Group meeting – November 6, 2013

# Stable Multi-Scale Signatures for Shapes using Topological Persistence

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$\rightarrow$ joint projects with Fred, Vin, Leo, David, etc.

# Nomenclature

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# Nomenclature

In our work and throughout the talk:

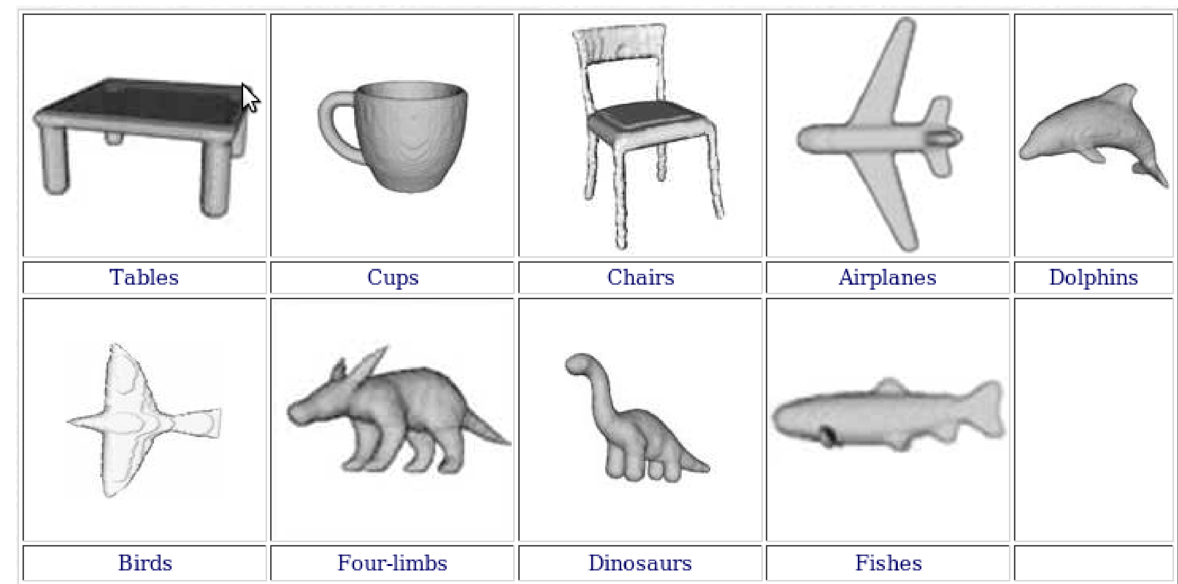
- *shape* $\equiv$ compact metric space (sometimes assumed finite or manifold)
- *distance between shapes* $\equiv$ Gromov-Hausdorff (GH) distance
- *signature* $\equiv$ persistence diagram (choose the filtration)
  - *multi-scale* $\equiv$ reflects the structure of the shape across scales
  - *global/local* $\equiv$ attached to the whole shape / to a (set of) base point(s)
  - *stable* $\equiv$ variations with GH-distance and base point location are controlled

# Why Compare Shapes

Comparisons between shapes occur in various contexts, including:

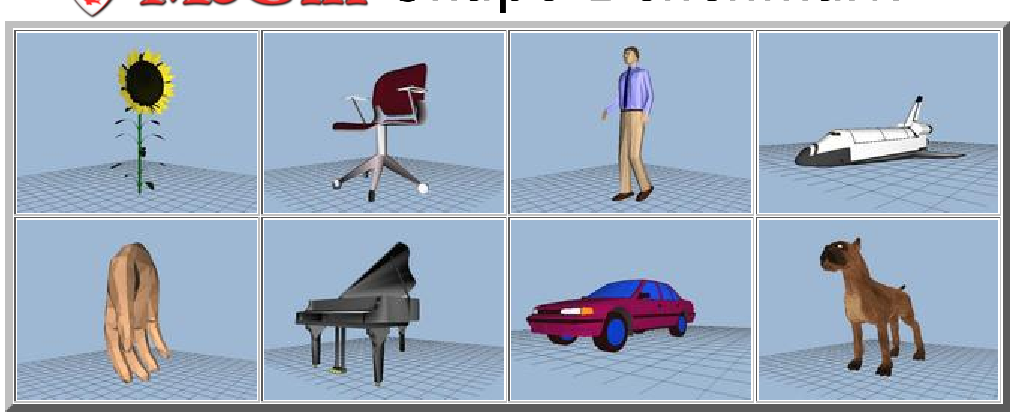
- shape classification (organizing large databases of shapes)

