Reconstructing 3D compact sets

F. Cazals; ABS; D. Cohen-Steiner, Geometrica; INRIA Sophia-Antipolis

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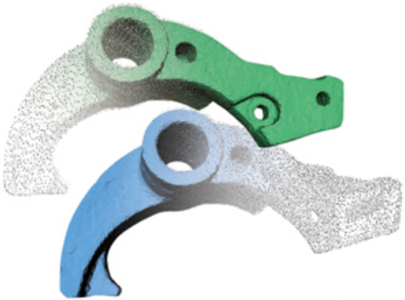
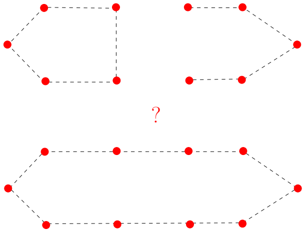
Outline

Surface Reconstruction

▷Pb.: Compute a (piece-wise) surface from a point cloud
▷Context: Reverse Engineering, Medical data processing, Geology, Cultural Heritage

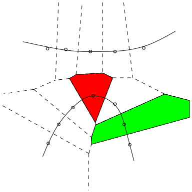


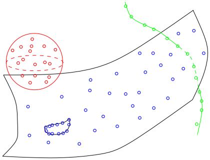


▷Sc. Challenge(s): Sampling models –minimal amount of information required to reconstruct *faithfully (geometry, topology)*
▷Previous work: heuristics before Amenta - Bern (98): *local Feature Size* —aka *Local Width*

Reconstruction Difficulties

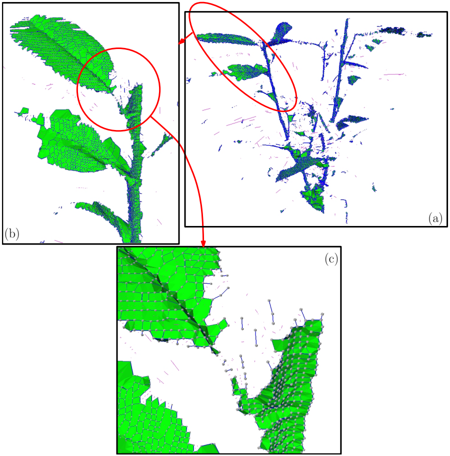
- Using an a priori model: classical surface-based reconstruction
  - (Compact) Smooth surfaces versus sharps features, boundaries
  - Samples may not comply with the model:
    - Unstable estimates for differential geometry based quantities:
      - normals, principal curvatures
    - The sampling may not be a surface at all
  - Unique versus plausible reconstructions
- Calls for a general reconstruction strategy:
  - No a priori: handling compact sets
  - Providing multi-scale reconstruction i.e. plausible shapes





Practically: Reconstructing the Boundary of a Solid?

▷ Thin parts might just be too thin



▷ Calls for a general reconstruction strategy

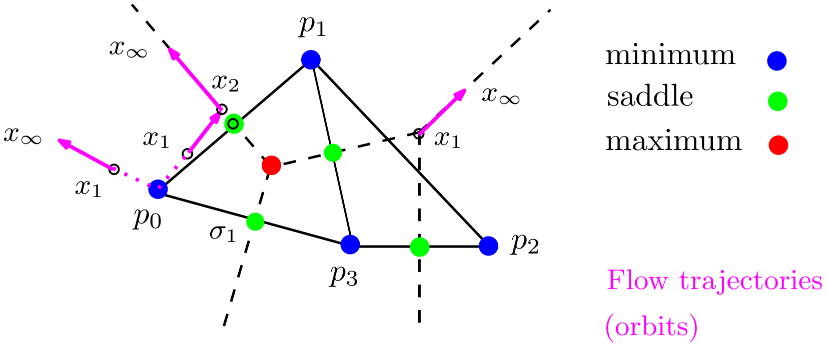
Beech tree reconstruction, with parameters $t_r = 2.2$, $t_p = 1.1$ (a) Overview of this noisy and under-sampled model (b,c) Zoom near a an under-sampled peduncle. The point cloud is courtesy of J-C. Chambelland et al, UMR 547 PIAF - INRA/UBP.

