# Persistence Theory: From Quiver Representations to Data Analysis

## Comments and Corrections

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## Comments

- page viii, bottom of page: the following names should be added to the acknowledgements:

  - Peter Landweber had an invaluable contribution to these notes. First, by reading the final manuscript carefully, chasing potential mistakes and omissions. Second, by making many suggestions to improve the exposition locally. Unfortunately, he came to know the existence of the book only after it was sent to press, therefore his feedback only made it to these notes so far—perhaps they will make it to a future edition of the book. I cannot thank him enough for his help.
  - Peter Bubenik, Alen Đurić, Elizabeth Munch and Rachel Roca also took close looks at the book after publication and pointed out a few minor errors, which I have reported in these notes. I want to thank them likewise.
  - Gunnar Carlsson, Jeff Erickson and Eric Goubault kindly served as reviewers for an early version of the manuscript, then my 'habilitation' thesis. I am in debt to them for their enthusiasm and encouragements to turn the manuscript into a book.


- pages 6-7, section *Connection to other theories*: let me complete this section with the following short historical account:

  The history of persistence is an organic one, with several independent developments at different points in time. The most ancient roots can be traced back to the 1940's and the extension of Morse theory beyond the setting of differentiable functions on manifolds. In particular, Morse's work on *ranks and spans of F-homology classes* [M] introduces ideas that are remarkably close to the ones underlying persistence theory, with *F-homology classes* referring to compatible homology basis elements in a filtration, *ranks* being the corresponding intervals, and *spans* being their lengths. Morse's ideas have echoed in several subsequent developments, most notably in the theory of spectral sequences, with the work of Deheuvels [D]. The 1990's have seen similar ideas rediscovered and phrased out in different languages, first in the work of Barannikov [B] on *canonical forms*, then later on in the work of Robins [R] on *persistent Betti numbers*. Meanwhile, an analogous theory was developed in a purely set-theoretic context, called *size theory*, where homology is replaced by connected components, and where ranks and spans are encoded implicitly in a functional called *size function* [131]. The added value of size theory was to introduce a notion of distance between size functions, which enabled their use as stable descriptors for geometric data (compared in the so-called *natural pseudo-distance*), with applications to shape comparison, computer vision and pattern recognition [97]. Finally, the modern formulation of the theory appeared in the years 2000, with the introduction of the *persistence algorithm* [118], then of *persistence modules* [243], then of the *interleaving* and *bottleneck* distances [71,87]. Around the same time, homology was introduced in size theory through the concept of *size functor* [C], and the two theories (size and persistence) eventually merged together.

Corresponding references to be added to the book:


[M] M. Morse, Rank and Span in Functional Topology, *Annals of Mathematics*, 40(1):419–454, April 1940.

[D] R. Deheuvels, Topologie d'une fonctionnelle, *Annals of Mathematics*, 61(2):13–72, January 1955.

[B] S. Barannikov, Framed Morse complex and its invariants, *Advances in Soviet Mathematics*, 21:93–115, 1994.

[R] V. Robins, Towards computing homology from finite approximations, *Topology Proceedings*, 24:503–532, 1999.

[C] F. Cagliari, M. Ferri, P. Pozzi, Size functions from a categorical viewpoint, *Acta Applicandae Mathematicae*, 67(3):225–235, 2001.


- page 17, 7 lines above Theorem 1.3: the product between two paths is defined to be zero in case the paths cannot be concatenated (see Definition A.31 in the Appendix, page 192).

- page 32: line 11: for Proposition 2.3 (i), see also F. Cagliari and C. Landi, Finiteness of rank invariants of multidimensional persistent homology groups, *Applied Mathematical Letters*, 24(4):516–518, 2011.

- page 38, line -2: by *$x$-monotone path* is meant a path in the diagram that connects the left and right boundaries of the pyramid and that is crossed at most once by every vertical line.

- page 51: line -5: for the existence of an optimal matching $M$ in (3.2), see also [96].

- page 53: line -6: a corresponding universality property for the bottleneck distance was shown in [96].

- page 83, Definition 4.14, line 4: the Delaunay triangulation of $P$ is defined as having one $k$-simplex per $(k+1)$-tuple of points of $P$ circumscribed by a ball containing no point of $P$ in its interior. When the filtration parameter $i$ becomes large enough, typically larger than the radii of all the (finitely many) empty circumscribing balls, the $i$-Delaunay complex becomes equal to the Delaunay triangulation of $P$ and therefore stops growing. For background material on Delaunay triangulations, see e.g. J.-D. Boissonnat and M. Yvinec: *Algorithmic geometry*, Cambridge University Press, UK, 1998.

