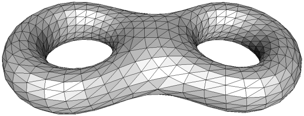
# Discrete Systolic Inequalities and Decompositions of Triangulated Surfaces

Éric Colin de Verdière [1,2]
Alfredo Hubard [1,3]    Arnaud de Mesmay [1]

[1]DIENS, équipe Talgo
École normale supérieure, Paris

[2]CNRS

[3]Institut Gaspard Monge, Université Paris-Est Marne-la-Vallée

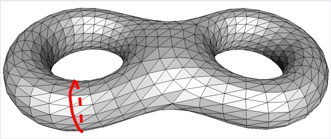


# A primer on surfaces

We deal with ***connected***, ***compact*** and ***orientable*** surfaces of ***genus*** $g$ without boundary.

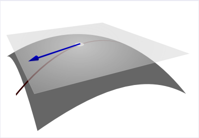
## Discrete metric

Triangulation $G$.
Length of a curve $|\gamma|_G$:
Number of edges.



## Riemannian metric

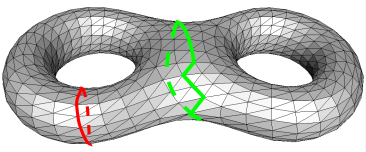
Scalar product $m$ on the tangent space.
Riemannian length $|\gamma|_m$.



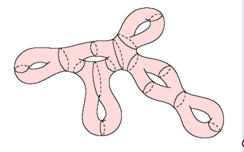
# Systoles and surface decompositions

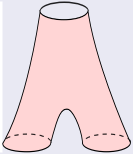
We study the length of topologically interesting curves and graphs, for discrete and continuous metrics.

## 1.Non-contractible curves

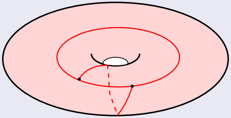


## 2.Pants decompositions





## 3.Cut-graphs

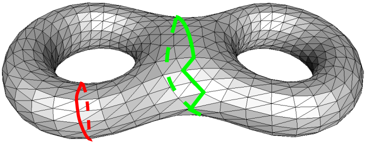


# Motivations

Why should we care ?

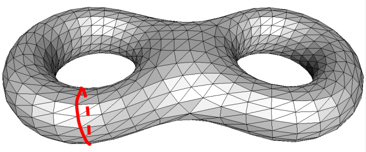
- ***Topological graph theory:*** If the shortest non-contractible cycle is ***long***, the surface is ***planar-like***.
  $\Rightarrow$ Uniqueness of embeddings, colourability, spanning trees.
- ***Riemannian geometry:***
  René Thom: ***"Mais c'est fondamental !"***.
  Links with isoperimetry, topological dimension theory, number theory.
- ***Algorithms for surface-embedded graphs:*** Cookie-cutter algorithm for surface-embedded graphs: Decompose the surface, solve the planar case, recover the solution.
- More practical sides: ***texture mapping***, ***parameterization***, ***meshing*** ...

## *Part 1:*
## *Cutting along curves*

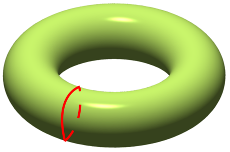


# Curves with prescribed topological properties

**Discrete setting**



**Continuous setting**



What is the length of the red curve?

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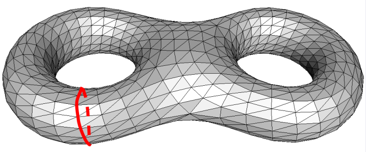

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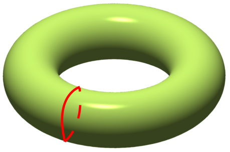

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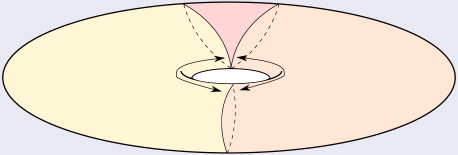
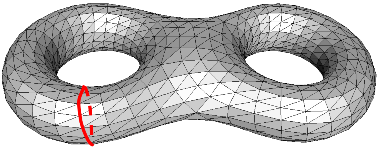
## Intuition



It should have length $O(\sqrt{A})$ or $O(\sqrt{n})$, but what is the dependency on $g$ ?

# Discrete Setting: Topological graph theory

The ***edgewidth*** of a triangulated surface is the length of the shortest ***noncontractible*** cycle.

