Manifold Reconstruction

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Q What do you see?

Why?
Reconstruction Paradigm

**Input:** point cloud $P \subset \mathbb{R}^d$ finite

**Prior:** points of $P$ are sampled along some *unknown shape* $M$ (manifold, compact set etc.), according to some *unknown measure* $\mu$.

**Goal:** (support estimation) build an *approximation* (implicit, PL, simplicial, etc.) that is *structurally faithful* (homotopic, homeomorphic, isotopic, etc.) and *close* (in Hausdorff distance, in $\ell^2$-distance, etc.) to $M$. 
Reconstruction Paradigm

Reconstruction problem is ill-posed by nature.
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→ make *regularity assumptions* on $M$ (fixed dimension, topological type, differentiability, etc.) and *sampling assumptions* (uniform measure, growth rate, etc.)
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→ for a suitable choice of hypotheses, the solution becomes unique up to a set of deformations (solution never unique!)
Reconstruction problem is ill-posed by nature.

→ make *regularity assumptions* on $M$ (fixed dimension, topological type, differentiability, etc.) and *sampling assumptions* (uniform measure, growth rate, etc.)

→ for a suitable choice of hypotheses, the solution becomes unique up to a set of deformations (solution never unique!)
Various forms of inference
Various forms of inference

clustering
Various forms of inference

clustering

topological inference
Various forms of inference

- clustering
- topological inference
- reconstruction
Where do the data come from?

3D scans

Sources

- LASER
- stereo vision
- mechanical sensor

Applications

- Reverse engineering
- Prototyping
- Quality control
- Cultural heritage

Stanford Michelangelo Project
(raw data with 2 billion polygons, sampling with a precision of 0.25 mm)
Where do the data come from?

Medical Imaging

Sources
- MRI scan
- echograph
- ...

Applications
- Diagnostic
- Endoscopy simulation
- Chirurgical intervention planning
Where do the data come from?

Geography, Geology

Sources

- satellite/aerial images
- ground probing
- seismograph

Applications

- Maps making / Terrain modeling
- Prospection (tunnels, oil)
Where do the data come from?

**Higher-Dimensions**

**Sources**
- Databases
- Simulations

**Applications**
- Machine Learning
- Robotics
- Image processing
- Biocomputing

conformation space of cyclo-octane

C$_8$H$_{16}$
Topological Criteria

These three surfaces are *homeomorphic* (they all have genus 1)
There exists a continuous bijection between surfaces, whose inverse is also continuous (formal definition given on the board)

*isotopic* surfaces (unknotted torus)

There exists a family of homeomorphisms which continuously transform the surfaces

knotted torus
Geometric Criteria

Hausdorff distance (order 0):\[ d_H(P, M) = \inf\{\varepsilon \mid P^\varepsilon \supseteq M \text{ and } M^\varepsilon \supseteq P\}\]

Normals (order 1):

Curvature (order 2):
Geometric simplicial complexes

vertex set: $V = \{v_0, v_1, \ldots, v_{n-1}\} \subset \mathbb{R}^d$

$k$-simplex: $\sigma = \text{CH}\{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\}$

*Inclusion property* ($\tau$ face of $\sigma$):

$\sigma \in K$ and $V(\tau) \subseteq V(\sigma) \implies \tau \in K$

*Intersection property*:

$\sigma_1, \sigma_2 \in K$ and $\sigma_1 \cap \sigma_2 \neq \emptyset \implies \\
\sigma_1 \cap \sigma_2 \in K$ and is a face of both
Reconstruction using Delaunay
What Delaunay has to do with reconstruction
What Delaunay has to do with reconstruction

→ faithful approximation of the curve appears as a subcomplex of the Delaunay
→ should hold whenever the point cloud is sufficiently densely sampled
What Delaunay has to do with reconstruction

→ faithful approximation of the curve appears as a subcomplex of the Delaunay
→ should hold whenever the point cloud is sufficiently densely sampled

Q What is this *good* subcomplex? Can it be defined in some canonical way?
Restricted Delaunay triangulation
Restricted Delaunay triangulation

Def: $\mathcal{D}^M(P) := \{\sigma \in \mathcal{D}(P) \mid \sigma^* \cap M \neq \emptyset\}$
Restricted Delaunay triangulation

Def: $D^M(P) := \{\sigma \in D(P) \mid \sigma^* \cap M \neq \emptyset\}$
Def: $\mathcal{D}^M(P) := \{\sigma \in \mathcal{D}(P) \mid \sigma^* \cap M \neq \emptyset\}$
**Sampling Condition**

**Def:** \( P \) is an \( \varepsilon \)-sample of \( M \) if \( \forall x \in M, \min\{\|x - p\| \mid p \in P\} \leq \varepsilon \).
Regularity Condition

**Medial axis:** \( \Gamma_M = \text{cl}\{x \in \mathbb{R}^d \mid |\text{NN}_M(x)| \geq 2\} \)

**Local feature size:** \( \forall x \in \mathbb{R}^d, \ lfs(x) = \min\{\|x - m\| \mid m \in \Gamma_M\} \)

**Reach:** \( \varrho_M = \min\{lfs(x) \mid x \in M\} \)
Regularity Condition

**Medial axis:** \( \Gamma_M = \text{cl}\{x \in \mathbb{R}^d \mid |\text{NN}_M(x)| \geq 2\} \)

**Local feature size:** \( \forall x \in \mathbb{R}^d, \ lfs(x) = \min\{\|x - m\| \mid m \in \Gamma_M\} \)

**Reach:** \( \varrho_M = \min\{\text{lfs}(x) \mid x \in M\} \)

\( x \mapsto x^3 \sin \frac{1}{x} \)

\( \varrho_M = +\infty \) (convex)

\( \varrho_M = r \) \( C^{1,1} \) but not \( C^2 \)

\( \varrho_M = 0 \) \( C^1 \) but not \( C^{1,1} \)
Regularities Condition

→ Fundamental properties: (see [Federer 1958])

**Tangent Ball Lemma:** \( \forall x \in M, \forall c \in M^\perp(x), \|x - c\| \leq \text{lfs}(x) \Rightarrow \nabla \left( B^o(c, \|x - c\|) \cap M = \emptyset \right). \)
Regularity Condition

→ Fundamental properties: (see [Federer 1958])

**Tangent Ball Lemma:** \( \forall x \in M, \forall c \in M^\perp(x), \|x - c\| \leq \text{lfs}(x) \Rightarrow B^o(c, \|x - c\|) \cap M = \emptyset. \)

**Topological Ball Lemma:**
If \( M \) is a \( k \)-manifold, then \( \forall B(c, r) \) s.t. \( B(c, r) \cap \Gamma_M = \emptyset \), \( B(c, r) \cap M \) is either empty or a point or homeomorphic to the ball \( B^k \).
Approximation via Restricted Delaunay

**Theorem:** [Amenta et al. 1998-99]
If $M$ is a closed curve or surface with positive reach $\varrho_M$, and if $P$ is an $\varepsilon$-sample of $M$ with $\varepsilon < \varrho_M$ (curve) or $\varepsilon < 0.1 \varrho_M$ (surface), then:

- $\hat{D}^M(P)$ is homeomorphic to $M$ (denoted $\hat{D}^M(P) \simeq M$),
- $d_H(\hat{D}^M(P), M) \in O(\varepsilon^2)$,
- $\forall \sigma \in \hat{D}^M(P), \forall p \in V(\sigma), \angle_{\sigma \perp M \perp}(p) \in O(\varepsilon)$,
- $\cdots$ (similar areas, curvature estimation, etc.)
Approximation via Restricted Delaunay

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- $\forall \sigma \in \hat{D}^M(P), \forall p \in V(\sigma), \angle \sigma \perp M \perp(p) \in O(\varepsilon)$,
- $\cdots$ (similar areas, curvature estimation, etc.)

