Functional maps and shape matching problems

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Underlying problem

We have two manifolds $\mathcal{A}$ and $\mathcal{B}$ and we want to find correspondences between them

Problem to solve

We are looking for a diffeomorphism T

$$T : \mathcal{A} \rightarrow \mathcal{B}$$

with small distortion.

We assume T nearly isometric.

Functional map properties

We see T as an operator

$$\begin{array}{rccc} C_T : & L^2(\mathbb{B}) & \rightarrow & L^2(\mathbb{A}) \\ & f & \mapsto & f \circ T \end{array}$$

Properties

- C_T is a linear operator and can be express as a matrix when decomposed in a Hilbert basis

$$C_T^{i,j} = \langle \varphi_i^{\mathbb{A}}, \varphi_j^{\mathbb{B}} \circ T \rangle$$

- If T is an isometry then

$$C_T^t C_T f = f$$

- Composition are expressed by matrix multiplication

$$C_{T \circ R} = C_R C_T$$

Functional map properties

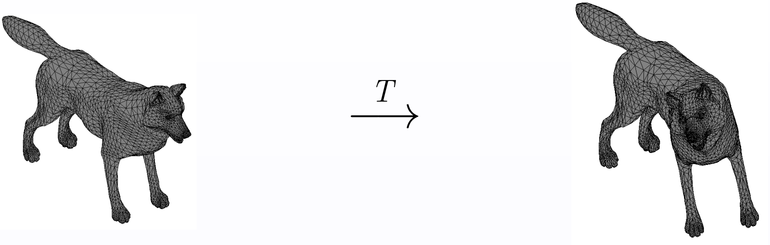
$$\begin{array}{rccc} C_T : & L^2(\mathbb{B}) & \rightarrow & L^2(\mathbb{A}) \\ & f & \mapsto & f \circ T \end{array}$$

Difficulties

- C_T is composition operator if and only if

$$\forall f, g \in L^2(\mathbb{B}) \text{ with } fg \in L^2(\mathbb{B}), \quad C_T(fg) = (C_T f)(C_T g)$$

Example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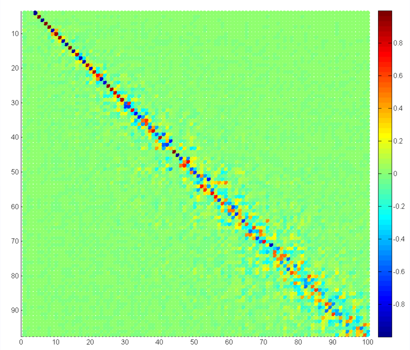
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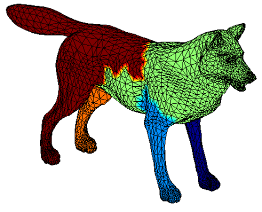
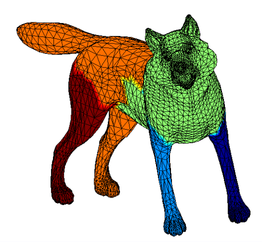
Functional map to solve shape matching problem

We compute a segmentation of the shapes





Segmentations are invariant by isometry but we do not know the correspondences

$$\forall i, \exists j, \quad C_T \mathbf{1}_i^{\mathbb{B}} = \mathbf{1}_j^{\mathbb{A}}$$

T is supposed nearly isometric

$$C_T^t C_T \approx I$$

An optimization problem

Finally we want to solve

$$\min_{C,\pi} \frac{1}{2}\|C\mathbf{1}^{\mathbb{B}} - \mathbf{1}^{\mathbb{A}}\pi\|_F^2 + \alpha\|C - I\|_1$$

with π a permutation matrix

The functional maps can be seen as a (huge) convex relaxation of the problem

Problems

- The segmentation can be assigned to their symmetric
- We lost the notion of continuity
- The functional map we find is not necessarily a composition operator. Often we have

$$C = \alpha C_T + (1-\alpha)C_{S\circ T}$$

The continuity problem

Imagine that one of our shape has an internal symmetry S

$$S : \mathcal{A} \to \mathcal{A}$$

In the Fourier basis we cannot make the difference between one part of the shape and his symmetric.

How to make the difference between C_T and $C_{S \circ T}$?

How to create orientation preserving functional map?

Orientation preserving transformations

$$\psi_t : \mathcal{A} \to \mathcal{A}$$

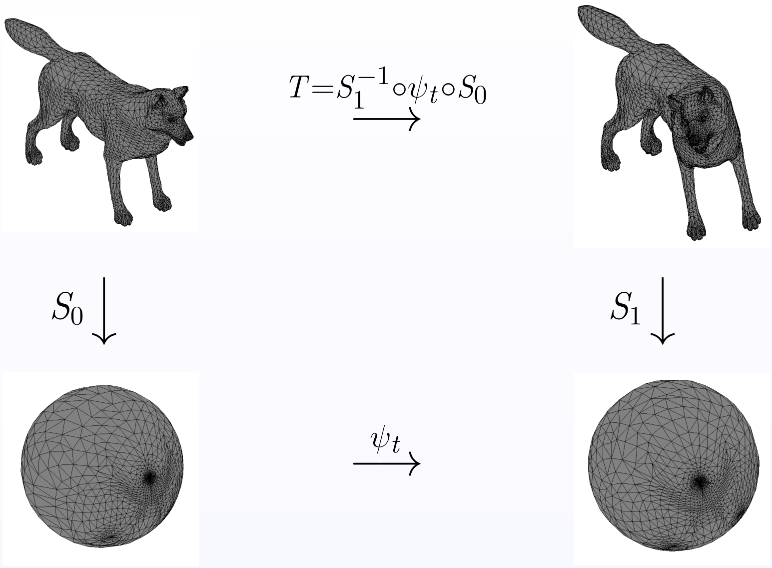
Solution of

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}t} = V(\psi_t)$$

is an orientation preserving diffeomorphism.

