^M(x, C) = \mathbb{P}(W_x^t \in C)$$

# Heat Kernel Properties

- Robust:

  $k_t(x, \cdot)$ is a probability density function, a weighted average over all paths, which should not be sensitive to local perturbations.




$$k_t^{\tilde{M}}(x, C) = \mathbb{P}(\tilde{W}_x^t \in C)$$

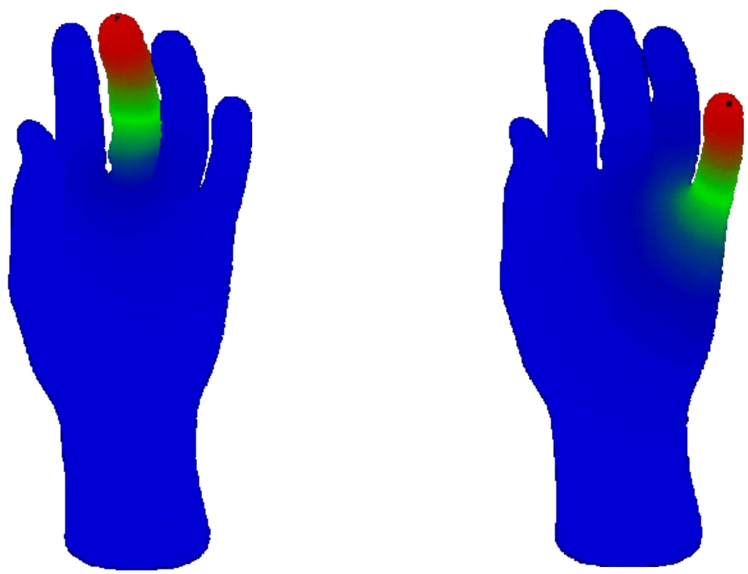
Only paths through the modified area $P$ will change.

# Defining a signature

Let $k_t(x, \cdot)$ be the signature of $x$ at scale $t$.

The heat kernel has all the properties we want.

Except easy comparison!



$k_t(x, \cdot)$ is a function on the entire manifold.

Nontrivial to align the domains of such functions across shapes, or even for different points of the same shape.

# Heat Kernel Signature

# Heat Kernel Signature

Define:

$\mathsf{HKS}(x,t) = k_t(x,x)$ signature of $x$ at scale $t$ .

Now HKSs of two points can be easily compared since they are defined on a common domain (time)

HKS is a restriction of the heat kernel, and thus:

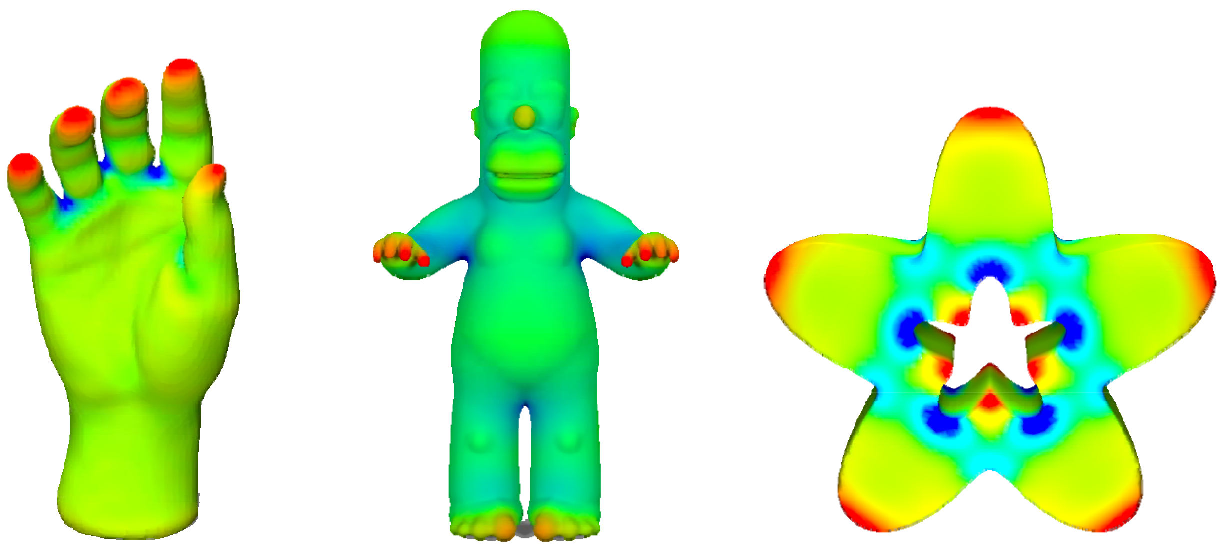
- isometry-invariant
- multi-scale
- robust