Reconstruction is uncertain if $\varepsilon$ is not small enough compared to $\varrho_M$. 
Approximation via Restricted Delaunay

Proof for curves:

show that every edge of $D^M(P)$ connects consecutive points of $P$ along $M$, and vice-versa
Approximation via Restricted Delaunay

Proof for curves:

show that every edge of \( D^M(P) \) connects consecutive points of \( P \) along \( M \), and vice-versa

Let \( c \in pq^* \cap M \).

\[
r = \|c - p\| = \|c - q\| = d(c, P) \leq \varepsilon < \rho_M \leq lfs(c)
\]

\( \Rightarrow B(c, r) \cap M \) is a topological arc
Proof for curves:

show that every edge of $D^M(P)$ connects consecutive points of $P$ along $M$, and vice-versa

Let $c \in pq^* \cap M$.

$r = \|c - p\| = \|c - q\| = d(c, P) \leq \varepsilon < \rho_M \leq lfs(c)$

$\Rightarrow B(c, r) \cap M$ is a topological arc

if $s \in P \setminus \{p, q\}$ belongs to this arc, then the arc is tangent to $\partial B(c, r)$ at the middle point (say $s$)

$q \Rightarrow d(c, P) = r = \|c - s\| \geq lfs(s) > \varepsilon.$

(contradiction with the hypothesis of the theorem)
Approximation via Restricted Delaunay

Proof for curves:

show that every edge of $\mathcal{D}^M(P)$ connects consecutive points of $P$ along $M$, and vice-versa

Let $c \in \text{arc}_M(pq) \cap \partial p^*$. $c \in ps^*$ for some $s \in P \setminus \{p\}$

$\Rightarrow ps \in \mathcal{D}^M(P)$

$\Rightarrow p, s$ consecutive along $M$, with $c \in \text{arc}_M(ps)$

(by previous part of the proof)

$\Rightarrow s = q$
Approximation via Restricted Delaunay

Proof for curves:
show that every edge of $D^M(P)$ connects consecutive points of $P$ along $M$, and vice-versa

$\Rightarrow D^M(P)$ is homeomorphic to $M$ between each pair of consecutive points of $P$

Since $D^M(P)$ is embedded in $D(P)$, it does not self-intersect $\Rightarrow$ global homeomorphism
Computing the Restricted Delaunay

Q How to compute $D^M(P)$ when $M$ is unknown?

→ a whole family of algorithms use various Delaunay extraction criteria:

- crust
- power crust
- cocone
- tight cocone
- ...
Crust Algorithm
Crust algorithm
[Amenta et al. 1997-98]
Crust algorithm

1. Compute Delaunay triangulation of $P$

[Amenta et al. 1997-98]
Crust algorithm

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[Amenta et al. 1997-98]
Crust algorithm

2. Compute poles (furthest Voronoi vertices)

[Amenta et al. 1997-98]
Crust algorithm

[Amenta et al. 1997-98]

3. Add poles to the set of vertices
Crust algorithm

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[Amenta et al. 1997-98]
Crust algorithm

4. Keep Delaunay simplices whose vertices are in $P$

[Amenta et al. 1997-98]
Crust algorithm

in 2-d, crust = \( \mathcal{D}^M(P) \approx M \)

[Amenta et al. 1997-98]
Crust algorithm

in 2-d, crust = \( D^M(P) \approx M \)

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Crust algorithm

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in 2-d, crust $= \mathcal{D}^M(P) \approx M$

in 3-d, crust $\supseteq \mathcal{D}^M(P) \approx M$
Crust algorithm
[Amenta et al. 1997-98]

in 2-d, crust $= \mathcal{D}^M(P) \approx M$

in 3-d, crust $\supseteq \mathcal{D}^M(P) \approx M$

$\Rightarrow$ manifold extraction step in post-processing
Witness Complex
Motivation: effect of scale / dimensionality

What is the reconstruction?
Multi-scale reconstruction

[Guibas, O. 07]

- build a sequence of complexes approximating the input at various scales
- long stable sub-sequences correspond to plausible reconstructions

→ the witness complex enables the use of the Delaunay paradigm
Multi-scale reconstruction algorithm

[Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^n$

→ resample $W$ iteratively, and maintain a simplicial complex:
Multi-scale reconstruction algorithm

[Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^n$

→ resample $W$ iteratively, and maintain a simplicial complex:

Let $L := \{p\}$, for some $p \in W$;
Multi-scale reconstruction algorithm

[Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^n$

→ resample $W$ iteratively, and maintain a simplicial complex:

Let $L := \{p\}$, for some $p \in W$;

WHILE $L \not\subseteq W$

Let $q := \arg \max_{w \in W} d(w, L)$;
Multi-scale reconstruction algorithm

[Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^n$

→ resample $W$ iteratively, and maintain a simplicial complex:

Let $L := \{p\}$, for some $p \in W$;

\[\text{WHILE } L \subsetneq W \text{ \WHILE}\]

Let $q := \arg\max_{w \in W} d(w, L)$;

$L := L \cup \{q\}$;

update simplicial complex;

END_WHERE
Multi-scale reconstruction algorithm

[Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^n$

→ resample $W$ iteratively, and maintain a simplicial complex:

Let $L := \{p\}$, for some $p \in W$;

WHILE $L \subset W$

    Let $q := \arg\max_{w \in W} d(w, L)$;
    $L := L \cup \{q\}$;
    update simplicial complex;

END_WHILE
Multi-scale reconstruction algorithm

[Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^n$

→ resample $W$ iteratively, and maintain a simplicial complex:

Let $L := \{p\}$, for some $p \in W$;

**WHILE** $L \subset W$

Let $q := \text{argmax}_{w \in W} d(w, L)$;
$L := L \cup \{q\}$;
update simplicial complex;

**END WHILE**

Output: the sequence of simplicial complexes
Witness complex
(definition)

Let $L \subseteq \mathbb{R}^d$ (landmarks) s.t. $|L| < +\infty$ and $W \subseteq \mathbb{R}^d$ (witnesses)
Witness complex
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**Def.** $w \in W$ strongly witnesses $[v_0, \ldots, v_k]$ if

$$\|w - v_i\| = \|w - v_j\| \leq \|w - u\|$$

for all $i, j = 0, \ldots, k$ and all $u \in L \setminus \{v_0, \ldots, v_k\}$. 