Distance Function to Sample Points

- Given a collection of points $\{p_i\}_{i=1,\ldots,n}$, consider:

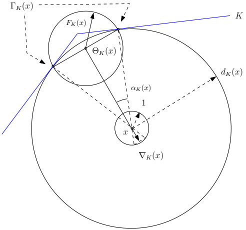
$$d(p) = \min_i \| pp_i \|$$

- Increasing the value of d:



Distance function to a compact set K

▷ Distance function to a compact set K



– Distance function :
$d_K(x) = \min_{y \in K} d(x, y)$

– Gradient : $\nabla d_K(x) = \frac{x - \Theta_K(x)}{d_K(x)}$
– Its norm:
$|| \nabla d_K(x) || = \cos \alpha_K(x)$

▷ Key properties

∇d_K can be integrated to define a continuous flow (Euler scheme converges)

(Non degenerate) Critical point:
(interior) of the convex hull of $\Gamma_K(x)$

Easy interpretation if $K =$ *point cloud*:
cf Delaunay-Voronoi diagrams

▷Ref: Lieutier; Computer Aided Design, 2004

The Morse Puzzle of the Distance Function

Flow Complex

Critical points

Stable manifolds

Unstable manifolds

Morse-Smale diagram

regions of homogeneous flow:

$\bigcap$ of (un)stable manifolds

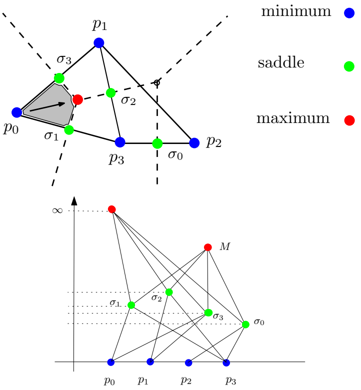
Hasse diagram

orbits of the distance function

through consecutive crit. pts

▷Ref: Giesen, John; ACM SODA; 2003

▷Ref: Cazals, Pion; ACM SoCG; 2008

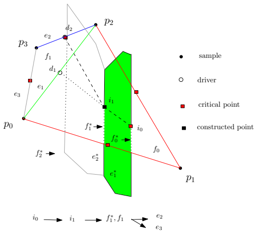


Construction of S2 and U1

- Stable manifolds, generically:

  Recursive structure
  Complicated case S2:
  neither in Delaunay nor in Voronoi

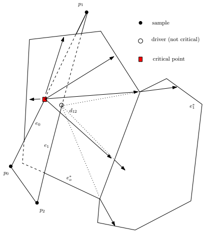
- S2:



- Ref: Giesen and John, ACM SODA 03
- Ref: Giesen, Ramos, Sadri; ACM SoCG 06
- Ref: Cazals, Parameswaran, Pion; ACM SoCG 08

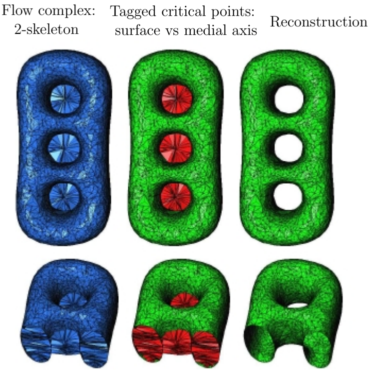
- Stable manifolds, generically:

  U1: dissection of the two-skeleton



Flow Based Surface Reconstruction: Separating Critical Points

▷ ε-samples accommodate a separation of critical points:



Algo. based on the separation of c.p.:
- surface c.p.: angle criterion
- MA c.p.: distance criterion wrt poles