Question: How informative is it?

# Heat Kernel Signature

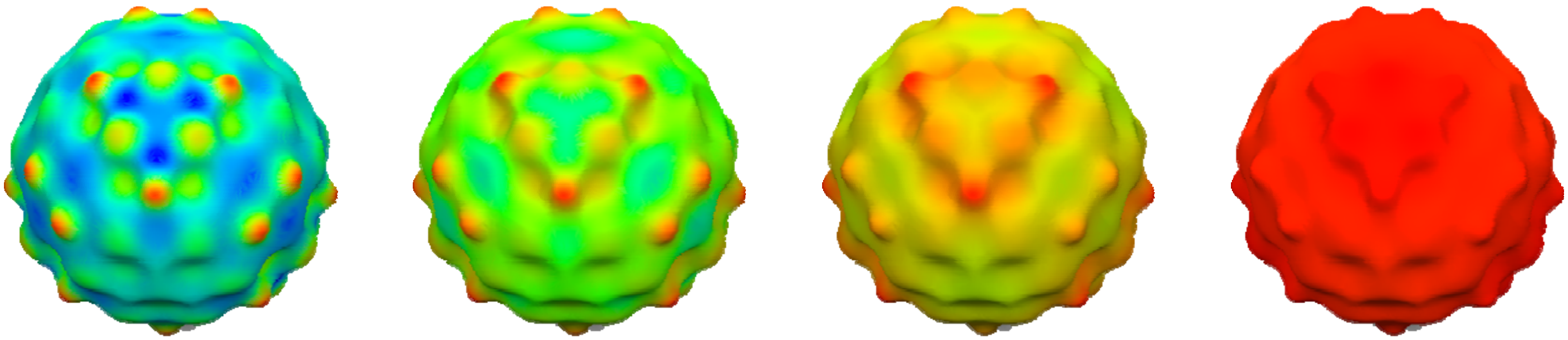
Relation to scalar curvature for small $t$ :

$$k_t(x,x) = \frac{1}{4\pi t}\sum_{i=0}^{\infty} a_i t^i \quad a_0 = 1, a_1 = \frac{1}{6}K$$



# Heat Kernel Signature

Can be interpreted as multi-scale intrinsic curvature.



$t = 0.004$ $t = 0.008$ $t = 0.02$ $t = 2$

# Informative Theorem

The set of all HKS on the shape almost always defines it up to isometry!

Theorem: If $X$ and $Y$ are two compact manifolds, such that $\Delta_X$ and $\Delta_Y$ have only non-repeating eigenvalues. Then a homeomorphim $T : X \to Y$ is an isometry if and only if

$$\mathsf{HKS}(x) = \mathsf{HKS}(T(x)) \ \forall \ x$$

# Informative Theorem

Intuition: Heat Kernel is related to the eigenvalues and eigenfunctions of the LB-operator:

$$\mathsf{HKS}(x,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i^2(x)$$

If eigenvalues do not repeat, we can recover $\{\lambda_i\}$ and $\{\phi_i^2(x)\}$ from $\mathsf{HKS}(x)$. E.g. $\lambda_0 = 0$, and

$$\phi_0^2(x) = \lim_{t \downarrow 0} \mathsf{HKS}(x,t)$$

and $\lambda_1 = \inf \left\{ a \text{ s.t.} \lim_{t \downarrow 0} e^{at} (\mathsf{HKS}(x,t) - \phi_0^2(x)) \neq 0 \right\}$

# Informative Theorem

Intuition: Heat Kernel is related to the eigenvalues and eigenfunctions of the LB-operator:

$$\mathrm{HKS}(x,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i^2(x)$$

After recovering the eigenvalues, and squared eigenfunctions, we only need to recover their signs.