![Diagram of witness complex](image)
Witness complex
(definition)

Let $L \subseteq \mathbb{R}^d$ (landmarks) s.t. $|L| < +\infty$ and $W \subseteq \mathbb{R}^d$ (witnesses)

**Def.** $w \in W$ *strongly witnesses* $[v_0, \cdots, v_k]$ if $\|w - v_i\| = \|w - v_j\| \leq \|w - u\|$ for all $i, j = 0, \cdots, k$ and all $u \in L \setminus \{v_0, \cdots, v_k\}$.

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Let $L \subseteq \mathbb{R}^d$ (landmarks) s.t. $|L| < +\infty$ and $W \subseteq \mathbb{R}^d$ (witnesses)

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**Def.** $w \in W$ **weakly witnesses** $[v_0, \cdots, v_k]$ if $\|w - v_i\| \leq \|w - u\|$ for all $i = 0, \cdots, k$ and all $u \in L \setminus \{v_0, \cdots, v_k\}$.

**Def.** $C^W(L)$ is the largest abstract simplicial complex built over $L$, whose faces are weakly witnessed by points of $W$.
**Witness complex**

*(properties)*

**Thm. 1** [de Silva 2003] \( \forall W, L, \forall \sigma \in C^W(L), \exists c \in \mathbb{R}^d \) that strongly witnesses \( \sigma \).

\[ \Rightarrow C^W(L) \text{ is a subcomplex of } \mathcal{D}(L) \]

\[ \Rightarrow C^W(L) \text{ is embedded in } \mathbb{R}^d \]

*(if \( L \) lies in general position)*
Weak witness theorem

**Thm. 1** \( \forall W \subseteq \mathbb{R}^d, \forall L \subset \mathbb{R}^d \text{ s.t. } |L| < \infty, \forall \sigma \in C^W(L), \exists c \in \mathbb{R}^d \) that \textit{strongly witnesses} \( \sigma \).

\[ \sigma \in C^W(L) \text{ iff } \forall \tau \subseteq \sigma, \tau \text{ weakly witnessed} \]
Weak witness theorem

\textbf{Thm. 1} \quad \forall W \subseteq \mathbb{R}^d, \forall L \subseteq \mathbb{R}^d \text{ s.t. } |L| < \infty, \forall \sigma \in C^W(L), \exists c \in \mathbb{R}^d \text{ that strongly witnesses } \sigma.

\sigma \in C^W(L) \text{ iff } \forall \tau \subseteq \sigma, \tau \text{ weakly witnessed}
Weak witness theorem

**Thm. 1** \( \forall W \subseteq \mathbb{R}^d, \forall L \subset \mathbb{R}^d \) s.t. \( |L| < \infty \), \( \forall \sigma \in \mathcal{C}^W(L), \exists c \in \mathbb{R}^d \) that strongly witnesses \( \sigma \).

Proof. [Attali, Edelsbrunner, Mileyko 2007]

\( \rightarrow \) induction on the dimension of \( \sigma \):

- Case \( \sigma = [v_0] \): trivial (all witnesses of \( v_0 \) are strong)

\( \sigma \in \mathcal{C}^W(L) \) iff \( \forall \tau \subseteq \sigma, \tau \) weakly witnessed
Weak witness theorem

**Thm. 1** \( \forall W \subseteq \mathbb{R}^d, \forall L \subset \mathbb{R}^d \) s.t. \( |L| < \infty \), \( \forall \sigma \in C^W(L) \), \( \exists c \in \mathbb{R}^d \) that *strongly* witnesses \( \sigma \).

Proof. [Attali, Edelsbrunner, Mileyko 2007]

\( \rightarrow \) induction on the dimension of \( \sigma \):

- Case \( \sigma = [v_0, \cdots, v_k] (k > 0) \):

\( \rightarrow \) induction on \( \# \{ \text{v}_i \text{'s equidistant to w} \} \)

assume that \( \|w - v_0\| = \cdots = \|w - v_{l-1}\| \)

\[ \geq \|w - v_i\| \ \forall i \geq l \]
Weak witness theorem

Thm. 1 $\forall W \subseteq \mathbb{R}^d, \forall L \subset \mathbb{R}^d$ s.t. $|L| < \infty$, $\forall \sigma \in C^W(L)$, $\exists c \in \mathbb{R}^d$ that strongly witnesses $\sigma$.

Proof. [Attali, Edelsbrunner, Mileyko 2007]

→ induction on the dimension of $\sigma$:

• Case $\sigma = [v_0, \cdots, v_k]$ ($k > 0$):

→ induction on $\#\{ v_i's \text{ equidistant to } w\}$

assume that $\|w - v_0\| = \cdots = \|w - v_{l-1}\|$

$\geq \|w - v_i\| \\forall i \geq l$

let $w_l$ be a strong witness of $[v_0, \cdots, v_{l-1}]$
Weak witness theorem

**Thm. 1** \( \forall W \subseteq \mathbb{R}^d, \forall L \subset \mathbb{R}^d \) s.t. \( |L| < \infty \), \( \forall \sigma \in C^W(L) \), \( \exists c \in \mathbb{R}^d \) that strongly witnesses \( \sigma \).

**Proof.** [Attali, Edelsbrunner, Mileyko 2007]

\( \rightarrow \) induction on the dimension of \( \sigma \):

- **Case** \( \sigma = [v_0, \cdots, v_k] \) \( (k > 0) \):

  \( \rightarrow \) induction on \( \# \{ v_i ' s equidistant to w \} \)

  assume that \( ||w - v_0|| = \cdots = ||w - v_{l-1}|| \)

  \( \geq ||w - v_i|| \) \( \forall i \geq l \)

  let \( w_l \) be a strong witness of \( [v_0, \cdots, v_{l-1}] \)

  \( \rightarrow \forall w' \in [w, w_l], B_{w'} \subseteq B_w \cup B_{w_l} \)
Weak witness theorem

**Thm. 1** \[ \forall W \subseteq \mathbb{R}^d, \forall L \subset \mathbb{R}^d \text{ s.t. } |L| < \infty, \forall \sigma \in C^W(L), \exists c \in \mathbb{R}^d \] that strongly witnesses \( \sigma \).