- page 160, Section 3, lines 4 and 31: a positive semidefinite kernel over a space $\mathcal{X}$ is a map $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that $\sum_{i,j=1}^{n} c_i c_j k(x_i, x_j) \geq 0$ for any $n \in \mathbb{N}$, $x_1, \cdots, x_n \in \mathcal{X}$, and

$c_1, \cdots, c_n \in \mathbb{R}$. It is positive definite if, furthermore, the inequality is strict except when $c_1 = \cdots = c_n = 0$. The two notions are often identified with each other in the literature, because the kernel trick applies to each one of them indifferently. However, there are some differences, for instance the fact that a positive semidefinite kernel may imply a loss in discriminative power (think of the extreme case where $k = 0$, which is positive semidefinite). For background material on the kernel trick, see e.g. M. Cuturi: *Positive Definite Kernels in Machine Learning*, arXiv preprint *arXiv:0911.5367*, 2009.

- page 167, footnote 1: part of the exposition in Appendix A follows [104], most notably: the definitions, notations and figures in Sections 1 through 3 (with minor adaptations); the description of the reflection functors construction in Section 4.1; the description of Tits' geometric argument in Section 5.2.2.

- page 177, lower half of page: on every occurrence of "op $v_a + v_b$", where op is either ker or im or coker, one should read "op $(v_a + v_b)$".

- page 188, line 7: the map $\mathbb{V} \mapsto \underline{\dim}\mathbb{V}$ is from the isomorphism classes of indecomposable representations to the postive roots of the Tits form.

- page 218: the following entries should be added to the index:

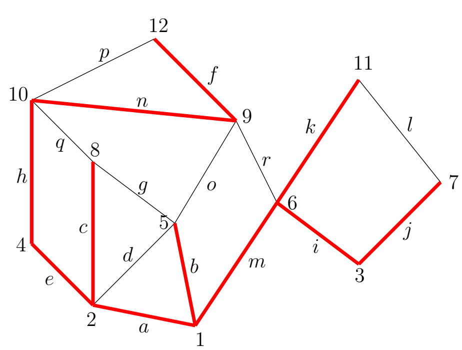
  

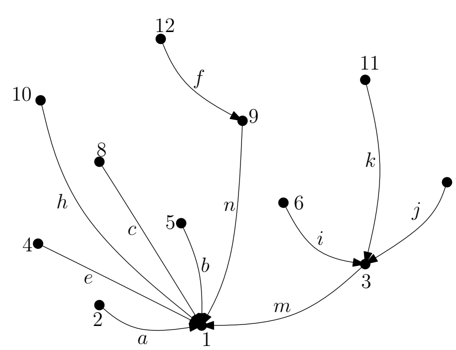
## Corrections

No major mistake has been found in the book so far. Below I list the minor errors, sorted by type.

### Pictures

- page 47, Figure 2.6: the spanning tree and associated hierarchy are slightly wrong. Here are the correct pictures:



## Internal references

- page 79, footnote 4, line 2: “The theorem" ⟶“Corollary 4.8"
- page 217, diamond principle: “178” ⟶“179”
- page 217, elder rule: the two lines should be combined into a single one
- page 218: signatures, persistence-based: “138” ⟶“135”
- page 218: “totally bounded space” ⟶“totally bounded metric space"

## External references

- page 198, [16]: the volume of the proceedings has now appeared, the article is on pages 484–490
- page 198, [17]: this entry should be removed, and the corresponding citations on pages 45 and 156 should be replaced by references to [18]
- page 198, [20]: the full version of this article is published in the *Journal of Computational Geometry*, 6(2):162–191, 2015.
- page 202, [83]: this is volume 5 in the SIAM series Classics in Applied Mathematics.
- page 203, [93]: this article is now published in the *Journal of Algebra and Its Applications*, Vol. 14, No. 5, 2015.
- page 203, [96]: this article is now published in *Acta Applicandae Mathematicae*, 109(2):527–554, 2010.
- page 204, [111]: “Vol. 1999" should be removed
- page 208, [179]: this article is now published in *Foundations of Computational Mathematics*, 15(3):613–650, 2015.
- page 209, [202]: “Vol. 2" should be removed.
- page 210, [220]: “p. 328” should be replaced by “pp. 328–334”
- page 211, [228]: “Vol. 55” should be removed.
- page 211, [235]: this is volume 58 in the series Graduate Studies in Mathematics.
- page 212, [245]: this is volume 16 in the series Cambridge Monographs in Applied and Computational Mathematics.