One idea is to map $\mathcal{A}$ and $\mathcal{B}$ into a sphere

Algorithm pipeline



Vector fields as operators

We define the linear operator D_V as

$$D_V : \begin{array}{ccc} \mathcal{C}^\infty(\mathbb{A}) & \to & \mathcal{C}^\infty(\mathbb{A}) \\ f & \mapsto & V.\nabla f \end{array}$$

Properties

- D_V as a matrix

$$D_V^{i,j} = \langle \varphi_i^{\mathbb{A}}, V.\nabla\varphi_j^{\mathbb{A}} \rangle$$

- Decomposition of D_V on a basis of vector fields V_i

$$D_V = \sum_i a_i D_{V_i}$$

- Integration of a vector fields

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}t} = V(\psi_t) \quad \Leftrightarrow \quad C_{\psi_t} = e^{-tD_V}$$

- Divergence free vector fields

$$\mathrm{div}(V) = 0 \quad \Leftrightarrow \quad D_V^{i,j} = -D_V^{j,i}$$

Difficulty

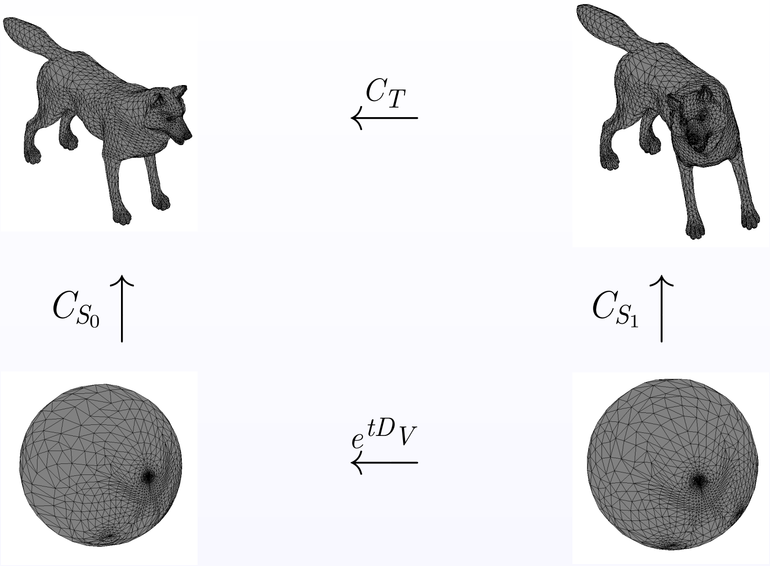
$$
\begin{array}{rccc}
D_V : & \mathcal{C}^\infty(\mathbb{A}) & \to & \mathcal{C}^\infty(\mathbb{A}) \\
 & f & \mapsto & V.\nabla f
\end{array}
$$

Properties

- D_V represents a vector field if and only if

$$\forall f, g \in \mathcal{C}^\infty(\mathbb{A}), \quad D_V(fg) = D_V(f)g + fD_V(g)$$

Algorithm pipeline



Optimization problem

We have a functional map

$$C = \alpha C_T + (1-\alpha) C_{S \circ T}$$

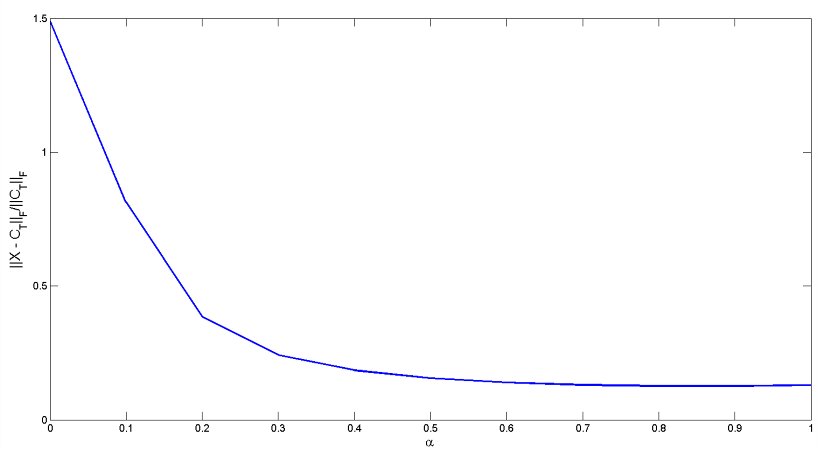
We want to retrieve an orientation preserving map by solving

$$\min_a \frac{1}{2} \| \exp(\sum_i a_i D_{V_i}) - C_{S_1^{-1}} C C_{S_0} \|_F^2$$

Algorithm

$$\begin{cases} a^{n+1} & \in & \arg\min_a \frac{1}{2} \| X^n (I + \sum_i a_i D_{V_i}) - C_{S_1^{-1}} C C_{S_0} \|_F^2 \\ t^{n+1} & \in & \arg\min_t \frac{1}{2} \| X^n \exp(t \sum_i a_i^n D_{V_i}) - C_{S_1^{-1}} C C_{S_0} \|_F^2 \\ X^{n+1} & = & X^n \exp(t^{n+1} \sum_i a_i^{n+1} D_{V_i}) \end{cases}$$

Results



Conclusion

What works

- Works for small deformations when the linearisation is a good approximation

What need to be improved

- The projection into the sphere creates problems
- The step
$$X^{n+1} = X^n \exp(t^{n+1} \sum_i a_i^{n+1} D_{V_i})$$
implies that the error grows exponentially during the iterations
- The method lacks of theoretical proof

The end

Thank you!