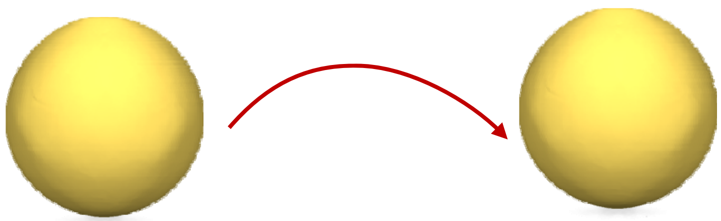
We use the properties of nodal domains of eigenfunctions.

Since the eigenvalues + eigenfunctions define the manifold, the theorem follows.

# Informative Theorem

How general is the theorem?

If there are repeated eigenvalues, it does not hold:



On the sphere, $\mathsf{HKS}(x) = \mathsf{HKS}(y)\ \forall\ x, y$ but there are non-isometric maps between spheres.

Uhlenbeck's Theorem (1976): for "almost any" metric on a 2-manifold $X$, the eigenvalues of $\Delta_X$ are non-repeating.

# Computing HKS

On a compact manifold:

$$k_t(x,y) = \sum_{i=0}^{N} e^{-t\lambda_i}\phi_i(x)\phi_i(y)$$

$\lambda_i, \phi_i$ : $i^{\text{th}}$ eigen value/function of Laplace-Beltrami operator.

Heat equation on a mesh:

$$\frac{\partial u}{\partial t} = Lu \quad \Rightarrow \quad u(t) = e^{-tL}u_0$$

We use Belkin et al. mesh Laplace: $L = A^{-1}W$

$A$ : diagonal area matrix

$W$ : symmetric positive definite weight matrix.

# Computing HKS

If $L = A^{-1}W$ :

$$u(t) = e^{-tL}u_0 = \sum_y \boxed{\sum_{i=0}^{N} e^{-\lambda_i t}\phi_i(x)\phi_i(y)} u(y)A(y)$$

$\lambda_i, \phi_i$ : $i^{\text{th}}$ eigen value/vector of $L$ .

Discretization of $f(x,t) = \int_M \boxed{k_t(x,y)} f(y)dy$ .

Use $k_t(x,y) = \sum_{i=0}^{N} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$

Especially suitable for large $t$ .

Once the eigen-decomposition is computed can obtain HKS at any scale.

# Computing HKS

Taylor approximation:

$$k_t(x,y) = \frac{e^{-tL}(x,y)}{A(y)} = \frac{1}{A(y)} \left( \sum_{j=0}^{\infty} \frac{(-t)^j}{j!} L^j \right)(x,y)$$

Can use this for small $t$.

Questions:

- Better approximation of matrix exponential (e.g. Padé)?
- Can compute the diagonal of exponential only?

Moler, C. B. and C. F. Van Loan, "Nineteen Dubious Ways to Compute the Exponential of a Matrix," SIAM Review 20, 1978, pp. 801-836.

# Sampling in Time

Two heuristics for making HKSs commensurable:

- For a fixed point $x$, sample HKS on a logarithmic scale at times $t_i$.
- For a fixed time $t$ scale each HKS, by the sum over all points of *M*.

$$\mathsf{HKS}(x) = \left\{ \frac{k_{t_i}(x,x)}{\sum_j e^{-t_i \lambda_j}}, i \in 1, 2, .., 100 \right\}$$