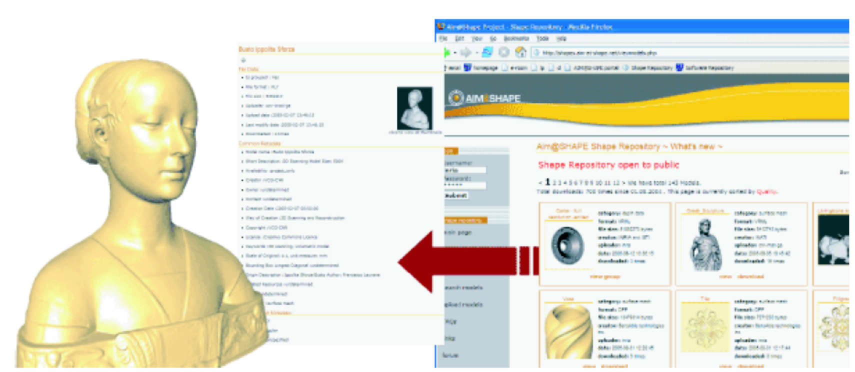
TOSCA




Princeton Shape Retrieval and Analysis Group
Princeton Shape Benchmark

McGill Shape Benchmark





AIM@SHAPE
Digital Shape WorkBench

# Why Compare Shapes

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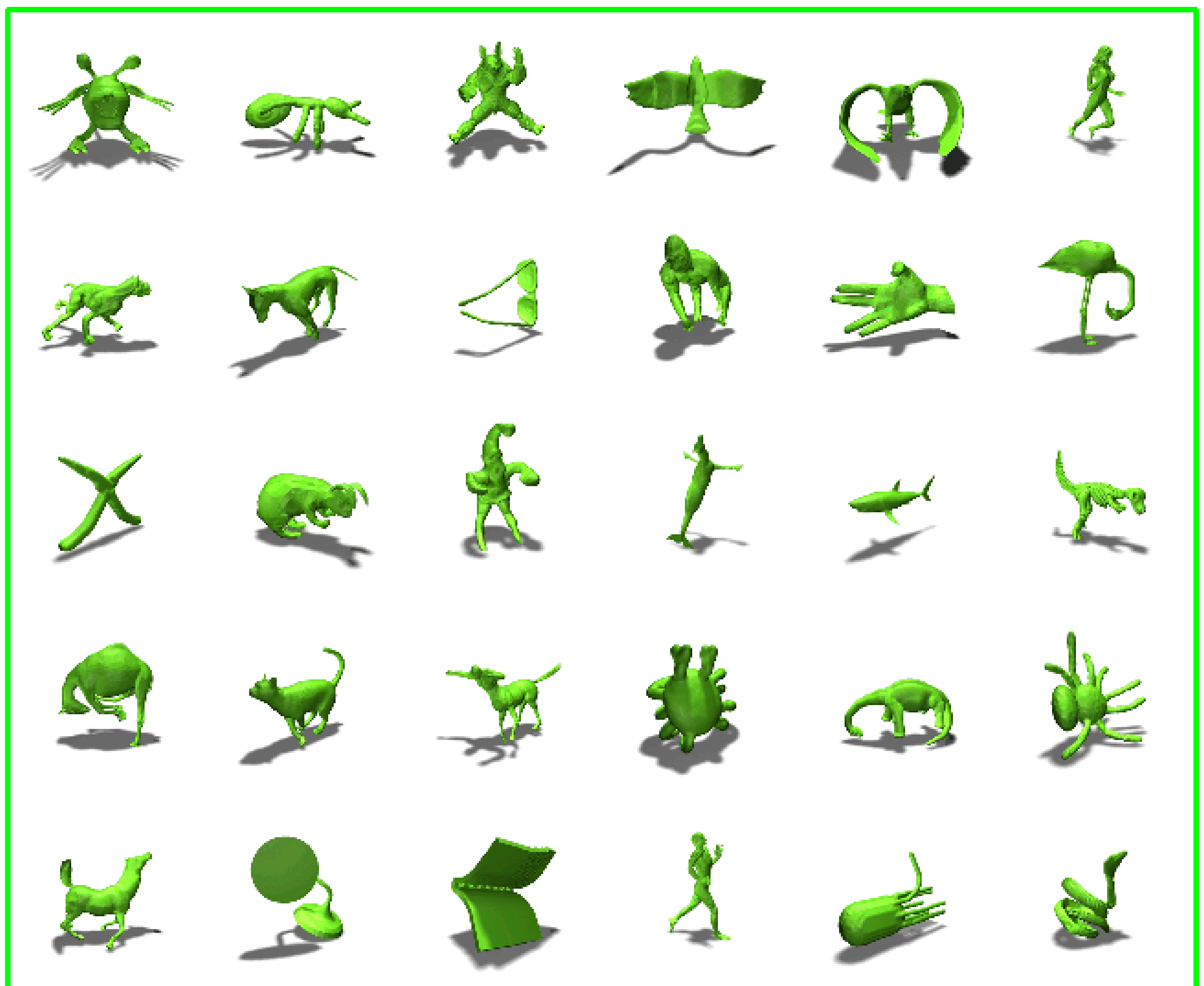


source: SHREC (Shape Retrieval Contest) 2011

# Why Compare Shapes

Comparisons between shapes occur in various contexts, including:

- shape classification (organizing large databases of shapes)
- shape retrieval (searching in databases of shapes)
- partial/global shape matching (finding the *best* mapping between shapes)

source: *Image Processing and Analysing With Graphs: Theory and Practice*, CRC Press, 2011

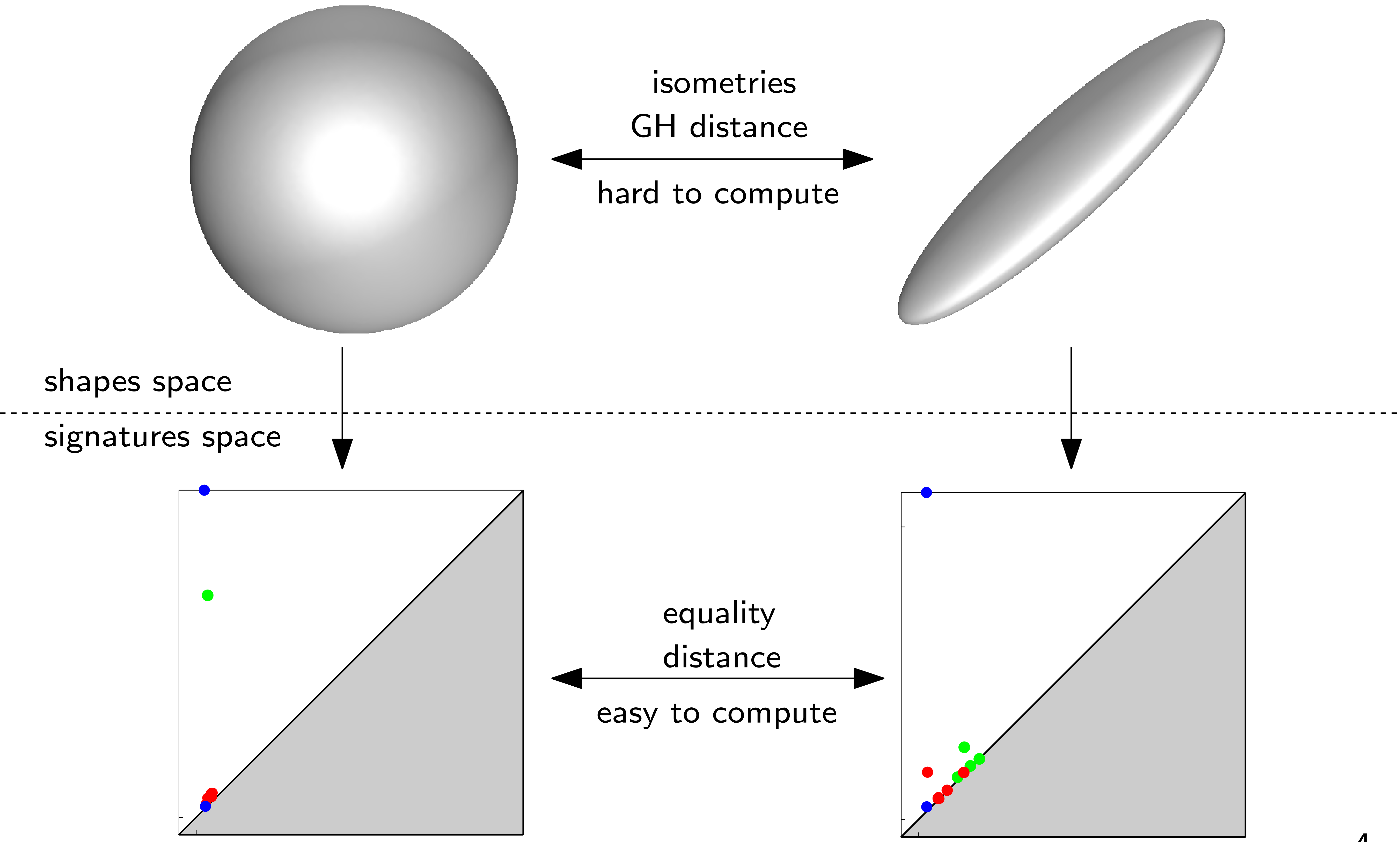
# Why Compare Shapes

Comparisons between shapes occur in various contexts, including:

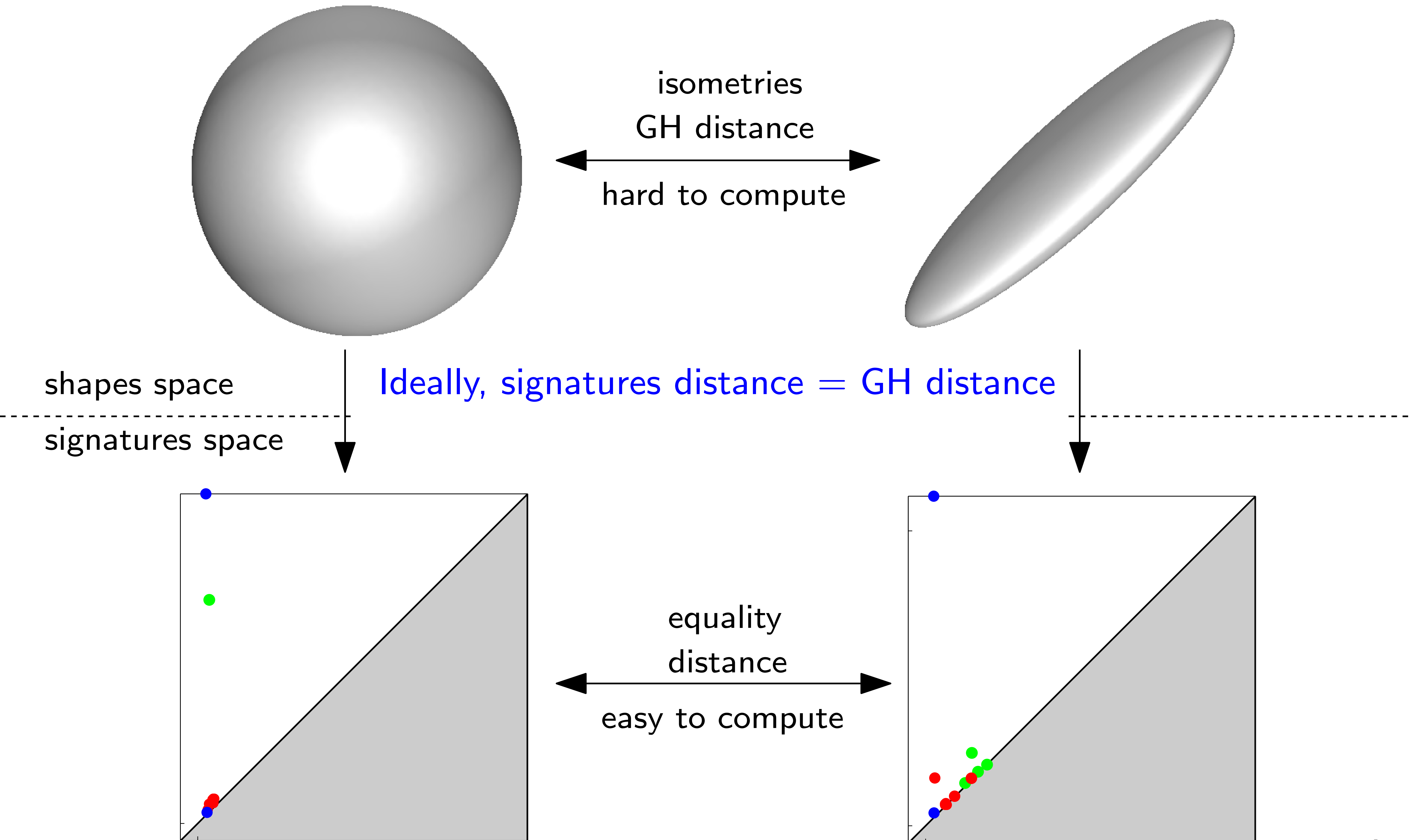
- shape classification (organizing large databases of shapes)
- shape retrieval (searching in databases of shapes)
- partial/global shape matching (finding the *best* mapping between shapes)

shape comparison is but one piece of the whole process, yet it is a crucial piece

# Why use Signatures



# Why use Signatures



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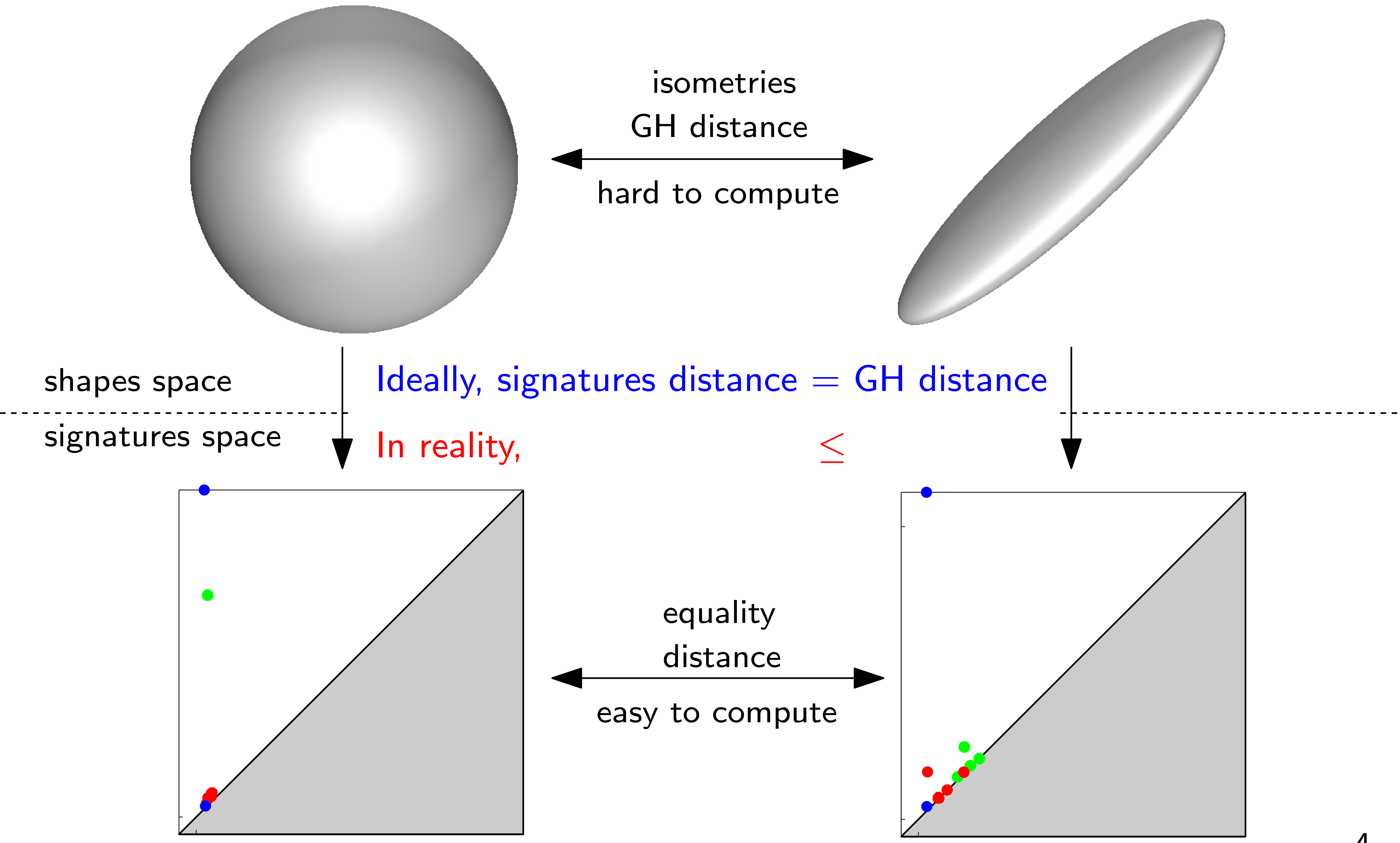

# Rips-Based Signatures

Input: a compact metric space $(X, \mathrm{d}_X)$

Signature: $\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X)$, where $\mathcal{R}(X, \mathrm{d}_X)$ is the *Rips filtration* of $(X, \mathrm{d}_X)$

