About optimal domains for Laplace eigenvalues

a numerical approach

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1) geometric problem;
2) optimization problem;
3) discretization for the numerical processing.

Geometric problem

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Consider the problem: find a non-zero map $u : \Omega \to \mathbb{R}$ and a scalar λ (both depending on Ω) such that

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Theoretical Question : Existence of a solution (λ, u)?

Geometric problem

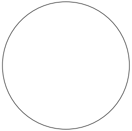
Answer: Yes!

Theorem (Spectral Theorem)

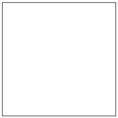
Let $(H, (\cdot|\cdot))$ be a separable Hilbert space of infinite dimension and T a positive (that is $(Tx|x) \geq 0$ for all $x \in H$), self-adjoint and compact operator on H.
Then, there exist a sequence of real positive eigenvalues $(\mu_n)_{n\geq 1}$, converging to 0 and a sequence of eigenvectors $(x_n)_{n\geq 1}$, defining a Hilbert basis of H such that $Tx_n = \mu_n x_n$ for all $n \geq 1$.

Geometric problem

Theoretically known examples:



$\lambda_{1,\text{Disc}_1} \simeq 18.168$



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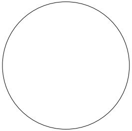

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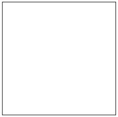

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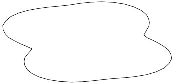
Example computed numerically



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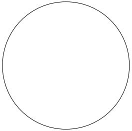

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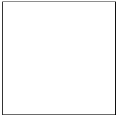

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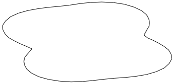

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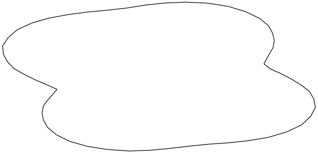

$\lambda_{1,h}(\sqrt{2}\Omega) \simeq 10.513$

Optimization problem

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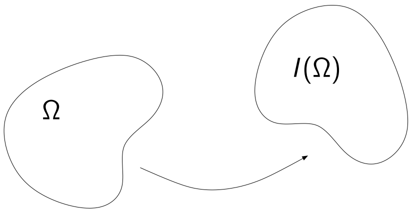

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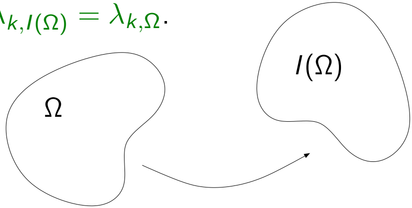
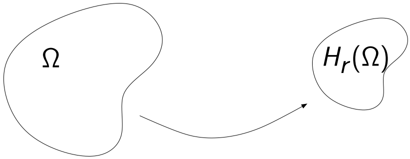
1) If I is an isometry in $\mathbb{R}^2$, then $\lambda_{k,I(\Omega)} = \lambda_{k,\Omega}$.

$$u_{k,I(\Omega)}(x) = u_{k,\Omega}(I^{-1}(x))$$



Optimization problem

2) If H_r is the homothety of factor r centred at the origin,



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2) If H_r is the homothety of factor r centred at the origin, then
$\lambda_{k,H_r(\Omega)} = \frac{1}{r^2}\lambda_{k,\Omega}$.

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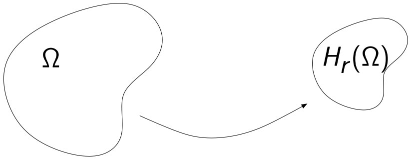

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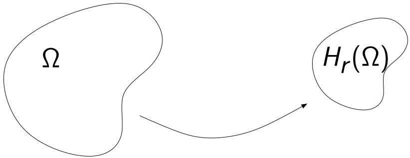

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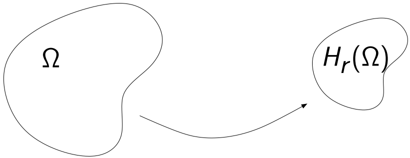