Under mild hypothesis on the sampling:
- Reconstruction $\hat{S}$:
  - $\hat{S} \subset$ tube around S
  - $\hat{S}$ isotopic to S

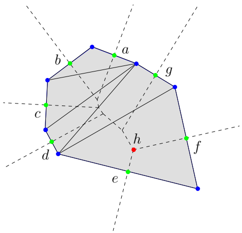
▷Ref: Dey, Giesen, Ramos, Sadri; ACM SoCG 2005

Generalization to Compact Sets:
Finding Cuts from the Hasse Diagram

- ▷ Reconstruction:
  - Collection of Stable Manifolds (SM) of all indices i.e. 0 to 3
  - Governed by a parameter $t_r > 0$
  - Abusing terminology: inserting a Hasse node $\sim$ inserting its SM

- ▷ Key ingredient: ratio $V(b)/V(a)$
  between crit. values of incicdent crit. pts in the Hasse diagram

- ▷ Initialization: selected Gabriel edges
- ▷ Upflow extension:
  a in reconstruction; $index(b) = index(a) + 1$
  a sponsors b with priority $r_u = V(b)/V(a)$
- ▷ Regularization:
  h in reconstruction
  h triggers insertion of a, b, c, d, e, f, g; $r_r = 1$
- ▷ Horizontal extension:
  a in reconstruction; $index(b) = index(a)$
  a sponsors b with priority $r_h = \max(V(a)/V(b), V(b)/V(a))$

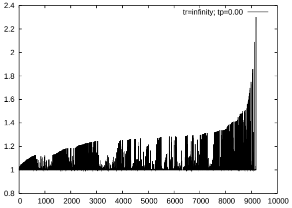


The Three Steps: Canonical Ordering of Insertions

Reconstruction Profiles

- ▷ Initialization: start from selected one-dimensional SM
  - – Gabriel edge $e = (v_0, v_1)$ is an init edge iff
    v_1 if the nearest neighbor of v_0 or vice-versa

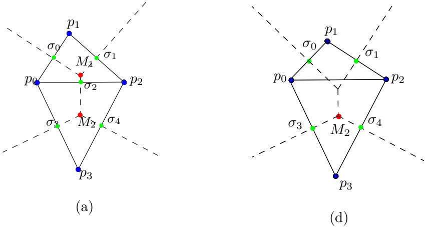
- ▷ Sponsoring as an iterative process:
  - – sponsored nodes placed into a priority queue Q
    - • priority : least r_u, r_r, r_h ratio from any sponsor
    - • requires insertion and/or updates of nodes already in Q
  - – induces a canonical insertion of nodes: take the easiest step
    i.e. pop node with least priority provided it is $< t_r$
- ▷ Reconstruction profile: $(r_i)_{i \geq 1}$ for $t_r = \infty$
  - – encompasses all possible reconstructions



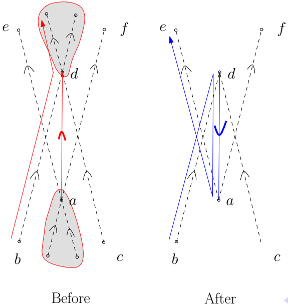
Example recons. profile (more later!)
Note: tail of diagram $\sim$ features
(significant maxima) of the model

Widening the Gaps: Persistence on the Hasse Diagram

- ▷ Insufficient sampling: widen the gaps
- ▷ Cancelling pairs of *nearby* critical points: reverting the flow



- ▷ Hasse simplification: multiplexing and redistribution of SMs

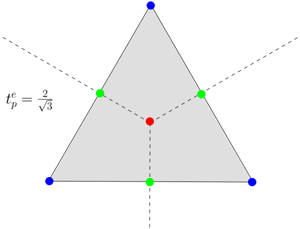


Hasse Simplification Cont'd

▷ Algorithm

– Iterative process based on the persistence priority $p(a,b) = V(b)/V(a)$
– Incremental cancellation of all pairs up to priority $\leq t_p$