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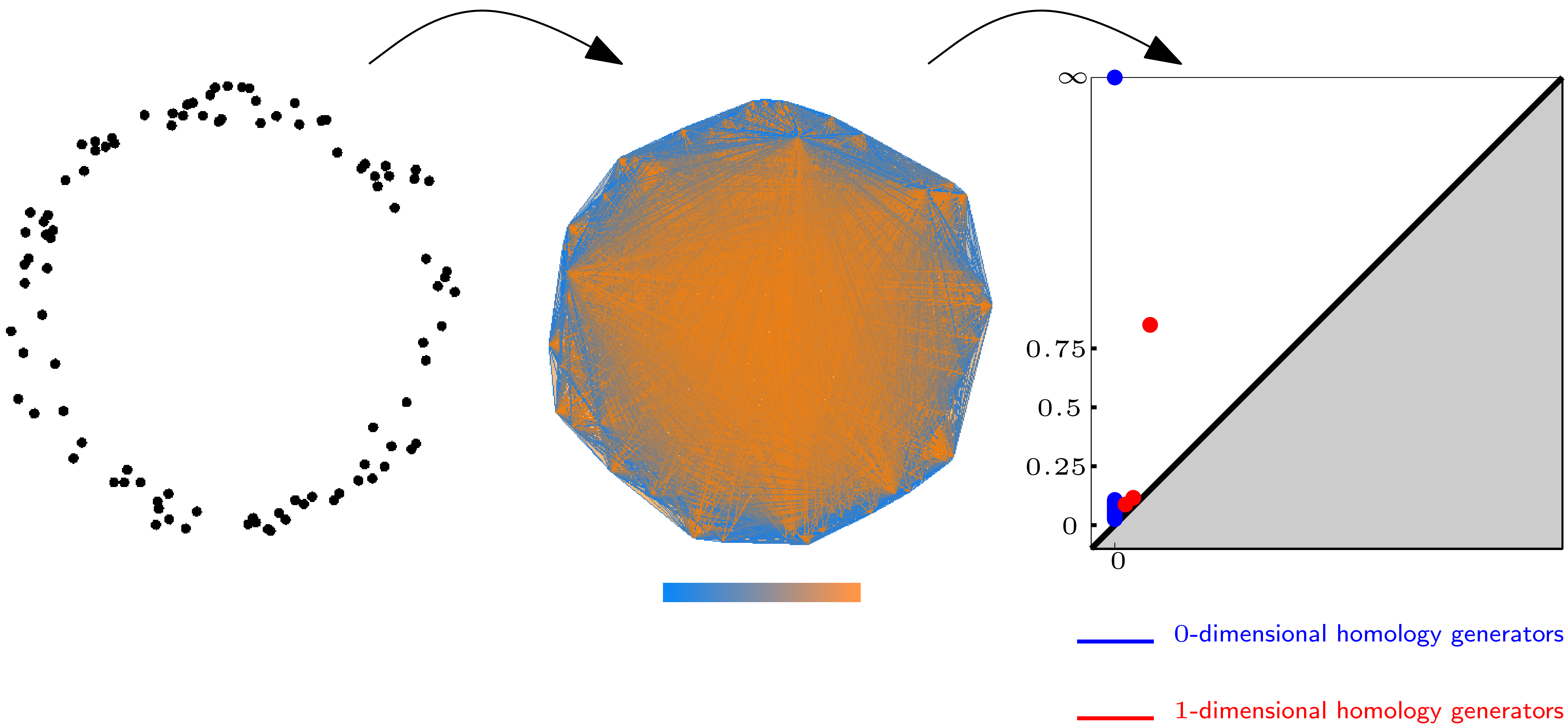


# Rips-Based Signatures (continued)

**Theorem (Stability):** For any compact metric spaces $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$, $\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X),\ \mathrm{Dg}\,\mathcal{R}(Y, \mathrm{d}_Y)) \leq 2\mathrm{d}_{\mathrm{GH}}(X, Y)$ (plus diags are well-defined).

Variants [Chazal, de Silva, Oudot '12/'13/'14?]:

- Čech complexe/ *extrinsic* Čech complex filtrations

- Witness complex filtrations (landmarks fixed)

- precompact metric spaces

- (dis-)similarity measures

# Rips-Based Signatures (continued)

finite

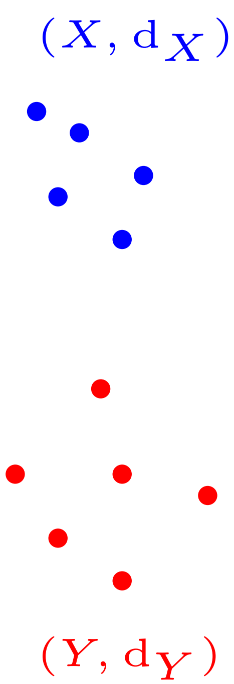
**Theorem (Stability):** For any ~~compact~~ metric spaces $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$, $\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X),\ \mathrm{Dg}\,\mathcal{R}(Y, \mathrm{d}_Y)) \leq 2\mathrm{d}_{\mathrm{GH}}(X, Y)$ (plus diags are well-defined).

**Proof:**

# Rips-Based Signatures (continued)

**Theorem (Stability):** For any ~~compact~~ finite metric spaces $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$, $\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X),\ \mathrm{Dg}\,\mathcal{R}(Y, \mathrm{d}_Y)) \leq 2\mathrm{d}_\mathrm{GH}(X, Y)$ (plus diags are well-defined).

**Proof:** Take any $\varepsilon > \mathrm{d}_\mathrm{GH}(X, Y)$.




$\mathrm{d}_\mathrm{GH}(X, Y) < \varepsilon$

$\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X),\ \mathrm{Dg}\,\mathcal{R}(Y, \mathrm{d}_Y))$

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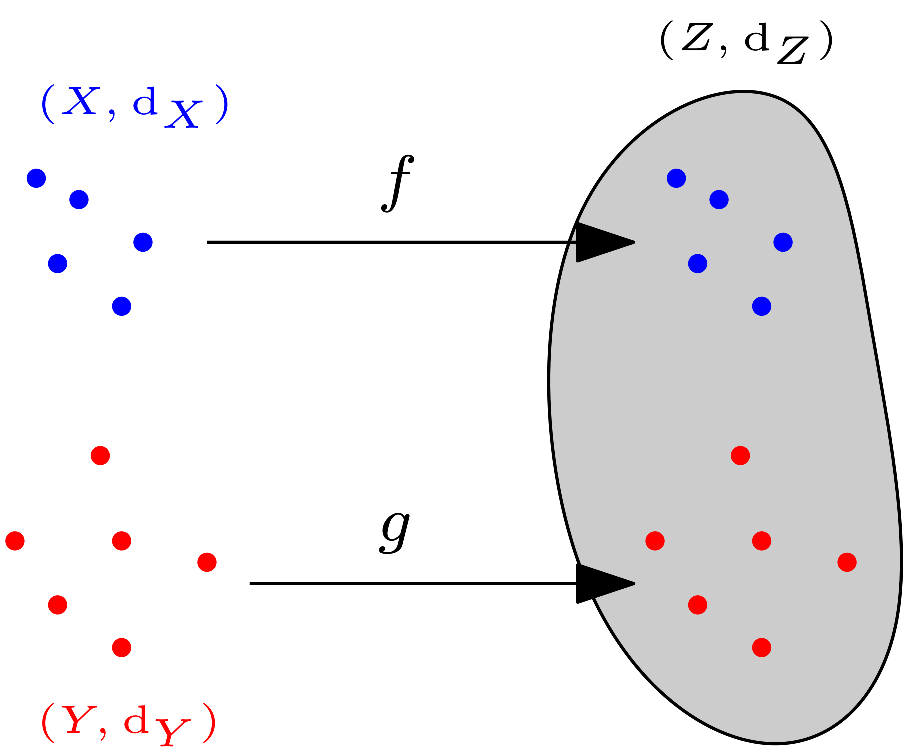


$\mathrm{d}_{\mathrm{GH}}(X, Y) < \varepsilon$ $\quad\quad$ $\mathrm{d}_{\mathrm{H}}(f(X), g(Y)) \leq \varepsilon$