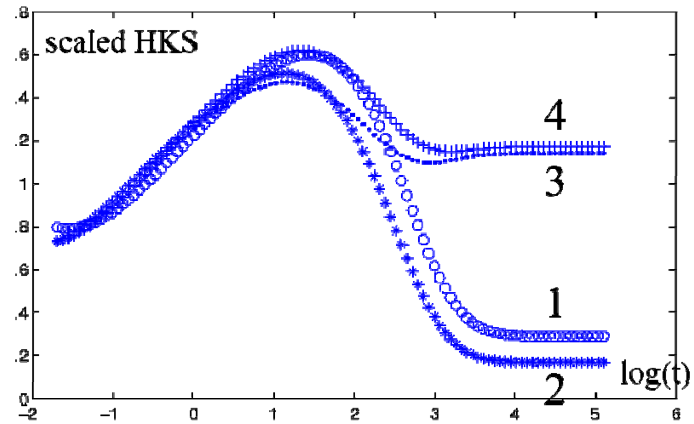
$$t_i = \alpha^i t_0$$

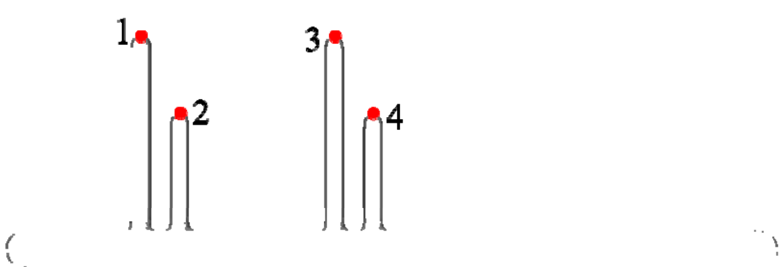
Compare using L2 norm of the HKS vectors.

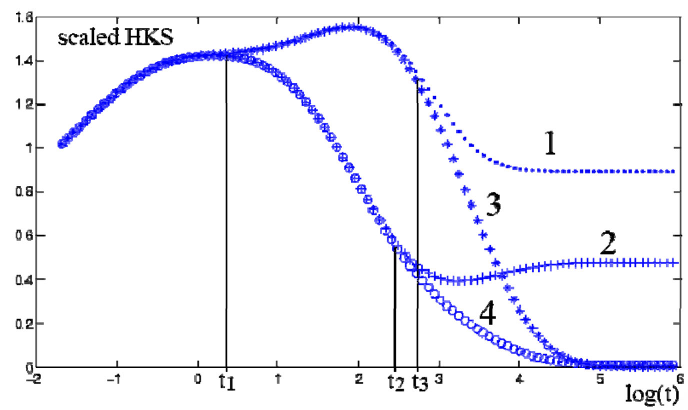
# Applications

# Multiscale Matching

Comparing points through their HKS signatures:

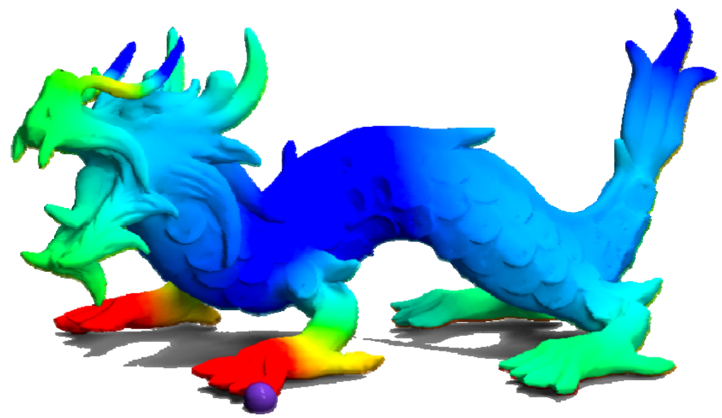



# Multiscale Matching

Comparing points through their HKS signatures:

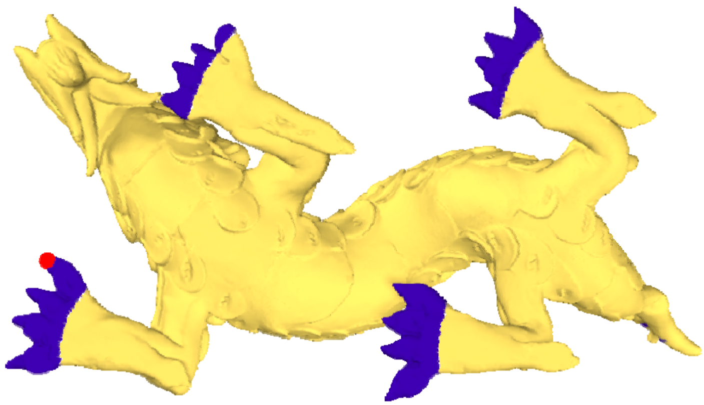
Medium scale



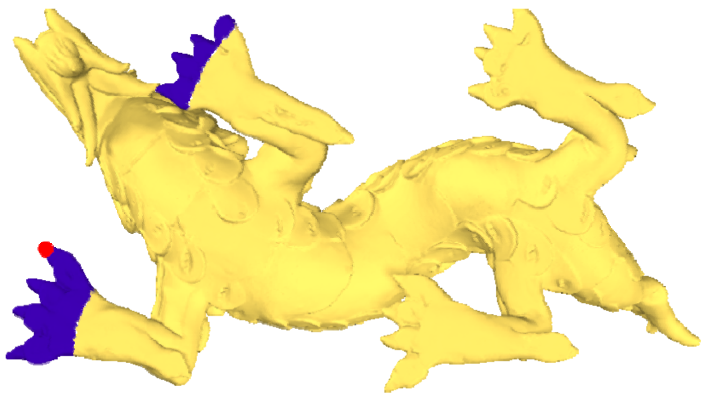
Full scale

# Multiscale Matching

Finding similar points:



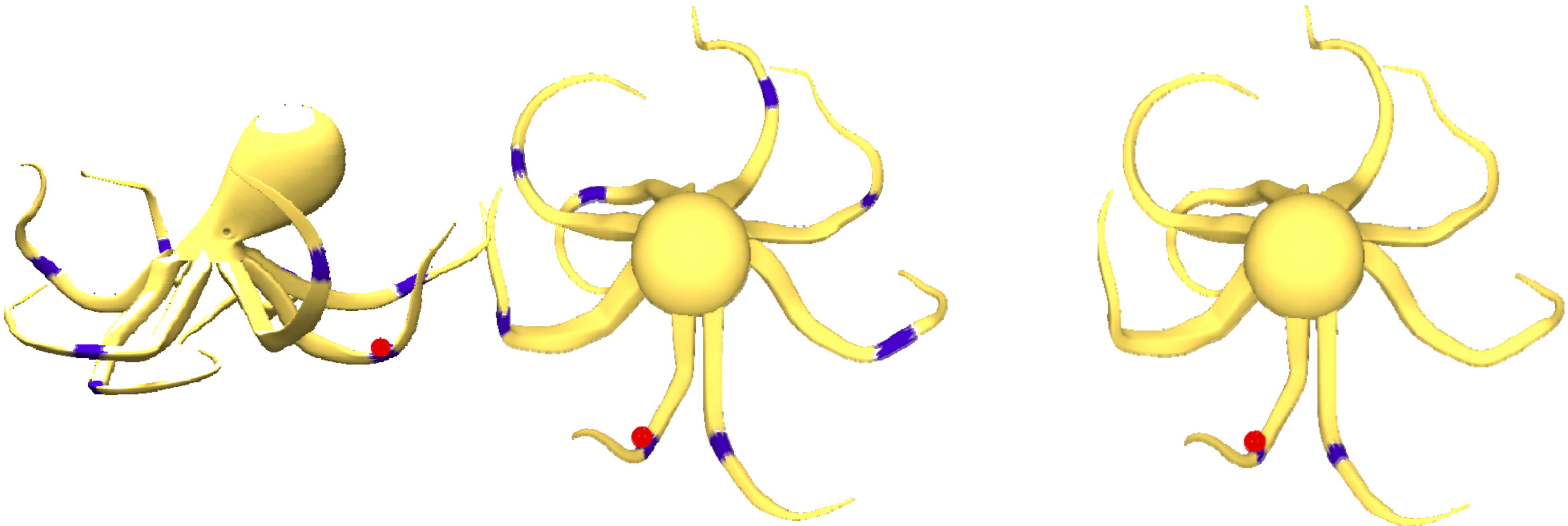
Medium scale



Full scale

# Multiscale Matching

Finding similar points:



Medium scale　　　　　　　　　　Full scale

# Multiscale Matching

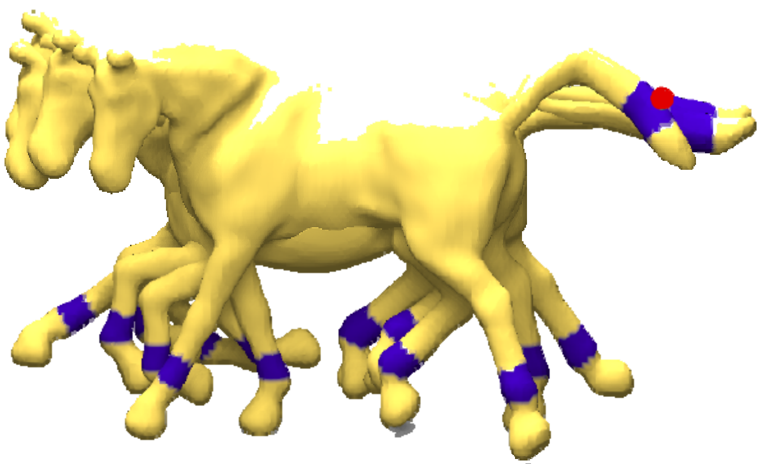
Finding similar points – robustly:

Medium scale

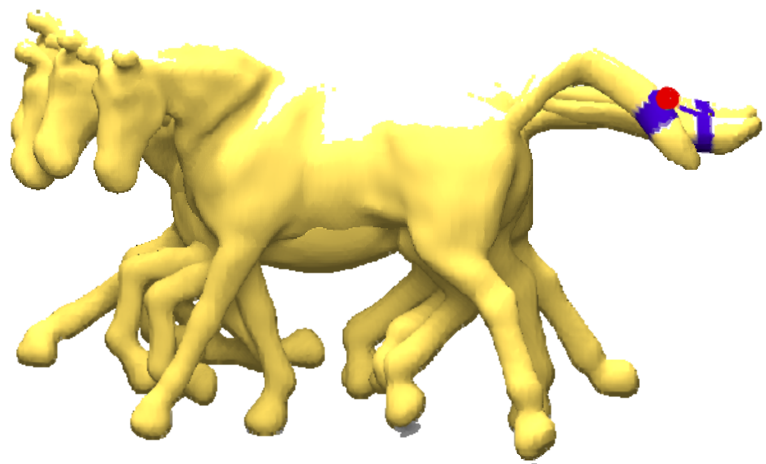
Full scale

# Multiscale Matching

Finding similar points across multiple shapes:



Medium scale



Full scale

# Feature Detection

Persistent feature detection:

Find points that are maxima of HKS:

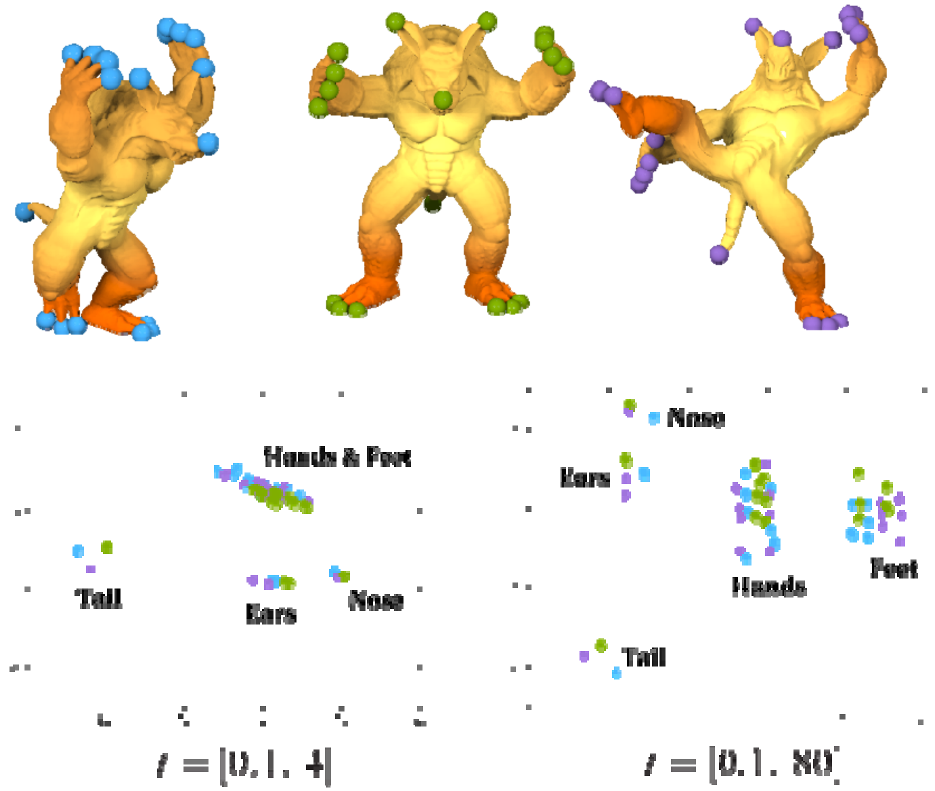
$$k_t(x, x)$$

for large enough $t$.

Motivation: high curvature points at large scale.

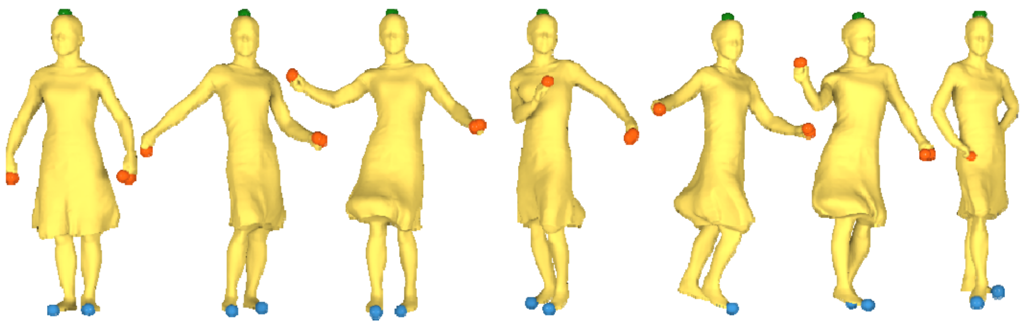
# Shared Structure

2D MDS embedding of feature points on three shapes according to distances of their HKS

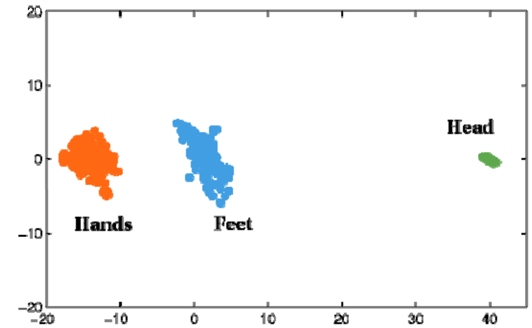


# Shared Structure

2D MDS embedding of feature points on 175 shapes according to distances of their HKS.



Feature points found on a few poses of the dancer model by Vlasic *et al.*




MDS of features from all 175 poses using a full range of scales

# Open Questions

- Convergence of the discrete heat kernel to the continuous one?
- Precise robustness statements (geometric/topological).
- Precise multi-scale analysis.
- Less restrictive Informative Theorem?

# Thank You

- Acknowledgements: NSF grants ITR 0205671, FRG 0354543, FODAVA 808515, NIH grant GM-072970, and DARPA grant HR0011-05-1-0007
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