▷ Persistence: the obtuse triangle



$t_p > t_p^e$ cancels edges-triangle

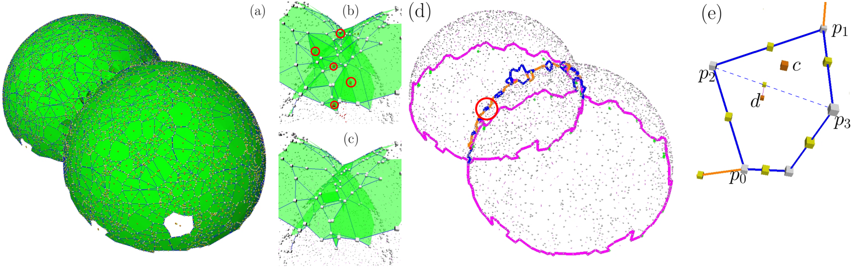
▷ Key steps

– Update of the Hasse diagram
remove edges incident on nodes cancelled
add edges of bipartite graph
$\mathrm{In}(b) \times \mathrm{Out}(a)$
– Redistribution of SM

– surface with boundaries:
may create leaks of SM
– but not systematic:
SM rescued thanks to multiplexing

Non Manifold Reconstruction

▷ Reconstructing two intersecting spheres

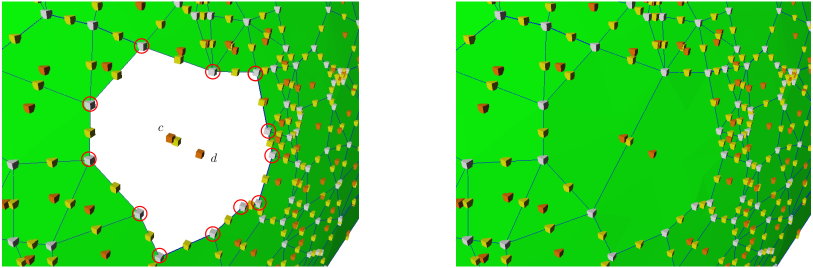


(a) $t_r = 1.9,\ t_p = 0$
(b) Transparent view of (a)
(c) $t_r = 2.5,\ t_p = 1.05$:
maxima cancelled

(d) $t_r = 2.5,\ t_p = 1.05$,
multiplicity of Gabriel edges:
zero one three four five
(e) Circled region of Fig. (d):
intersection curve stretched to a disk

Enumerating Plausible Reconstructions

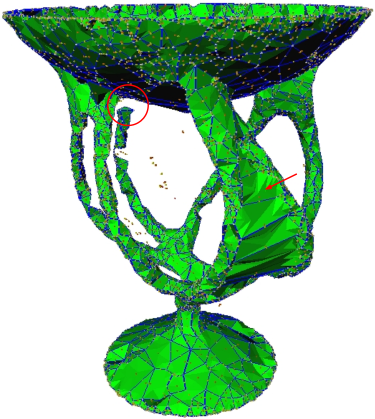
- Do the circled points punch a hole or not?