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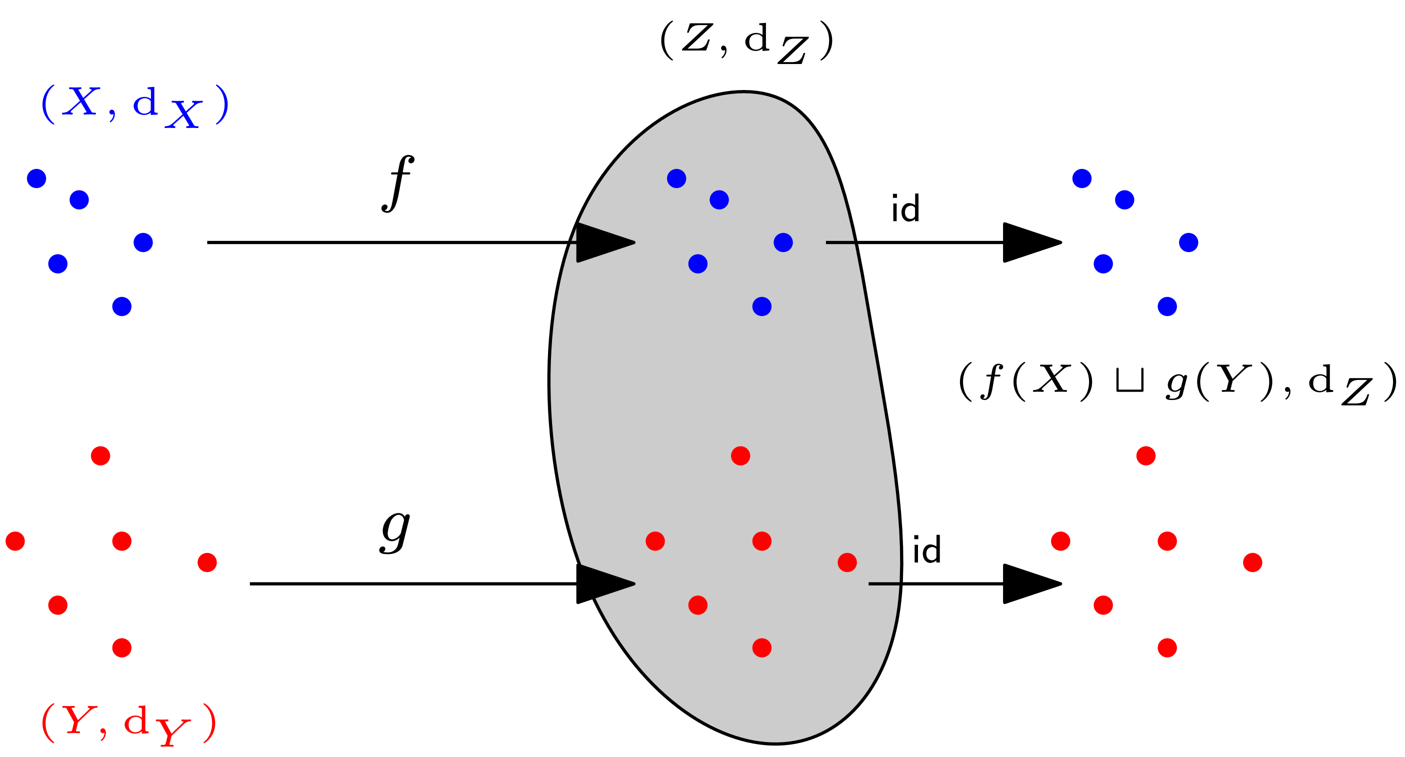


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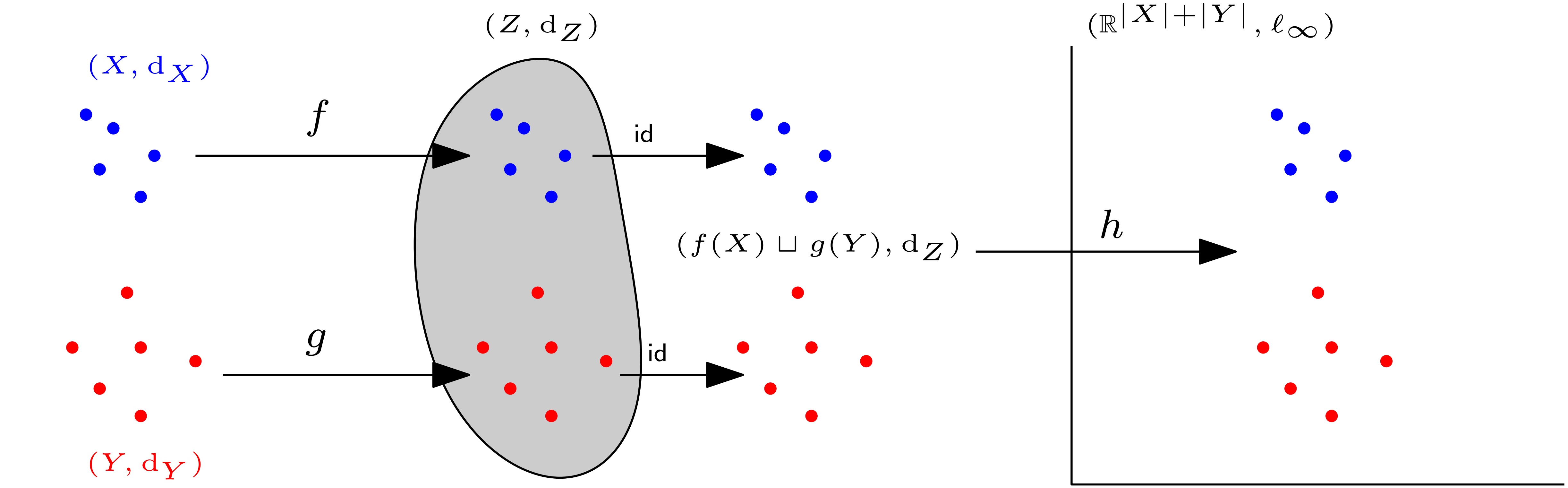


$\mathrm{d}_\mathrm{GH}(X, Y) < \varepsilon$ $\quad$ $\mathrm{d}_\mathrm{H}(f(X), g(Y)) \leq \varepsilon$ $\quad$ $\mathrm{d}_\mathrm{H}(h \circ f(X), h \circ g(Y)) \leq \varepsilon$

$$\begin{aligned}\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X),\ \mathrm{Dg}\,\mathcal{R}(Y, \mathrm{d}_Y)) &= \mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\,\mathcal{R}(f(X), \mathrm{d}_Z),\ \mathrm{Dg}\,\mathcal{R}(g(Y), \mathrm{d}_Z)) \\ &= \mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\,\mathcal{R}(h \circ f(X), \ell_\infty),\ \mathrm{Dg}\,\mathcal{R}(h \circ g(Y), \ell_\infty))\end{aligned}$$