## Mathematics

- page 6, line -6: "$C^\infty$-continuous" $\longrightarrow$"$C^2$ (twice continuously differentiable)"
- page 23, Eq. (1.16): "$[p, s]$" $\longrightarrow$"$(p, s)$"
- page 36, Theorem 2.9 (EP Symmetry): in the second bullet, $d - p - 1$ should be $d - p + 1$; meanwhile, in the third bullet, the reflection is not across the minor diagonal but through the origin. This is according to an erratum published by the authors of the article [86] on extended persistence:
  D. Cohen-Steiner, H. Edelsbrunner and J. Harer. Erratum: Extending Persistence Using Poincaré and Lefschetz Duality. *J. Found. Comput. Math.*, 9(1):133–134, Feb. 2009.
- page 71, Figure 4.3, line 4 in the caption: "$C^1$-continuous" $\longrightarrow$"$C^1$"
- page 78, lines 3-4: "$C^\infty$-continuous" $\longrightarrow$"$C^\infty$"
- page 81, footnote 6: "Čech homology" $\longrightarrow$"Čech cohomology"
- page 149, Figure 7.7: the $-1$ should be in subscript (i.e. one should read $\varepsilon_{k-1}$) in both occurrences.
- page 180, line 6: "$(s_1, \cdots, s_{i-1})\mathtt{Q}$" $\longrightarrow$"$(s_{i-1}, \cdots, s_1)\mathtt{Q}$"

## Notations

- page 95, line -8: $\varrho$ should be $\rho$.

## Spelling

- page viii, line -7: "manuscrit" $\longrightarrow$"manuscript"
- page 8, line -8 (ignoring footnote): "lense" $\longrightarrow$"lens"
- page 9, line -4: "adresses" $\longrightarrow$"addresses"
- page 15, 2 lines below display (1.8): "ismormophisms" $\longrightarrow$"isomorphisms"
- page 32, 8 lines below Proposition 2.3: "fom" $\longrightarrow$"from"
- page 34, paragraph on extended persistence, line 2: "as a mean" $\longrightarrow$"as a means"
- page 45, line 9: "repsective" $\longrightarrow$"respective"
- page 72, line below display (4.7): "infinimum" $\longrightarrow$"infimum"
- page 82, Lemma 4.12, line 5: "al l" $\longrightarrow$"all"
- page 86, line 8 of text: "indepent" $\longrightarrow$"independent"
- page 88, paragraph on simplifying the predicates, line 4: "vietoris" $\longrightarrow$"Vietoris"
- page 99, line 4: "we" should be removed

- page 117, Section 1, line 4: “a mean" ⟶“a means"
- page 139, Figure 7.3, line 1 in the caption: “Gromov-Gausdorff" ⟶“Gromov-Hausdorff"
- page 143, line -8: “theorems." ⟶“theorems”
- page 151, line 2: “continguous” ⟶“contiguous"
- page 158, line 9: “desireable” ⟶“desirable”
- page 164, paragraph starting in middle of page, line 3: “Eulclidean” ⟶“Euclidean”
- page 165, Section 2, second paragraph, line 4: “stablity” ⟶“stability”

### English

- page vii, footnote 1, line 2: “was holding" ⟶“held"
- page 25, 2 lines above Theorem 1.13: “a straight consequence" ⟶“a straightforward consequence"
- page 27, last paragraph, first line: “to shape" ⟶“to shaping"
- page 36, Theorems 2.8 and 2.9: “are mirror" ⟶“are mirror images" (repeated 6 times)
- page 39, paragraph on typical applications, first line: “has been" ⟶“was"
- page 52, first line below display (3.3): “diagram" ⟶“diagrams"
- page 52, line 3 below display (3.3): “a commutative diagram” ⟶“commutative diagrams”
- page 52, line 4 below display (3.3): “involves” ⟶“involve”
- page 53, line 12: “noted" ⟶“denoted"
- page 81, Section 3, line 5 of text: “notorious" ⟶“notable"
- page 83, line 1: “notorious" ⟶“notable"
- page 97, line -12: “justifies to run" ⟶“justifies running"
- page 106, 9 lines above item (1): “agrees" ⟶“agree"
- page 120, final sentence of paragraph ending near middle of page: “is stored ... it is treated” ⟶“are stored ... they are treated"
- page 128, second paragraph, line 2: “notorious” ⟶“notable"
- page 130, line -4: “to using" ⟶“to use"
- page 130, line -2: “to studying" ⟶“to study"
- page 182, footnote 7, line 2: “incidence" ⟶“bearing"