(a) At $t_r = 1.9, t_p = 0$: hole (b) At $t_r = 2.1, t_p = 0$: hole filled

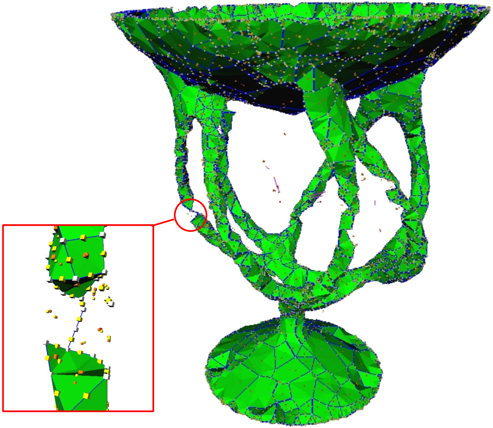
On the Importance of Persistence

- Undesirable extensions



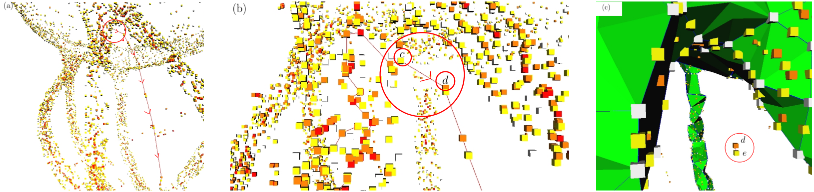
(a) At $t_r = 1.7, t_p = 0$: fin

- Fixed by persistence



(b) At $t_r = 2, t_p = 1.02$: fin is gone

Untangling the Role of Persistence

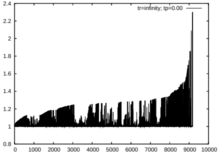


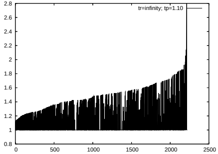
(a,b)The extension path followed by the algorithm–red arrows.
(c) Pairing by persistence: *d* paired to an index one critical point *e*

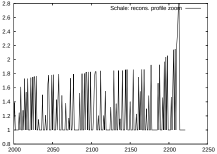
▷ Note: offers a local control so as to fix the sampling

Combining Parameters t_r and t_p: the Exple of Vase

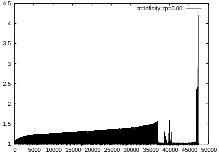
▷ Vase: (a) $t_p = 0, t_r = \infty$ (a) $t_p = 1.1, t_r = \infty$ (c) Tail of (b)

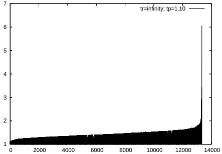


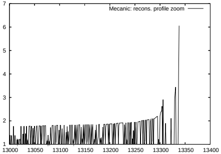




▷ Mechanic.: (a) $t_p = 0, t_r = \infty$ (b) $t_p = 1.1, t_r = \infty$ (c) Tail of (b)







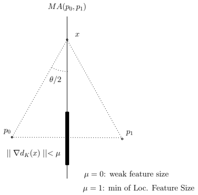
▷ Recommendations

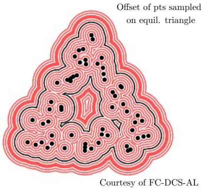
– values of $t_p \in [1.02, 1.2]$ and $t_r \in [1.9, 2.1]$ yield comparable results
– beyond $t_r \in [1.9, 2.1]$: number of <u>features</u> of the model (significant max.)

Reconstruction Guarantees: Background

▷ **Def.** The *μ-medial axis* of a compact set $K \subset \mathbb{R}^n$ is the set of points $x \notin K$ such that $||\nabla d_K(x)|| < \mu$. The *μ-reach* of K, denoted by $r_\mu(K)$, is the minimum distance between a point in K and a point in the closure of its μ-medial axis.

▷ **Def.** Given two non-negative real numbers κ and μ, we say that a compact set $P \subset \mathbb{R}^n$ is a *(κ, μ)-approximation* of a compact set $K \subset \mathbb{R}^n$ if the Hausdorff distance between K and P does not exceed κ times the μ-reach of K.





▷ **Thm.** Reconstructing with offsets:

– $P \subset \mathbb{R}^n$ is a *(κ, μ)-approximation* of a compact set K
– Under *suitable* conditions:
P^α is homotopy equivalent to K^η for sufficiently small η.

▷Ref: Chazal, Cohen-Steiner, Lieutier; ACM SoCG 06

Reconstruction Guarantees

▷ **Def.** Point cloud P is a ρ-*uniform* approximation of a compact set K if half the distance between the two closest sample points in P is at least ρ times the the Hausdorff distance between P and K, where $0 < \rho < 1$.