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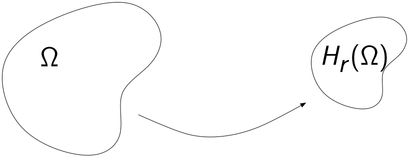

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Optimization problem : What is the bounded domain of volume 1 in $\mathbb{R}^2$ which minimizes $\lambda_{k,\Omega}$?

$$\min_{\substack{\text{vol}(\Omega)=1,\\ \Omega \text{ bounded}}} \lambda_{k,\Omega} \quad \Leftrightarrow \quad \min_{\Omega \text{ bounded}} \text{vol}(\Omega)\,\lambda_{k,\Omega}$$

Optimization problem

Known results:

Theorem (Faber-Krahn, 1923)

Let B be the ball of volume 1. Then,

$$\lambda_{1,B} = \min \left\{ \lambda_{1,\Omega} \,\middle|\, \Omega \subset \mathbb{R}^2, \mathrm{vol}(\Omega) = 1 \right\}.$$

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Theorem (Krahn-Szegö, 1926)

Let B_2 be the union of two identical balls, $\text{vol}(B_2) = 1$. *Then,*

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- These theorems also hold in $\mathbb{R}^n$, $n \geq 3$;

Optimization problem

Known results:

Theorem (Bucur 2012 & Mazzoleni, Pratelli 2013)

There exists a minimizer for $\lambda_{k,\Omega}$, $k \geq 3$, among all quasi-open sets Ω of given volume. Moreover, it is bounded and has finite perimeter.

Optimization problem

Known results:

Theorem (Bucur 2012 & Mazzoleni, Pratelli 2013)

There exists a minimizer for $\lambda_{k,\Omega}$*,* $k \geq 3$*, among all quasi-open sets* Ω *of given volume. Moreover, it is bounded and has finite perimeter.*

However, it does not provide the shape of the minimizing domain!

Open problem

For $k \geq 3$*, what is the bounded domain of volume 1 in* $\mathbb{R}^2$ *which minimizes* $\lambda_{k,\Omega}$*?*

Optimization problem

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Generally, for a given bounded domain Ω, it is quite impossible to find analytically the eigenvalues $\lambda_{k,\Omega}$.

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$\rightsquigarrow$ numerics !

Discretization for the numerical processing

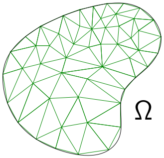
Weak formulation of problem $(\mathcal{P})$:

$$(\mathcal{WP})\begin{cases} \text{find } u \in H_0^1(\Omega) \text{ such that} \\ \int\limits_\Omega (\nabla u | \nabla v) = \int\limits_\Omega uv, \quad \forall v \in H_0^1(\Omega). \end{cases}$$

Discretization for the numerical processing

Galerkin approximation

Discretization of Ω into triangles K of type $\mathcal{P}_1 \rightsquigarrow$ we get a mesh $\mathcal{M}_h$ with N nodes inside Ω ;



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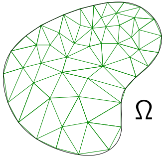

Instead of $H_0^1(\Omega)$ in $(\mathcal{WP})$, consider the finite dimensional space

$$V_h := \left\{\varphi \in \mathcal{C}^0(\overline{\Omega}) \,|\, \varphi_{|\partial\Omega} = 0,\ \varphi_{|K} \text{ linear } \forall K \in \mathcal{M}\right\} \ ;$$

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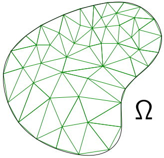

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A basis $\{\varphi_{h,i}\}_{i=1}^N$ of V_h is given by

$$\varphi_{h,i} \in V_h, \varphi_{h,i}(P_j) = \delta_{ij}, \quad i,j = 1,\ldots,N.$$

Discretization for the numerical processing

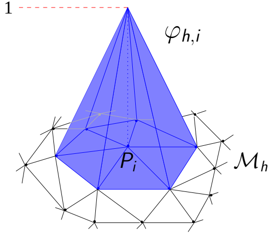

Figure: A basis function $\varphi_{h,i}$.