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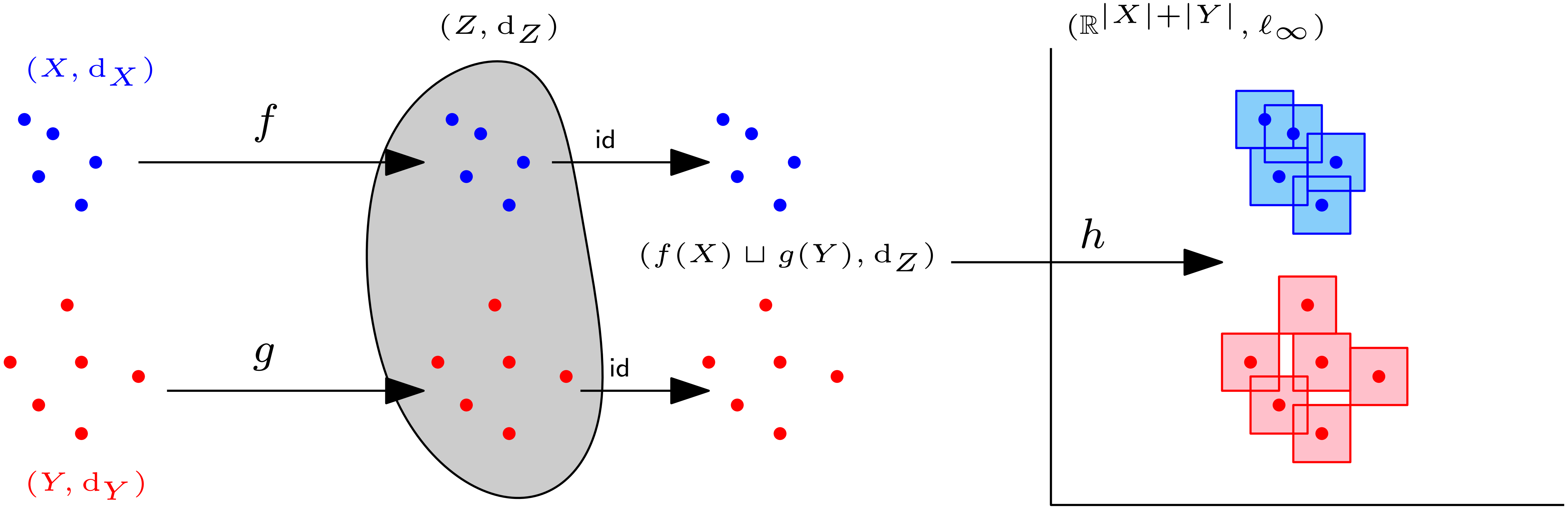


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\end{aligned}
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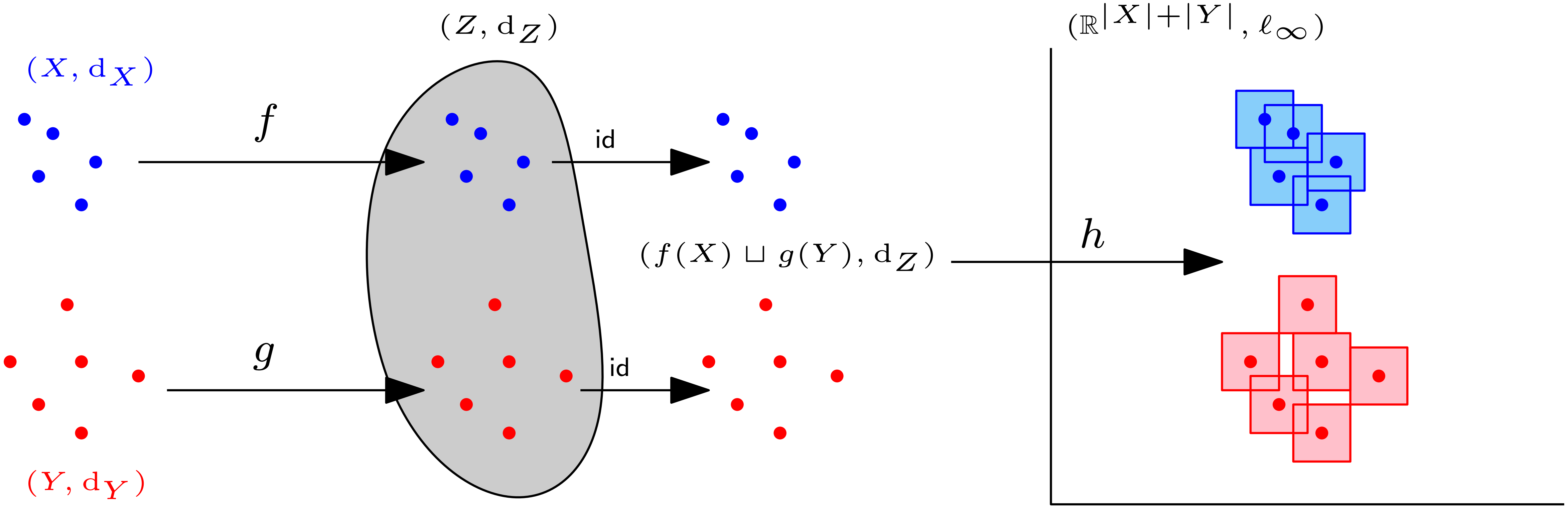


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&= 2\,\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\,\mathrm{d}_\infty(\cdot, h \circ f(X)),\ \mathrm{Dg}\,\mathrm{d}_\infty(\cdot, h \circ g(Y))) \leq 2\varepsilon. \quad \square
\end{aligned}$$

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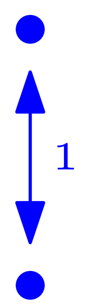
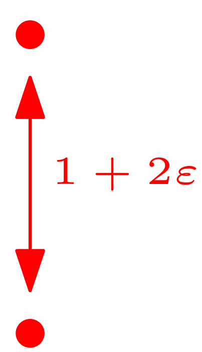
**Theorem (Stability):** For any ~~compact~~ finite metric spaces $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$, $\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X),\ \mathrm{Dg}\,\mathcal{R}(Y, \mathrm{d}_Y)) \leq 2\mathrm{d}_{\mathrm{GH}}(X, Y)$ (plus diags are well-defined).

The bound is worst-case tight...

$X =$ 


$Y =$ 


$$\mathrm{d}_{\mathrm{GH}}(X, Y) = \varepsilon$$

$$\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X) = \{(0, \infty),\ (0, 1)\}$$

$$\mathrm{Dg}\,\mathcal{R}(Y, \mathrm{d}_Y) = \{(0, \infty),\ (0, 1 + 2\varepsilon)\}$$

$$\Rightarrow \mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X),\ \mathrm{Dg}\,\mathcal{R}(Y, \mathrm{d}_Y)) = 2\varepsilon$$

# Rips-Based Signatures (continued)

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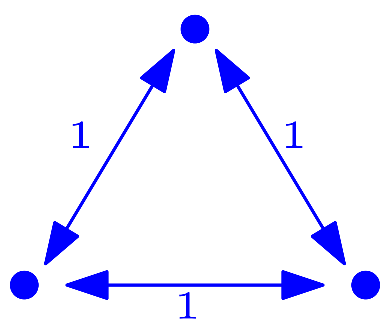
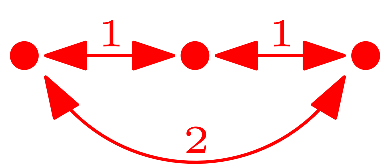
The bound is worst-case tight... but it is still only an upper bound