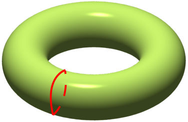


**Theorem (Hutchinson '88)**

*The edgewidth of a triangulated surface with n triangles of genus g is* $O(\sqrt{n/g}\log g)$.

- Hutchinson conjectured that the right bound is $\Theta(\sqrt{n/g})$.
- Disproved by Przytycka and Przytycki '90-97 who achieved $\Omega(\sqrt{n/g}\sqrt{\log g})$, and conjectured $\Theta(\sqrt{n/g}\log g)$.
- How about non-separating, or separating but non-contractible cycles ?

The ***systole*** of a Riemannian surface is the length of the shortest ***noncontractible*** cycle.



**Theorem (Gromov '83, Katz and Sabourau '04)**

*The systole of a Riemannian surface of genus $g$ and area $A$ is $O(\sqrt{A/g}\log g)$.*

# Continuous Setting: Systolic Geometry

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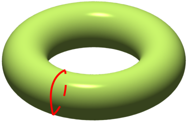

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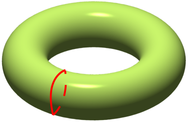

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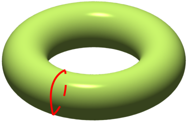

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- Buser and Sarnak '94 introduced ***arithmetic surfaces*** achieving the lower bound $\Omega(\sqrt{A/g}\log g)$.
- Larry Guth: "Arithmetic hyperbolic surfaces are remarkably hard to picture."

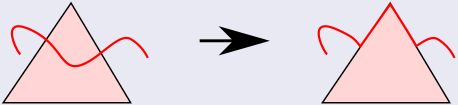
# A two way street: From discrete to continuous.

## Theorem (Colin de Verdière, Hubard, de Mesmay '14)

*Let $(S,G)$ be a triangulated surface of genus $g$, with $n$ triangles. There exists a Riemannian metric $m$ on $S$ with area $n$ such that for every closed curve $\gamma$ in $(S,m)$ there exists a homotopic closed curve $\gamma'$ on $(S,G)$ with*

$$|\gamma'|_G \leq (1+\delta)\sqrt[4]{3}\ |\gamma|_m \quad \text{for some arbitrarily small } \delta.$$

## Proof.

- Glue Euclidean triangles of area 1 (and thus side length $2/\sqrt[4]{3}$) on the triangles.
- Smooth the metric.
- 
- In the worst case the lengths double.

# Corollaries

**Corollary**

*Let $(S, G)$ be a triangulated surface with genus $g$ and $n$ triangles.*

1. *Some non-contractible cycle has length $O(\sqrt{n/g}\log g)$.*
2. *Some non-separating cycle has length $O(\sqrt{n/g}\log g)$.*
3. *Some separating and non-contractible cycle has length $O(\sqrt{n/g}\log g)$.*

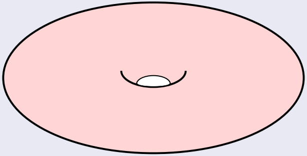
- (1) shows that Gromov $\Rightarrow$ Hutchinson and improves the best known constant.
- (2) and (3) are new.

**Theorem (Colin de Verdière, Hubard, de Mesmay '14)**

*Let $(S, m)$ be a Riemannian surface of genus g and area $A$. There exists a triangulated graph $G$ embedded on $S$ with $n$ triangles, such that every closed curve $\gamma$ in $(S, G)$ satisfies*

$|\gamma|_m \leq (1+\delta)\sqrt{\frac{32}{\pi}}\sqrt{A/n}\ |\gamma|_G$ *for some arbitrarily small* $\delta$.

**Proof.**



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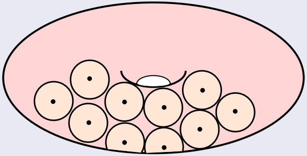
**Proof.**

Take a maximal set of balls of radius $\varepsilon$ and perturb them a little.



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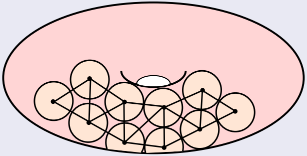

By [Dyer, Zhang and Möller '08], the Delaunay graph of the centers is a triangulation for $\varepsilon$ small enough.