**Proof.** [Attali, Edelsbrunner, Mileyko 2007]

→ induction on the dimension of \( \sigma \):

• Case \( \sigma = [v_0, \cdots, v_k] \) (\( k > 0 \)):

→ induction on \( \#\{ v_i \text{'s equidistant to } w \} \)

assume that \( ||w - v_0|| = \cdots = ||w - v_{l-1}|| \)

\[ \geq ||w - v_i|| \ \forall i \geq l \]

let \( w_l \) be a strong witness of \( [v_0, \cdots, v_{l-1}] \)

→ \( \forall w' \in [w, w_l], \ B_{w'} \subseteq B_w \cup B_{w_l} \)

move \( w \) to \( w' \) as shown opposite

→ \( B_{w'} \cap L = \{v_0, \cdots, v_k\} \)

→ \( |\partial B_{w'} \cap L| \geq l + 1 \)
**Thm. 1** [de Silva 2003] \[\forall W, L, \forall \sigma \in C^W(L), \exists c \in \mathbb{R}^d \text{ that strongly witnesses } \sigma.\]

\[\Rightarrow C^W(L) \text{ is a subcomplex of } D(L)\]

\[\Rightarrow C^W(L) \text{ is embedded in } \mathbb{R}^d\]

(if \(L\) lies in general position)
Witness complex
(properties)

**Thm. 1** [de Silva 2003] \( \forall W, L, \forall \sigma \in C_W(L), \exists c \in \mathbb{R}^d \) that strongly witnesses \( \sigma \).

\[ \Rightarrow C_W(L) \text{ is a subcomplex of } D(L) \]
\[ \Rightarrow C_W(L) \text{ is embedded in } \mathbb{R}^d \]
\[ (\text{if } L \text{ lies in general position}) \]

**Thm. 2** [de Silva, Carlsson 2004]
- The size of \( C_W(L) \) is \( O(d|W|) \)
- The time to compute is \( \text{Poly}(d, |W|, |L|) \)
**Witness complex**

(properties)

**Thm. 1** [de Silva 2003] \( \forall W, L, \forall \sigma \in C_W^W (L), \exists c \in \mathbb{R}^d \) that strongly witnesses \( \sigma \).

\[ \Rightarrow C_W^W (L) \text{ is a subcomplex of } D(L) \]
\[ \Rightarrow C_W^W (L) \text{ is embedded in } \mathbb{R}^d \]
(if \( L \) lies in general position)

**Thm. 2** [de Silva, Carlsson 2004]
- The size of \( C_W^W (L) \) is \( O(d|W|) \)
- The time to compute is \( \text{Poly}(d, |W|, |L|) \)

→ What if \( W, L \) lie on or near a submanifold \( M \)?

**Thm. 3** [Guibas, Oudot 2007]
- [Attali, Edelsbrunner, Mileyko 2007]
Under some conditions, \( C_W^W (L) = D^M (L) \simeq M \)
Witness complex
(connection to reconstruction)

- $W \subseteq \mathbb{R}^d$ is given as input
- $L \subseteq W$ is generated
- underlying manifold $M$ unknown
- only distance comparisons

$\Rightarrow$ algorithm is applicable
  in any metric space

- In $\mathbb{R}^d$, $C_W(L)$ can be maintained by
  updating, for each witness $w$, the list of
  $d+1$ nearest landmarks of $w$.

$\Rightarrow$
  space $\leq O(d|W|)$
  time $\leq O(d|W|^2)$
The full algorithm

Input: a finite point set \( W \subset \mathbb{R}^d \).
The full algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.

Init: $L := \{p\}$; construct lists of nearest landmarks; $C_W(L) = \{[p]\}$;

Invariant: $\forall w \in W$, the list of $d + 1$ nearest landmarks of $w$ is maintained throughout the process.
The full algorithm

Input: a finite point set \( W \subset \mathbb{R}^d \).

Init: \( L := \{p\} \); construct lists of nearest landmarks; \( C^W(L) = \{[p]\} \);

Invariant: \( \forall w \in W \), the list of \( d+1 \) nearest landmarks of \( w \) is maintained throughout the process.

\[ \text{while } L \subsetneq W \]
The full algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.

Init: $L := \{p\}$; construct lists of nearest landmarks; $C^W(L) = \{[p]\}$;

Invariant: $\forall w \in W$, the list of $d+1$ nearest landmarks of $w$ is maintained throughout the process.

\textbf{while} $L \subset W$

\hspace{1cm} insert $\arg\max_{w \in W} d(w, L)$ in $L$;

\hspace{1cm} update the lists of nearest neighbors;
The full algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.

Init: $L := \{p\}$; construct lists of nearest landmarks; $C^W(L) = \{[p]\}$;

Invariant: $\forall w \in W$, the list of $d+1$ nearest landmarks of $w$ is maintained throughout the process.

**WHILE** $L \subsetneq W$

insert $\text{argmax}_{w \in W} d(w, L)$ in $L$;
update the lists of nearest neighbors;
update $C^W(L)$;

**END WHILE**
The full algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.

Init: $L := \{p\}$; construct lists of nearest landmarks; $C^W(L) = \{[p]\}$;

Invariant: $\forall w \in W$, the list of $d+1$ nearest landmarks of $w$ is maintained throughout the process.

\[\text{while } L \subsetneq W \\text{ do}\]

insert $\text{argmax}_{w \in W} d(w, L)$ in $L$;
update the lists of nearest neighbors;
update $C^W(L)$;

\[\text{end while}\]
The full algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.

Init: $L := \{p\}$; construct lists of nearest landmarks; $C^W(L) = \{[p]\}$;

Invariant: $\forall w \in W$, the list of $d+1$ nearest landmarks of $w$ is maintained throughout the process.

\[ \text{while } L \subsetneq W \]
\[ \quad \text{insert } \arg\max_{w \in W} d(w, L) \text{ in } L; \]
\[ \quad \text{update the lists of nearest neighbors;} \]
\[ \quad \text{update } C^W(L); \]
\[ \text{END\_WHILE} \]

Output: the sequence of complexes $C^W(L)$
Relation with the restricted Delaunay

If $M$ is a closed $k$-manifold smoothly embedded in $\mathbb{R}^d$, then, under sufficient sampling conditions, $C^W(L) = D^M(L) \simeq M$
Relation with the restricted Delaunay

If $M$ is a closed $k$-manifold smoothly embedded in $\mathbb{R}^d$, then, under sufficient sampling conditions, $C^W(L) = D^M(L) \simeq M$

- Case $k = 1$:
  - $C^W(L) = D^M(L) \simeq M$

[Guibas, O. 07]
[Attali, Edelsbrunner, Mileyko 07]
Relation with the restricted Delaunay

If $M$ is a closed $k$-manifold smoothly embedded in $\mathbb{R}^d$, then, under sufficient sampling conditions, $C^W(L) = D^M(L) \simeq M$

- **Case $k = 1$:**
  - $C^W(L) = D^M(L) \simeq M$

- **Case $k = 2$:**
  - $C^W(L) \subseteq D^M(L) \simeq M$
  - $C^W(L) \nsubseteq D^M(L)$

[Amenta, Bern 98]
[Attali, Edelsbrunner, Mileyko 07]
[de Silva, Carlsson 04]
[Guibas, O. 07]
Relation with the restricted Delaunay

If $M$ is a closed $k$-manifold smoothly embedded in $\mathbb{R}^d$, then, under sufficient sampling conditions, $\mathcal{C}^W(L) = \mathcal{D}^M(L) \simeq M$

- **Case $k = 1$:**
  - $\mathcal{C}^W(L) = \mathcal{D}^M(L) \simeq M$

- **Case $k = 2$:**
  - $\mathcal{C}^W(L) \subseteq \mathcal{D}^M(L) \simeq M$
  - $\mathcal{C}^W(L) \not\subseteq \mathcal{D}^M(L)$

- **Case $k \geq 3$:**
  - $\mathcal{C}^W(L) \not\subseteq \mathcal{D}^M(L)$
  - $\mathcal{D}^M(L) \not\simeq M$

[Cheng, Dey, Ramos 05]
[O. 07]
Relation with the restricted Delaunay
(case of curves)

**Conjecture** [Carlsson, de Silva 2004]
$C^W(L)$ coincides with $D^M(L)$...
Relation with the restricted Delaunay
(case of curves)

Conjecture [Carlsson, de Silva 2004]
$C^W(L)$ coincides with $D^M(L)$...