Discretization for the numerical processing

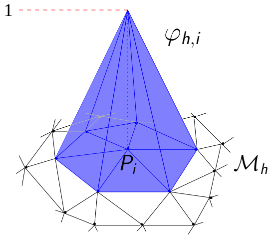

Figure: A basis function $\varphi_{h,i}$.

Approximation of $u \in H_0^1(\Omega)$ by $u_h = \sum_{j=1}^{N} u_j \varphi_{h,j} \in V_h$.

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$$(\mathcal{WP}_h)\left\{\begin{array}{l}\text{find } u_h \in V_h, u_h \not\equiv 0, \text{ and } \lambda > 0 \text{ such that} \\ \int\limits_\Omega (\nabla u_h | \nabla \varphi_{h,i}) = \lambda \int\limits_\Omega u_h \varphi_{h,i}, \quad \forall i = 1, \ldots, N.\end{array}\right.$$

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Pluging $u_h = \sum\limits_{j=1}^{N} u_j \varphi_{h,j} \in V_h$ into $(\mathcal{WP}_h)$:

$$\sum_{j=1}^{N} \underbrace{\int_\Omega (\nabla \varphi_{h,j} | \nabla \varphi_{h,i})}_{S_{i,j}} u_j = \lambda \sum_{j=1}^{N} \underbrace{\int_\Omega \varphi_{h,j} \varphi_{h,i}}_{M_{i,j}} u_j, \quad \forall i = 1, \dots, N.$$

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$\rightsquigarrow$ Lanczos algorithm to solve $(\mathcal{WP}_h)$.

Discretization for the numerical processing

Shape optimization

The idea is to use a descent algorithm to minimize the *cost functional* $J(\Omega) = \lambda_k(\Omega)\,\mathrm{vol}(\Omega)$.

The first problem is to determine the domain of the functional J, that is the admissible shapes Ω.

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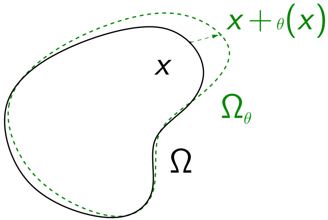
Given an initial domain Ω_0, we allow deformations of the form

$$\Omega_\theta = (\mathrm{id}+\theta)(\Omega_0),\ \theta \in W^{1,\infty}(\Omega).$$



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Now, we can compute the derivative with respect to the domain of J, that is the Fréchet derivative of $\theta \mapsto J(\Omega_\theta)$.

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$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \left(\lambda_k(\Omega_0) - \mathrm{vol}(\Omega_0) \left(\frac{\partial u_k}{\partial \vec{n}} \right)^2 \right) (\theta|\vec{n}) \, \mathrm{d}\sigma.$$

And for every node $P_i \in \partial\Omega$, we choose θ_i, and move P_i to

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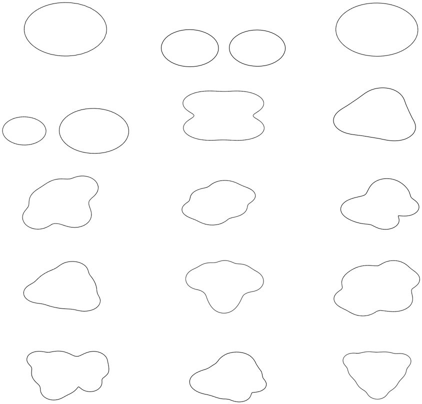
Then, we obtain a new domain, we can mesh it, compute the associated eigenvalues and eigenfunctions, move the new boundary, and so on...

Discretization for the numerical processing

15 first candidates to be minimizing domains of volume 1 in $\mathbb{R}^2$.