$X =$




$Y =$




$\mathrm{d}_\mathrm{GH}(X, Y) = \frac{1}{2}$

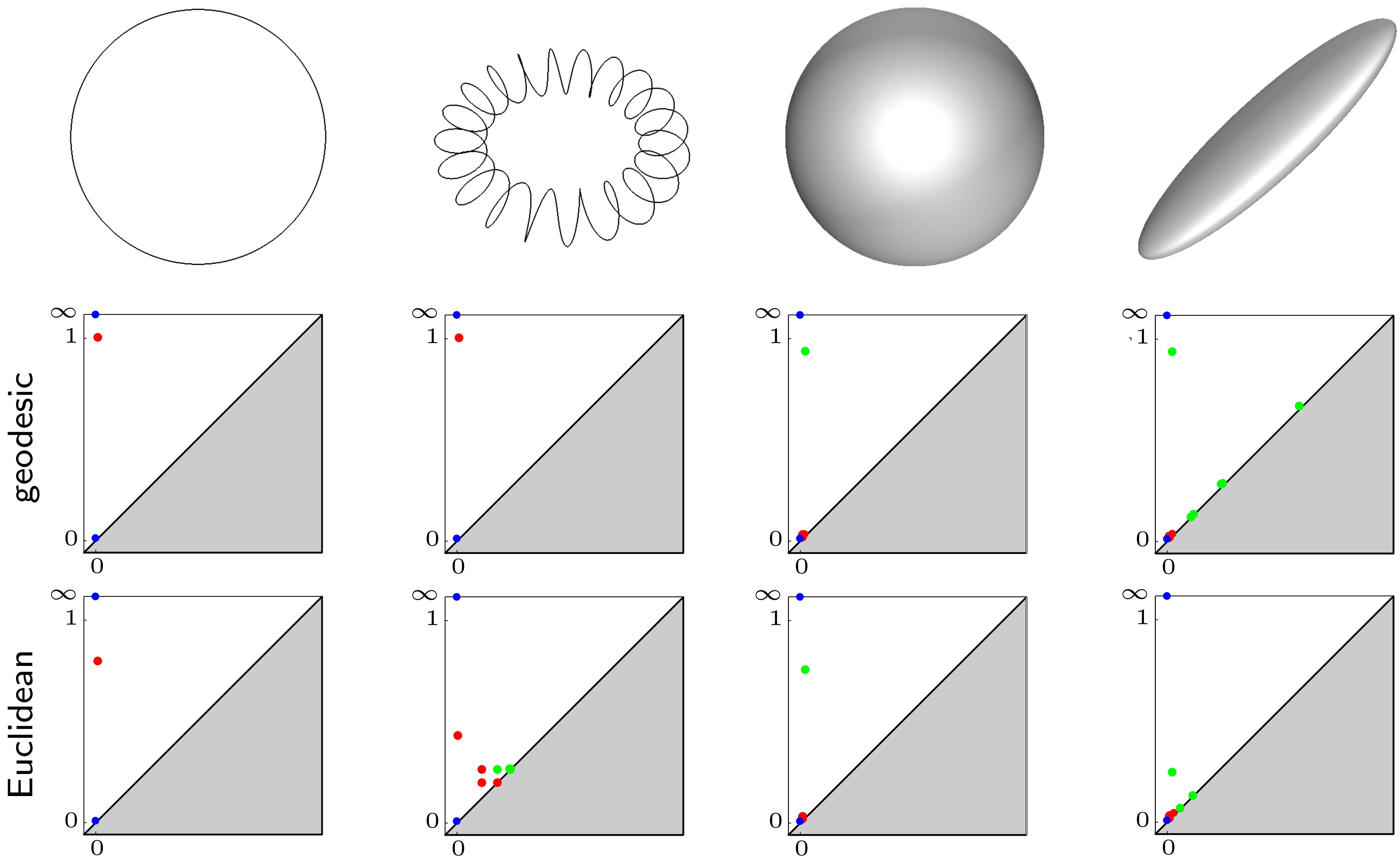
$\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X) = \{(0, \infty),\ (0, 1),\ (0.1)\}$

$\mathrm{Dg}\,\mathcal{R}(Y, \mathrm{d}_Y) = \{(0, \infty),\ (0, 1),\ (0, 1)\}$

$\Rightarrow \mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\,\mathcal{R}(X, \mathrm{d}_X),\ \mathrm{Dg}\,\mathcal{R}(Y, \mathrm{d}_Y)) = 0$

# Rips-Based Signatures (continued)

Signatures of some elementary shapes (approximated from finite samples):



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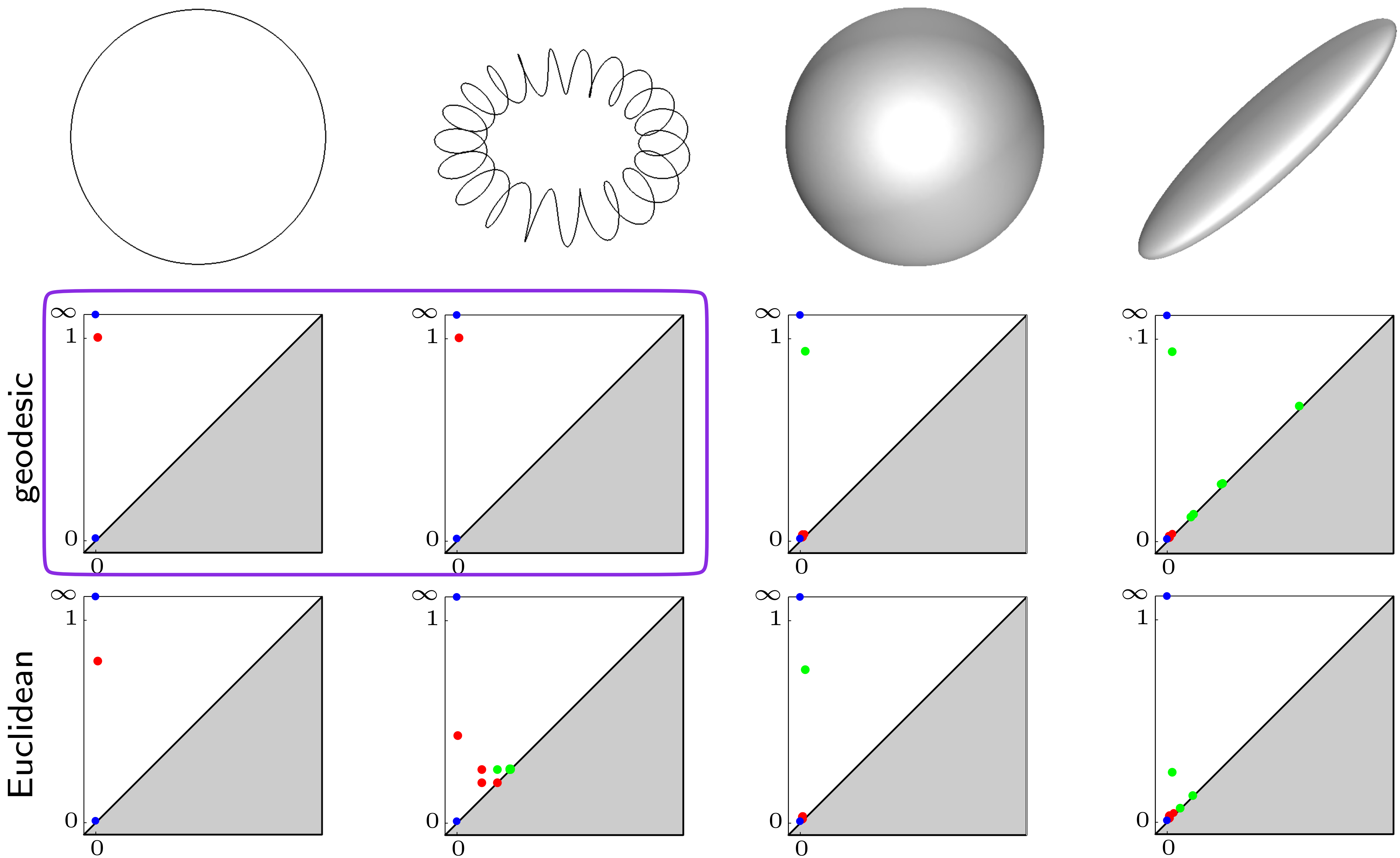