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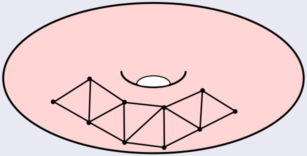

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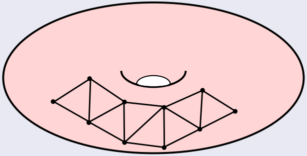

$$|\gamma_m| \leq 4\varepsilon|\gamma_G|.$$

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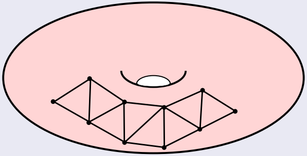


$$|\gamma_m| \leq 4\varepsilon|\gamma_G|.$$

Each ball has radius $\pi\varepsilon^2 + o(\varepsilon^2)$, thus $\varepsilon = O(\sqrt{A/n})$.

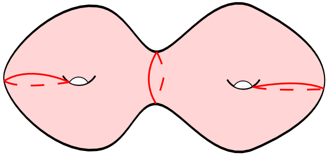
# Corollaries

- This shows that Hutchinson $\Rightarrow$ Gromov.
- Proof of the conjecture of Przytycka and Przytycki:

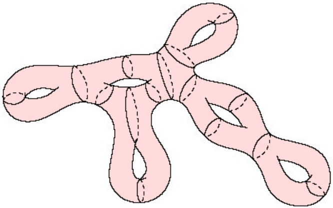
**Corollary**

*There exist arbitrarily large $g$ and $n$ such that the following holds: There exists a triangulated combinatorial surface of genus $g$, with $n$ triangles, of edgewidth at least $\frac{1-\delta}{6}\sqrt{n/g}\log g$* *for arbitrarily small $\delta$.*

## *Part 2: Pants decompositions*



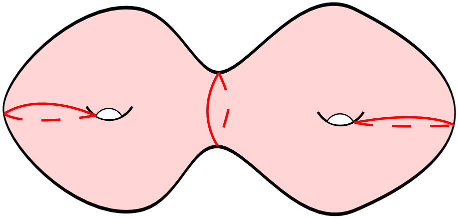
- A ***pants decomposition*** of a triangulated or Riemannian surface $S$ is a family of cycles $\Gamma$ such that cutting $S$ along $\Gamma$ gives pairs of pants, e.g., spheres with three holes.



- A pants decomposition has $3g - 3$ curves.
- Complexity of computing a shortest pants decomposition on a triangulated surface: in NP, not known to be NP-hard.

An algorithm to compute pants decompositions:

1. Pick a shortest non-contractible cycle.
2. Cut along it.
3. Glue a disk on the new boundaries.
4. Repeat $3g - 3$ times.



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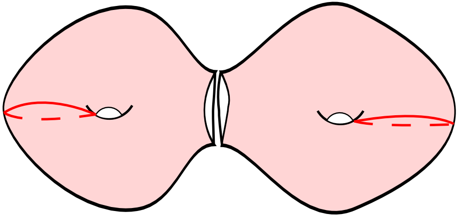

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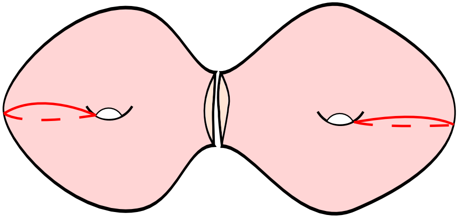

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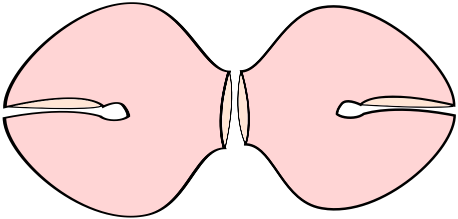


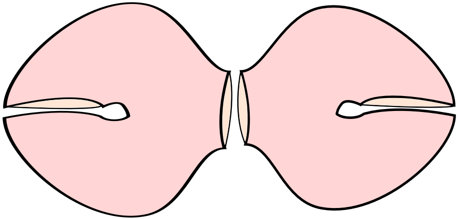
We obtain a pants decomposition of length

$$(3g - 3)O(\sqrt{n/g} \log g) = O(\sqrt{ng} \log g).$$

# Let us just use Hutchinson's bound

An algorithm to compute pants decompositions:

1. Pick a shortest non-contractible cycle.
2. Cut along it.
3. Glue a disk on the new boundaries. ***This increases the area!***
4. Repeat $3g-3$ times.



We obtain a pants decomposition of length

$$\sout{(3g-3)O(\sqrt{n/g}\log g) = O(\sqrt{ng}\log g).}\ \textbf{\textit{Wrong!}}$$

Doing the calculations correctly gives a subexponential bound.