... under some conditions on $W$ and $L
Relation with the restricted Delaunay  
(case of curves)

**Thm:** If $M$ is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $d_H(W, M) \leq \delta$, $L \subseteq W$ $\varepsilon$-sparse $\varepsilon$-sample of $W$ with $\delta \ll \varepsilon \ll \varrho_M$, then $C^W(L) = D^M(L) \simeq M$. 
Relation with the restricted Delaunay
(case of curves)

Thm: If $M$ is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $d_H(W, M) \leq \delta$, $L \subseteq W$ $\varepsilon$-sparse $\varepsilon$-sample of $W$ with $\delta \ll \varepsilon \ll \delta_M$, then $C^W(L) = D^M(L) \simeq M$.

→ There is a plateau in the diagram of Betti numbers of $C^W(L)$. 

There is a plateau in the diagram of Betti numbers of $CW(L)$. 

Relation with the restricted Delaunay (case of curves)
Thm: If $M$ is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $d_H(W, M) \leq \delta$, $L \subseteq W$ $\varepsilon$-sparse $\varepsilon$-sample of $W$ with $\delta \ll \varepsilon \ll \rho_M$, then $C^W(L) = D^M(L) \simeq M$.

- $D^M(L) \subseteq C^W(L)$
Relation with the restricted Delaunay
(case of curves)

**Thm:** If $M$ is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $d_H(W, M) \leq \delta$, $L \subseteq W$ $\varepsilon$-sparse $\varepsilon$-sample of $W$ with $\delta \ll \varepsilon \ll \varrho_M$, then $C^W(L) = D^M(L) \simeq M$.

- $D^M(L) \subseteq C^W(L)$
Relation with the restricted Delaunay
(case of curves)

**Thm:** If $M$ is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $d_H(W, M) \leq \delta$, $L \subseteq W$ $\epsilon$-sparse $\epsilon$-sample of $W$ with $\delta \ll \epsilon \ll \rho_M$, then $C^W(L) = D^M(L) \simeq M$.

- $D^M(L) \subseteq C^W(L)$
Relation with the restricted Delaunay
(case of curves)

**Thm:** If $M$ is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $d_H(W, M) \leq \delta$, $L \subseteq W$ $\varepsilon$-sparse $\varepsilon$-sample of $W$ with $\delta \ll \varepsilon \ll \rho_M$, then $C_W(L) = D^M(L) \simeq M$.

- $D^M(L) \subseteq C_W(L)$
- $C_W(L) \subseteq D^M(L)$
Thm: If $M$ is a closed curve with positive reach, $W \subset \mathbb{R}^d$ s.t. $d_H(W, M) \leq \delta$, $L \subseteq W$ $\varepsilon$-sparse $\varepsilon$-sample of $W$ with $\delta \ll \varepsilon \ll \varrho_M$, then $C^W(L) = D^M(L) \simeq M$.

- $D^M(L) \subseteq C^W(L)$
- $C^W(L) \subseteq D^M(L)$
Some results

Input:

Output:
Relation with the restricted Delaunay
(case of surfaces)

\[ D^M(L) \not\subseteq C^W(L) \text{ if } W \subsetneq M \]

\[ \varepsilon = 0.2, \ rch(M) \approx 0.25 \]
Relation with the restricted Delaunay
(case of surfaces)

\[ D^M(L) \nsubseteq C^W(L) \text{ if } W \subsetneq M \]

order-2 Voronoi diagram

\[ \varepsilon = 0.2, \ rch(M) \approx 0.25 \]
Relation with the restricted Delaunay
(case of surfaces)

\[ D^M(L) \not\subset C^W(L) \text{ if } W \subsetneq M \]

\[ \varepsilon = 0.2, \ rch(M) \approx 0.25 \]
Relation with the restricted Delaunay
(case of surfaces)

\[ \mathcal{D}^M(L) \not\subseteq C^W(L) \text{ if } W \subsetneq M \]

Solution  relax witness test.

\[ \Rightarrow C^W(L) = D^M(L) + \text{slivers} \]
\[ \Rightarrow C^W(L) \not\subseteq D(L) \]
\[ \Rightarrow C^W(L) \text{ not embedded.} \]

Post-process extract manifold \( M \)
from \( C^W(L) \cap D(L) \)
[Amenta, Choi, Dey, Leekha]
Some results
Some results
Some results
Some results
Some results

Tangle Cube (diam.=4, rch=0.25, genus=5, delta=0.02, noise=0, 78,088 witnesses)

1

2

3
Some results

Asklepios (diam.=120, lrl=4, genus=4, delta=1, noise=0, 40,888 witnesses)

input model provided courtesy of IMATI by the Aim@Shape repository
Some results

Asklepios (diam.=120, lr=4, genus=4, delta=1, noise=0, 48,888 witnesses)

input model provided courtesy of IMATI by the Aim@Shape repository
Some results

Asklepios (diam.=128, l1=4, genus=4, delta=1, noise=0, 48,888 witnesses)

input model provided courtesy of IMATI by the Aim@Shape repository
Some results

Aklepios (diam.=120, l1=4, genus=4, delta=1, noise=0, 48,888 witnesses)
Some results

Asklepios (diam.=120, lrl=4, genus=4, delta=1, noise=0, 48,888 witnesses)

input model provided courtesy of IMATI by the Aim@Shape repository
Some results

Asklepios (diam.=120, l1=4, genus=4, delta=1, noise=0, 48,888 witnesses)

input model provided courtesy of IMATI by the Aim@Shape repository
Some results

Asklepios (diam.=120, lrl=4, genus=4, delta=1, noise=0, 48,888 witnesses)

input model provided courtesy of IMATI by the Aim@Shape repository
Some results

Asklepios (diam=120, lr1=4, genus=4, delta=1, noise=0, 48,888 witnesses)

input model provided courtesy of IMATI by the Aim@Shape repository
Some results

Asklepios (diam.=120, l1=4, genus=4, delta=1, noise=0, 48,888 witnesses)

input model provided courtesy of IMATI by the Aim@Shape repository
Some results

Asklepios \( (\text{diam.} = 120, \text{lr1} = 4, \text{genus} = 4, \text{delta} = 1, \text{noise} = 0, \text{48,888 witnesses}) \)

input model provided courtesy of IMATI by the Aim@Shape repository
Some results

Filigree (diam.=1.2, rch=?, genus=65, delta=0.001, noise=0, 514,300 witnesses)

input model provided courtesy of Sensable Technologies by the Aim@Shape repository
Some results