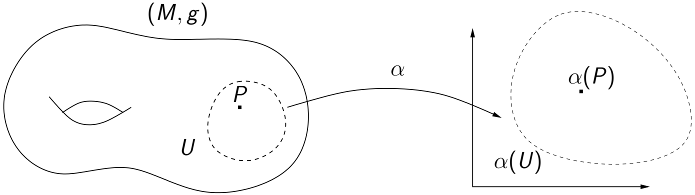


k	λ_k
1	18.17
2	36.39
3	46.30
4	64.78
5	78.53
6	89.05
7	106.51
8	120.01
9	134.06
10	144.82
11	160.55
12	174.37
13	188.84
14	202.22
15	211.16

Previously found by Oudet ('04, partly) and Antunes-Freitas ('12)

Generalization to surfaces

Let (M, g) be a Riemannian manifold of dimension 2



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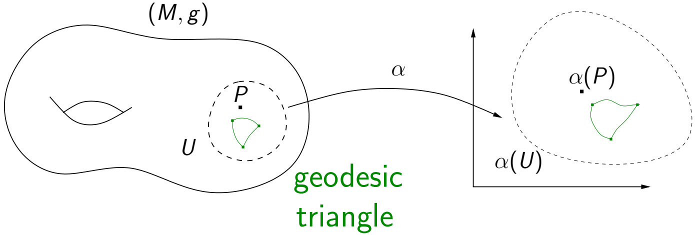

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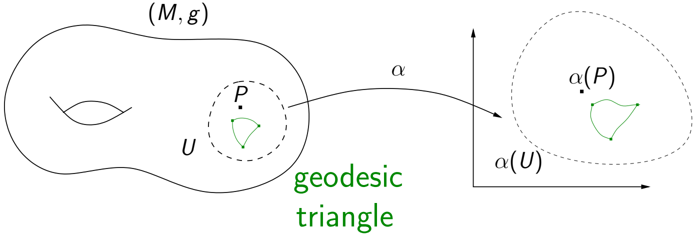

Mesh $\alpha(U)$ in order to consider manifold non embeddable in $\mathbb{R}^3$.
$\rightsquigarrow$ use the expression of the Laplacian in local coordinates:

$$\Delta f = \frac{1}{\sqrt{\det(G)}} \sum_{j,k=1}^{2} \partial x_j \left(G^{jk} \sqrt{\det(G)} \partial x_k f \right).$$

Generalization to surfaces

It implies several modifications. For instance,

- for the computation:

$$(\mathcal{WP}_h)\begin{cases}\text{find } u_h \in V_h, u_h \not\equiv 0, \text{ and } \lambda > 0 \text{ such that} \\ \int\limits_\Omega \nabla u_h^t G^{-1} \nabla \varphi_{h,i} \sqrt{\det G} = \lambda \int\limits_\Omega u_h \varphi_{h,i} \sqrt{\det G}, \\ \text{for all } i = 1, \dots, N.\end{cases}$$

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- for the optimization:
  There is no homothety any more! The volume constraint has to be taken into consideration. $\rightsquigarrow$ Lagrange multiplier.

Generalization to surfaces

We look for a saddle point of the functional

$$J(\mu, \Omega) = \lambda_k(\Omega) + \mu(\mathrm{vol}(\Omega) - V_0),$$

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$\rightsquigarrow$ We get a similar formula for the shape optimization.

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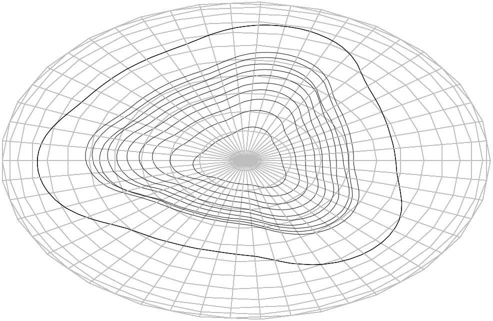
- in a hyperboloid H (curvature between 0 and 1);

Generalization to surfaces

Plot of the optimizers for $\lambda_{10}(\Omega^*_{10,\mathbb{S}^2})$ and $\mathrm{vol}(\Omega^*_{10,\mathbb{S}^2}) = 0.1, 0.2, \ldots, 0.9, 1$ and 2.



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End