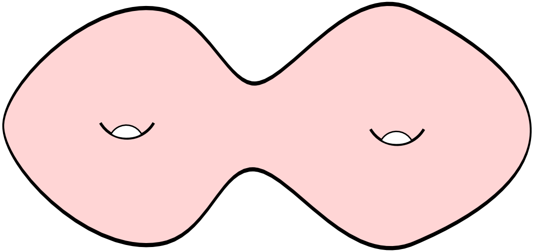
# A correct algorithm

Denote by *PantsDec* the shortest pants decomposition of a triangulated surface.

- ***Best previous bound:*** $\ell(PantsDec) = O(gn)$.
  [Colin de Verdière and Lazarus '07]
- ***New result:*** $\ell(PantsDec) = O(g^{3/2}\sqrt{n})$.
  [Colin de Verdière, Hubard and de Mesmay '14]
- Moreover, the proof is algorithmic.

We "combinatorialize" a continuous construction of Buser.

***First idea***



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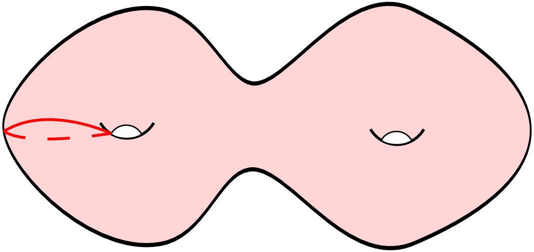

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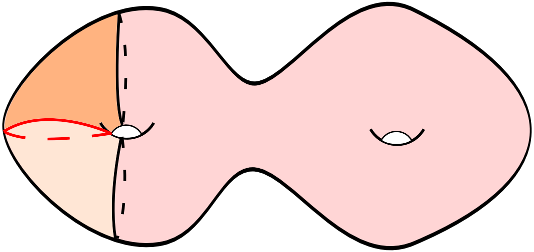

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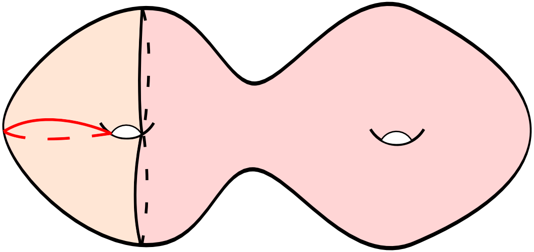
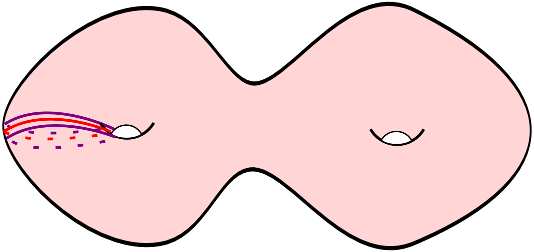


If the torus is fat, this is too long.

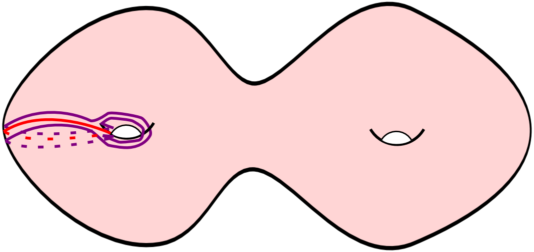
~~*First idea*~~

***Second idea***



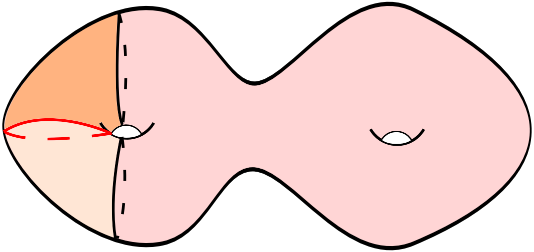
# How to compute a short pants decomposition

~~***First idea***~~
***Second idea***



If the torus is thin, this is too long.

~~*First idea*~~
~~*Second idea*~~
*Both at the same time*



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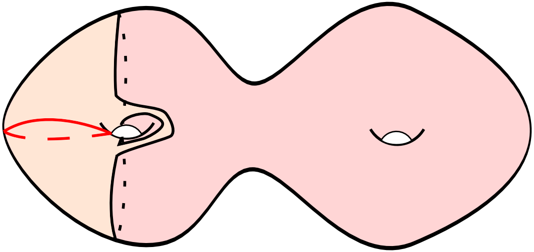
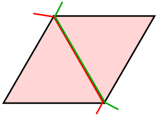
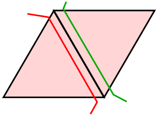


We take a trade-off between both approaches: As soon as the length of the curves with the first idea exceeds some bound, we switch to the second one.