Filigree (diam.=1.2, rch=?, genus=65, delta=0.001, noise=0, 514,300 witnesses)

input model provided courtesy of Sensable Technologies by the Aim@Shape repository
Some results

Filgree (diam.=1.2, rch=?, genus=65, delta=0.001, noise=0, 514,300 witnesses)

input model provided courtesy of Sensable Technologies by the Aim@Shape repository
Some results

Happy Buddha (diam.=0.1, rch=?, genus=104, delta=?, noise=?, 1,631,368 witnesses)

input model courtesy of the Computer Graphics Laboratory at Stanford University
Some results

Happy Buddha ($diam. = 0.1$, $rch = ?$, $genus = 104$, $delta = ?$, $noise = ?$, 1,631,360 witnesses)

input model courtesy of the Computer Graphics Laboratory at Stanford University
Some results

Happy Buddha (diam.=0.1, rch=?, genus=104, delta=?, noise=?, 1,631,368 witnesses)

input model courtesy of the Computer Graphics Laboratory at Stanford University
Relation with the restricted Delaunay

\( \partial[-\Delta, \Delta]^4 \subset \mathbb{R}^4 \)
\( \delta \ll 1 \ll \Delta \)

\( u(1, 0, 0, \Delta) \)
\( v(1, 1, 0, \Delta) \)
\( w(0, 1, 0, \Delta) \)
\( p_0(0, 0, \delta, \Delta) \)
\( c_0(\frac{1}{2}, \frac{1}{2}, \frac{\delta}{2}, \Delta) \)

\( \text{Relation with the restricted Delaunay} \)

(intrinsic dim. \( \geq 3 \))

[O. 2007]
Relation with the restricted Delaunay

\[ \partial [-\Delta, \Delta]^4 \subset \mathbb{R}^4 \]
\[ \delta \ll 1 \ll \Delta \]
\[ t = \Delta + \delta/2 \]

\[ \partial[-\Delta, \Delta]^4 \subset \mathbb{R}^4 \]
\[ \delta \ll 1 \ll \Delta \]
\[ t = \Delta + \delta/2 \]

[\(p, u, v, w\)]* is horizontal [CDR05]

\[ \text{[O. 2007]} \]

\[ \text{Relation with the restricted Delaunay (intrinsic dim. } \geq 3) \]

\[ D^M(L) \not\cong M \]

\[ u(1, 0, 0, \Delta) \]
\[ v(1, 1, 0, \Delta) \]
\[ w(0, 1, 0, \Delta) \]
\[ p(0, 0, 0, \Delta + \delta) \]
Relation with the restricted Delaunay

(intrinsic dim. $\geq 3$)  

\[ \partial[-\Delta, \Delta]^4 \subset \mathbb{R}^4 \]

\[ \delta \ll 1 \ll \Delta \]

\[ t = \Delta + \delta/2 \]

\[ [p, u, v]^* \cap M = \{ c \} \]

\[ [p, v, w]^* \cap M = \{ c \} \]

\[ D^M(L) \nsubseteq M \]

\[ u(1, 0, 0, \Delta) \]

\[ v(1, 1, 0, \Delta) \]

\[ w(0, 1, 0, \Delta) \]

\[ p(0, 0, 0, \Delta + \delta) \]

\[ c(\frac{1}{2}, \frac{1}{2}, \frac{\delta}{2}, \Delta + \frac{\delta}{2}) \]

\[ [p, u, v, w]^* \text{ is horizontal} \]  

[CDR05]

[O. 2007]
Relation with the restricted Delaunay

\[(\text{intrinsic dim. } \geq 3)\] [O. 2007]

\[\partial[-\Delta, \Delta]^4 \subset \mathbb{R}^4\]
\[\delta \ll 1 \ll \Delta\]

\[t = \Delta + \delta/2\]

\[\mathcal{D}^M(L) \not\cong M\]

\[u(1, 0, 0, \Delta)\]
\[v(1, 1, 0, \Delta)\]
\[w(0, 1, 0, \Delta)\]
\[p(0, 0, 0, \Delta + \delta)\]
\[c\left(\frac{1}{2}, \frac{1}{2}, \frac{\delta}{2}, \Delta + \frac{\delta}{2}\right)\]

\[[p, u, v, w]^* \text{ is horizontal} \ [\text{CDR05}]\]

\[[p, u, v]^* \cap M = \{c\} \Rightarrow \mathcal{D}^M(L) \text{ is no longer a closed hyper-surface if } c \text{ is moved downwards slightly}\]
Relation with the restricted Delaunay (arbitrary dimensions)

If $M$ is a closed $k$-manifold smoothly embedded in $\mathbb{R}^d$, then, under reasonable sampling conditions, $C^W(L) = D^M(L) \simeq M$

- Case $k = 1$:
  - $C^W(L) = D^M(L) \simeq M$

- Case $k = 2$:
  - $C^W(L) \subseteq D^M(L) \simeq M$
  - $C^W(L) \not\subseteq D^M(L)$

- Case $k \geq 3$:
  - $C^W(L) \not\subseteq D^M(L)$
  - $D^M(L) \not\simeq M$

→ Source of problems: slivers

assign weights to the landmarks to remove all slivers from the vicinity of $D^M(L)$ [Cheng et al. 00]
Weighted Voronoi / Delaunay

**Input:** point cloud \( P \), weight function \( \omega : P \rightarrow \mathbb{R}_{\geq 0} \)

**Metric:**
\[
d(x, (p, \omega(p)))^2 = \|x - p\|^2 - \omega(p)^2
\]
**Input:** point cloud $P$, weight function $\omega : P \to \mathbb{R}_{\geq 0}$

**Metric:** $d(x, (p, \omega(p)))^2 = \|x - p\|^2 - \omega(p)^2$

**Induced diagram:** $V(p) = \{x \in \mathbb{R}^d \mid d(x, (p, \omega(p))) \leq d(x, (q, \omega(q))) \ \forall q \in P\}$
Weighted Voronoi / Delaunay

**Input:** point cloud $P$, weight function $\omega : P \rightarrow \mathbb{R}_{\geq 0}$

**Metric:** $d(x, (p, \omega(p)))^2 = \|x - p\|^2 - \omega(p)^2$

**Induced diagram:** $\mathcal{V}(p) = \{x \in \mathbb{R}^d | d(x, (p, \omega(p))) \leq d(x, (q, \omega(q))) \forall q \in P\}$