▷ **Thm.** Let K be a compact subset of $\mathbb{R}^3$;
Assume P is a ρ-uniform (κ, μ)-approximation of K. If

$$\frac{4}{\rho\mu^2} < t_r < \frac{\mu^2}{4\kappa} - 1,$$

then the reconstruction is homotopy equivalent to K^η for small enough η.

▷ Proof sketch:

Dey-Gisen-John'03 Edelsbrunner'95 Recons. thm.

$$\begin{tikzcd} F_\alpha\text{: flow shape} \arrow[r, leftrightarrow, "\text{Homot. Equiv.}"] & \alpha\text{-shape} \arrow[r, leftrightarrow, "\text{Homot. Equiv.}"] & P^\alpha\text{: union balls} \arrow[r, leftrightarrow, "\text{Homot. Equiv.}"] & K^\eta\text{: compact set} \end{tikzcd}$$

($\cup$ stable manifolds) ($\cup$ of domains of simplices)

▷ **Note.** Thm uses uniform sampling; algorithm does not.

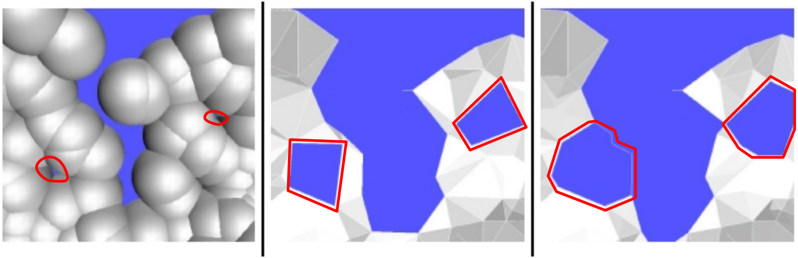
Union of balls, α-shapes, Flow-shapes

- **Topology of a union of balls**: union of balls $\sim$ flow shape $\sim$ α-shape

  union of balls and α-shapes are homotopy equivalent [Edelsbrunner, 92]
  α-shapes and flow shapes are homotopy equivalent [Dey, Giesen, John, 03]

- **Key differences**

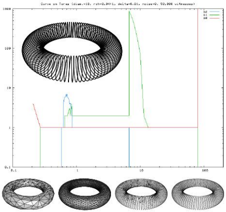
  Events triggering addition in α-shape: Gabriel simplices
  critical points



Union of balls — α-complex / α-shape — Flow shape

Connexions to Recent Previous Work

▷ Multi-scale reconstruction

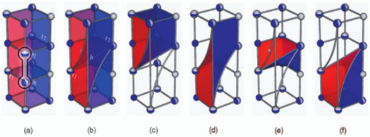


▷Ref: Guibas, Oudot; DCG 08

– Based on the witness complex
– Incremental construction of
subset of the Del. triangulation of $L \subset W$,
and associated Betti numbers
complex is the ν-witness complex $\mathcal{C}_\nu^W(L)$
– Provably correct reconstruction of curves and surfaces
contains a manifold isotopic to the surface

$\longrightarrow$ approach radically different; we handle non manifold shapes

▷ Topological simplification of 3D scalar functions



▷Ref: Gyulassy et al., IEEE TVCG, 06

Morse-Smale diagram simplification
– Operations: cancellations $(m, \sigma_1), (\sigma_1, \sigma_2), (\sigma_2, M)$
– Simplified complex is that of some scalar function:
topological constraints respected

$\longrightarrow$ Our simplified MS diagram is not *realized*

Conclusion - Outlook

- Flow complex based reconstruction
  versatile strategy (cf strata)
  complex to build, yet tractable for models of (moderate) size

- Developments called for
  computing stratifications
  handling noise
  approximation scheme based on Delaunay (some restricted Delaunay)
  gazing on the high dimensional side. . .