- Several curves may run along the same edge: 

# Pathologies

- Several curves may run along the same edge: 

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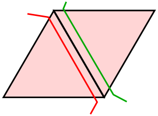
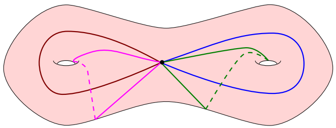


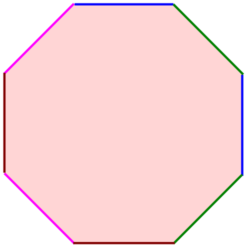
***Random surfaces***: Sample uniformly at random among the triangulated surfaces with $n$ triangles.
These run-alongs happen a lot for random triangulated surfaces:

**Theorem (Guth, Parlier and Young '11)**

*If $(S, G)$ is a random triangulated surface with $n$ triangles, and thus $O(n)$ edges, the length of the shortest pants decomposition of $(S, G)$ is $\Omega(n^{7/6-\delta})$ w.h.p.* *for arbitrarily small $\delta$*

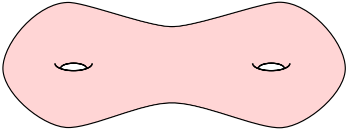
## *Part 3:*
## *Cut-graphs with fixed combinatorial map*

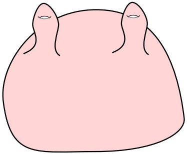




# Cut-graphs with fixed combinatorial map

- What is the length of the shortest cut-graph with a fixed shape (combinatorial map) ?
- Useful to compute a homeomorphism between two surfaces.





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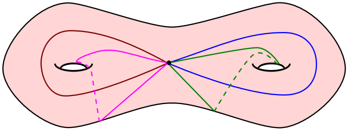

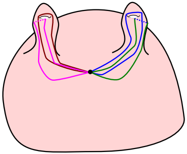

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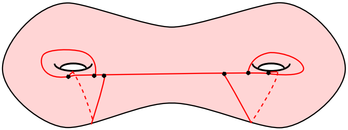

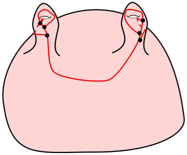

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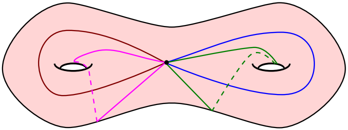

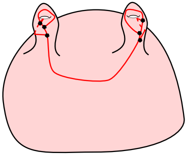

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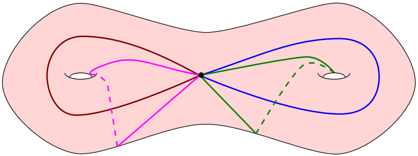
- Example: Canonical systems of loops [Lazarus et al '01] have $\Theta(gn)$ length.



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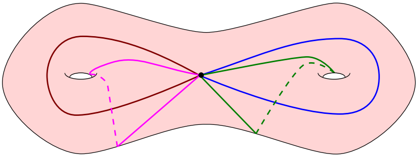

- Can one find a better map ?

**Theorem (Colin de Verdière, Hubard, de Mesmay '13)**

*If $(S, G)$ is a random triangulated surface with $n$ triangles and genus $g$, for any combinatorial map $M$, the length of the shortest cut-graph with combinatorial map $M$ is $\Omega(n^{7/6-\delta})$ w.h.p. for arbitrarily small $\delta$.*

***Idea of proof:***

- How many surfaces with $n$ triangles ?
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- How many surfaces of genus $g$ with $n$ triangles and cut-graph of length $L$? Roughly $L\,(L/g+1)^{12g-9}$.

- Restated in a dual setting: What is the minimal number of crossings between two cellularly embedded graphs $G_1$ and $G_2$ with specified combinatorial maps ?
- Related to questions of [Matoušek et al. '14] and [Geelen et al. '14].

**Corollary**

*For a fixed $G_1$, for most choices of trivalent $G_2$ with $n$ vertices, there are $\Omega(n^{7/6-\delta})$ crossings in any embedding of $G_1$ and $G_2$ for arbitrarily small $\delta$.*

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***Thank you ! Questions ?***