**Prop:** $x \in \mathcal{V}(p) \iff x$ center of sphere orthogonal to $B(p, \omega(p))$

and obtuse to $B(q, \omega(q))$ for all $q \in P \setminus \{p\}$
Point / sphere lifting

\[ p^* = (p_1, \cdots, p_d, p_{d+1} = \sum_{i=1}^{d} p_i^2 - \omega(p)^2) \]
Point / sphere lifting

\[ \Sigma^*_x : \sum_{i=1}^{d} (-x_i)y_i + y_{d+1} = r^2 - x^2 \]

\[ \Sigma_x : y^2 - 2x \cdot y + x^2 = r^2 \]
$p^* \in \Sigma^*_x$

$B(p, \omega(p)) \subseteq \Sigma_x$
Point / sphere lifting
Point / sphere lifting

\[ p^* \text{ above } \Sigma^*_x \]

\[ \Sigma^*_x \]

\[ \Sigma_x \]

\[ x \]

\[ \omega(p) \]

\[ B(p, \omega(p)) \cap \Sigma_x \]
Point / sphere lifting

Lower CH

Weighted Delaunay
Sliver Removal [CDEFT’00]

- Each landmark \( u \in L \) is assigned a weight \( 0 \leq \omega(u) < \frac{1}{2} d(u, L \setminus \{u\}) \).
- The Voronoi diagram of \( L \) is replaced by its weighted version, \( \mathcal{V}_\omega(L) \): 
  \[
  p \in \text{cell}(u) \text{ iff } \forall v \in L, \ d(p, u)^2 - \omega(u)^2 \leq d(p, v)^2 - \omega(v)^2
  \]
- \( \mathcal{V}_\omega(L) \) is an affine diagram, its dual complex \( \mathcal{D}_\omega(L) \) is a triangulation.
Sliver Removal

- Each landmark $u \in L$ is assigned a weight $0 \leq \omega(u) < \frac{1}{2} d(u, L \setminus \{u\})$.
- The Voronoi diagram of $L$ is replaced by its weighted version, $\mathcal{V}_\omega(L)$:
  \[ p \in \text{cell}(u) \text{ iff } \forall v \in L, \ d(p, u)^2 - \omega(u)^2 \leq d(p, v)^2 - \omega(v)^2 \]
- $\mathcal{V}_\omega(L)$ is an affine diagram, its dual complex $\mathcal{D}_\omega(L)$ is a triangulation.

**Thm** [Cheng, Dey, Ramos 05] If $L$ is an $\varepsilon$-sparse $\varepsilon$-sample of $M$, with $\varepsilon \ll rch(M)$, then $\exists \omega_0$ that removes slivers from the vicinity of $\mathcal{D}_{\omega_0}^M(L)$.

$\Rightarrow \mathcal{D}_{\omega_0}^M(L) \simeq M$

- $\omega_0$ removes slivers, thereby improving the normals
- Closed Ball Property
Sliver Removal

- Each landmark \( u \in L \) is assigned a weight \( 0 \leq \omega(u) < \frac{1}{2} d(u, L \setminus \{u\}) \).
- The Voronoi diagram of \( L \) is replaced by its weighted version, \( \mathcal{V}_\omega(L) \):
  \[
p \in \text{cell}(u) \text{ iff } \forall v \in L, \ d(p, u)^2 - \omega(u)^2 \leq d(p, v)^2 - \omega(v)^2
  \]
- \( \mathcal{V}_\omega(L) \) is an affine diagram, its dual complex \( \mathcal{D}_\omega(L) \) is a triangulation.

**Thm** [Cheng, Dey, Ramos 05] If \( L \) is an \( \varepsilon \)-sparse \( \varepsilon \)-sample of \( M \), with \( \varepsilon \ll \text{rch}(M) \), then \( \exists \omega_0 \) that removes slivers from the vicinity of \( \mathcal{D}^M_{\omega_0}(L) \).

\[
\Rightarrow \mathcal{D}^{M}_{\omega_0}(L) \simeq M
\]

- \( \omega_0 \) removes slivers, thereby improving the normals
- Closed Ball Property
Each landmark $u \in L$ is assigned a weight $0 \leq \omega(u) < \frac{1}{2} d(u, L \setminus \{u\})$.

The Voronoi diagram of $L$ is replaced by its weighted version, $\mathcal{V}_\omega(L)$:

$$p \in \text{cell}(u) \iff \forall v \in L, d(p, u)^2 - \omega(u)^2 \leq d(p, v)^2 - \omega(v)^2$$

$\mathcal{V}_\omega(L)$ is an affine diagram, its dual complex $\mathcal{D}_\omega(L)$ is a triangulation.

**Thm** [Cheng, Dey, Ramos 05] If $L$ is an $\varepsilon$-sparse $\varepsilon$-sample of $M$, with $\varepsilon \ll \text{rch}(M)$, then $\exists \omega_0$ that removes slivers from the vicinity of $\mathcal{D}_\omega^M(L)$.

$$\implies \mathcal{D}_\omega^M(L) \simeq M$$

- $\omega_0$ removes slivers, thereby improving the normals
- Closed Ball Property

**Thm** [Boissonnat, Guibas, O. 07] [Boissonnat, Dyer, Ghosh, O. 17]

- Under the same conditions on $L$, one has $\mathcal{C}_\omega^W(L) \subseteq \mathcal{D}_\omega^M(L)$ for all $W \subseteq M$.

- If $W$ is a $\delta$-sample of $M$, with $\delta \ll \varepsilon$, then $\mathcal{C}_\omega^W(L) = \mathcal{D}_\omega^M(L)$.
Application to reconstruction in arbitrary dimensions

[Guibas, O. 07] [Boissonnat, Guibas, O. 07]

Input: a finite point set \( W \subset \mathbb{R}^d \).

\( \rightarrow \) greedy: furthest-point resampling of \( L \)

maintain \( C^W_\omega (L) \) for some carefully-chosen weight function \( \omega \).

Init: \( L := \{p\} \), for some arbitrary \( p \in W \);
Application to reconstruction in arbitrary dimensions

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Input: a finite point set $W \subset \mathbb{R}^d$.

→ greedy: furthest-point resampling of $L$

maintain $C^W_\omega(L)$ for some carefully-chosen weight function $\omega$.

Init: $L := \{p\}$, for some arbitrary $p \in W$;

WHILE $L \subsetneq W$

insert $p = \arg\max_{w \in W} d(w, L)$ in $L$;
Application to reconstruction in arbitrary dimensions

[Guibas, O. 07] [Boissonnat, Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^d$.

→ greedy: furthest-point resampling of $L$

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while $L \subsetneq W$

insert $p = \arg\max_{w \in W} d(w, L)$ in $L$;
Application to reconstruction in arbitrary dimensions

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Init: $L := \{p\}$, for some arbitrary $p \in W$;

while $L \subsetneq W$

insert $p = \arg \max_{w \in W} d(w, L)$ in $L$;

assign weight to $p$;
Application to reconstruction in arbitrary dimensions

[Guibas, O. 07] [Boissonnat, Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^d$.

→ greedy: furthest-point resampling of $L$

\[
\text{maintain } C_w(L) \text{ for some carefully-chosen weight function } \omega.
\]

Init: $L := \{p\}$, for some arbitrary $p \in W$;

\textbf{while} $L \subsetneq W$

insert $p = \arg\max_{w \in W} d(w, L)$ in $L$;

\textbf{assign weight} to $p$;

update $C_w(L)$;

\textbf{end while}
Application to reconstruction in arbitrary dimensions

[Guibas, O. 07] [Boissonnat, Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^d$.

→ greedy: furthest-point resampling of $L$

    maintain $C^W_\omega (L)$ for some carefully-chosen weight function $\omega$.

Init: $L := \{p\}$, for some arbitrary $p \in W$;

WHILE $L \subsetneq W$

    insert $p = \arg\max_{w \in W} d(w, L)$ in $L$;

    assign weight to $p$;

    update $C^W_\omega (L)$;

END WHILE
Application to reconstruction in arbitrary dimensions

[Guibas, O. 07] [Boissonnat, Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^d$.

→ greedy: furthest-point resampling of $L$

\[ \text{maintain } C^W_\omega (L) \text{ for some carefully-chosen weight function } \omega. \]

Init: $L := \{p\}$, for some arbitrary $p \in W$;

\begin{algorithmic}
\WHILE{$L \subsetneq W$}
  \STATE insert $p = \arg\max_{w \in W} d(w, L)$ in $L$;
  \STATE assign weight to $p$;
  \STATE update $C^W_\omega (L)$;
\ENDWHILE
\end{algorithmic}
Application to reconstruction in arbitrary dimensions

[Guibas, O. 07] [Boissonnat, Guibas, O. 07]

Input: a finite point set \( W \subset \mathbb{R}^d \).

→ greedy: furthest-point resampling of \( L \)

    maintain \( C^W_\omega (L) \) for some carefully-chosen weight function \( \omega \).

Init: \( L := \{p\} \), for some arbitrary \( p \in W \);

\[ \text{while } L \subsetneq W \]

    insert \( p = \arg\max_{w \in W} d(w, L) \) in \( L \);

    assign weight to \( p \);

    update \( C^W_\omega (L) \);

END_WHILE
Application to reconstruction in arbitrary dimensions

[Guibas, O. 07] [Boissonnat, Guibas, O. 07]

Input: a finite point set \( W \subset \mathbb{R}^d \).

\[ \rightarrow \text{greedy}: \text{furthest-point resampling of } L \]

\[ \text{maintain } C^W_\omega(L) \text{ for some carefully-chosen weight function } \omega. \]

Init: \( L := \{p\} \), for some arbitrary \( p \in W \);

\begin{algorithmic}
  \While{$L \subsetneq W$}
    \State insert \( p = \arg\max_{w \in W} d(w, L) \) in \( L \);
    \State assign weight to \( p \);
    \State update \( C^W_\omega(L) \);
  \EndWhile
\end{algorithmic}

Output: sequence of simplicial complexes \( C^W_\omega(L) \) built throughout.
Weight Assignment

[Boissonnat, Dyer, Ghosh, O. 17]

Candidate simplices: (requires to know the intrinsic dimension $k$)

$$N(p) = \{66^k\text{-NN of } p \text{ in } L\}$$

$\sigma \in 2^N(p)$ is a candidate simplex if it is a sliver (flat + small radius)
Weight Assignment
[Boissonnat, Dyer, Ghosh, O. 17]

Candidate simplices: (requires to know the intrinsic dimension $k$)

$$N(p) = \{66^k\text{-NN of } p \text{ in } L\}$$

$\sigma \in 2^N(p)$ is a candidate simplex if it is a sliver (flat + small radius)

every candidate simplex $\sigma$ has a forbidden interval $I_\sigma$ of weights for $p$

(those for which $\sigma \in D_\omega(P)$)
Weight Assignment

[Boissonnat, Dyer, Ghosh, O. 17]

Candidate simplices: (requires to know the intrinsic dimension $k$)

$$N(p) = \{66^k\text{-NN of } p \text{ in } L\}$$

$\sigma \in 2N(p)$ is a **candidate simplex** if it is a *sliver* (flat + small radius)

every candidate simplex $\sigma$ has a **forbidden interval** $I_\sigma$ of weights for $p$

take $\omega(p) \in [0, \bar{\omega}] \setminus \bigcup_{\sigma:\text{candidate}} I_\sigma$ (those for which $\sigma \in D_\omega(P)$)
Candidate simplices: (requires to know the intrinsic dimension $k$)

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take \( \omega(p) \in [0, \bar{\omega}] \setminus \bigcup_{\sigma:\text{candidate}} I_\sigma \) (those for which \( \sigma \in \mathcal{D}_\omega(P) \))

**Claims:**

\[ [0, \bar{\omega}] \setminus \bigcup_{\sigma:\text{candidate}} I_\sigma \neq \emptyset \]

for every \( \sigma, I_\sigma \) depends only on weights of \( L \) and on radius & flatness of \( \sigma \)

(no need to maintain \( \mathcal{D}(L) \))
Application to reconstruction in arbitrary dimensions

[Guibas, O. 07] [Boissonnat, Guibas, O. 07] [Boissonnat, Dyer, Ghosh, O. 17]

Input: a finite point set $W \subset \mathbb{R}^d$.

**Thm** If $W$ is a $\delta$-sample of $M$, with $\delta \ll \text{rch}(M)$, then, at some stage of the process, the weight assignment removes all slivers from the vicinity of $D_M^\omega (L)$, therefore $C^W (L) = D_M^\omega (L) \simeq M$.

Init: $L := \{p\}$, for some arbitrary $p \in W$;

**while** $L \subsetneq W$

- insert $p = \arg\max_{w \in W} d(w, L)$ in $L$;
- assign weight to $p$;
- update $C^W (L)$;

**end while**

Output: sequence of simplicial complexes $C^W (L)$ built throughout.
Application to reconstruction in arbitrary dimensions

[Guibas, O. 07] [Boissonnat, Guibas, O. 07] [Boissonnat, Dyer, Ghosh, O. 17]

Input: a finite point set $W \subset \mathbb{R}^d$.

Running time: $dn(2^{O(k)}n + 2^{O(k^2)} + O(kn)) + O(k^3n)$

Space usage: $n(d + 2^{O(k^2)}) + O(kn^2)$

$\begin{align*}
\text{Init: } & L := \{p\}, \text{ for some arbitrary } p \in W; \\
\text{while } & L \subsetneq W \\
\text{insert } & p = \arg\max_{w \in W} d(w, L) \text{ in } L; \\
\text{assign weight } & \text{ to } p; \\
\text{update } & C^W_\omega (L); \\
\text{END WHILE} \\
\text{Output: sequence of simplicial complexes } & C^W_\omega (L) \text{ built throughout.}
\end{align*}$
Some results
Example of application: Sensor Networks

[Gao, Guibas, O., Wang '07]

Input: a set of nodes $W$ sampling some unknown planar domain $M$.

→ each node has:
  - no location capabilities,
  - limited computation power,
  - limited memory,
  - limited battery power,
  - communication radius $r$.

Q What is the topology of $X$?

How many nodes are needed to recover it?
Example of application: Sensor Networks

[Gao, Guibas, O., Wang ’07]

**Input:** a set of nodes $W$ sampling some unknown planar domain $M$.

→ the witness complex disregards the embedding (only approximate geodesic distances are used)