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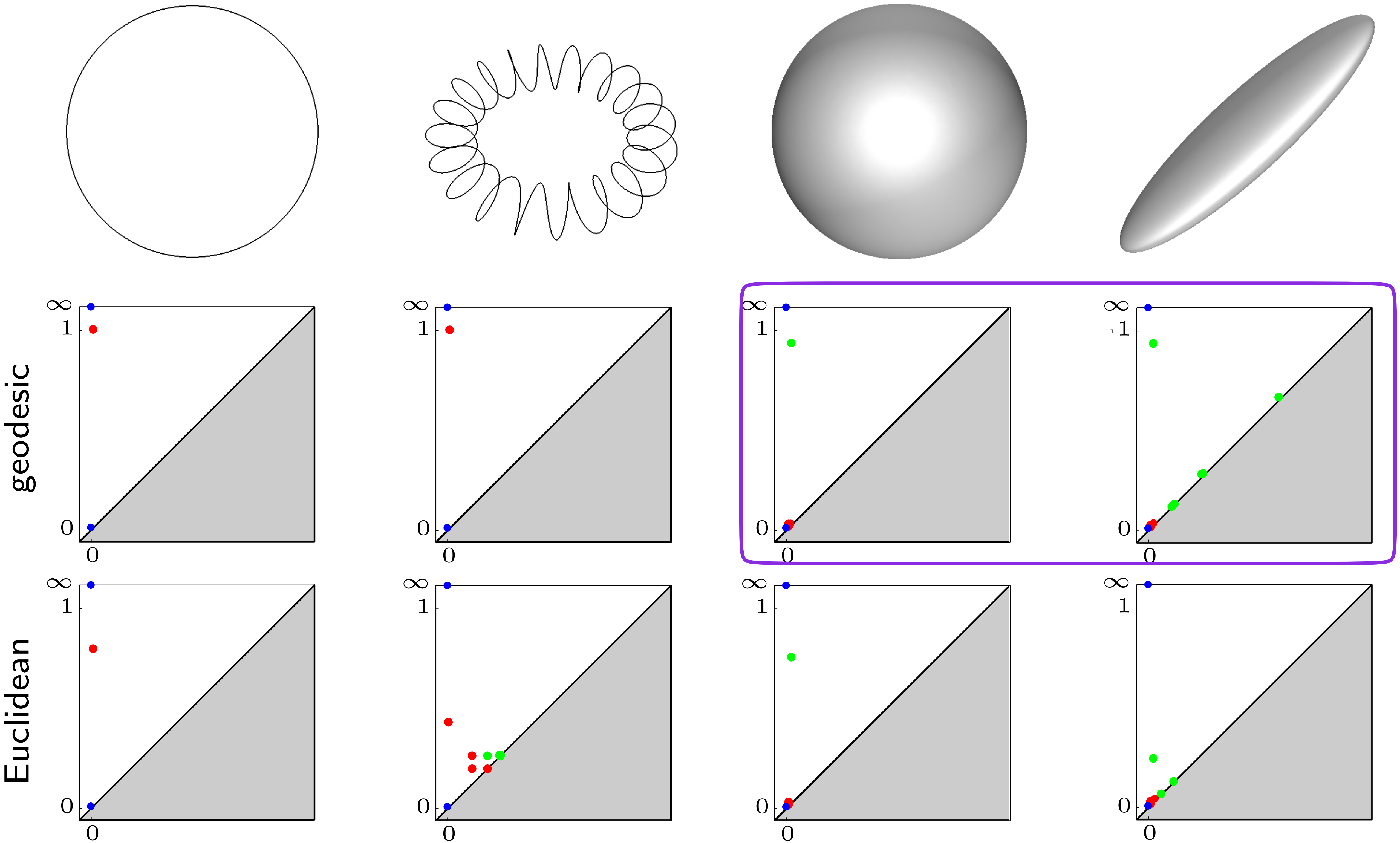
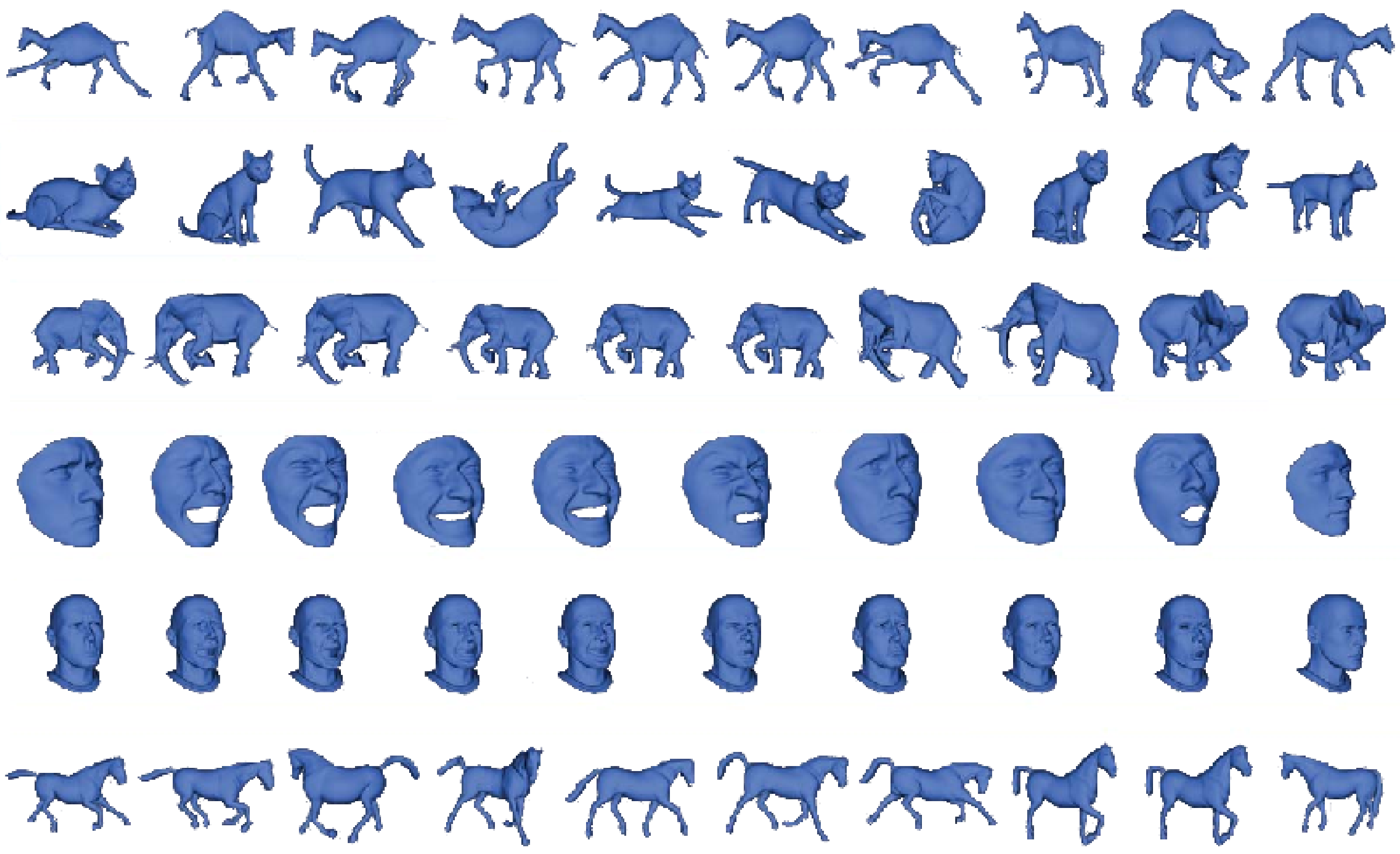


# Rips-Based Signatures (continued)

Experimental results:



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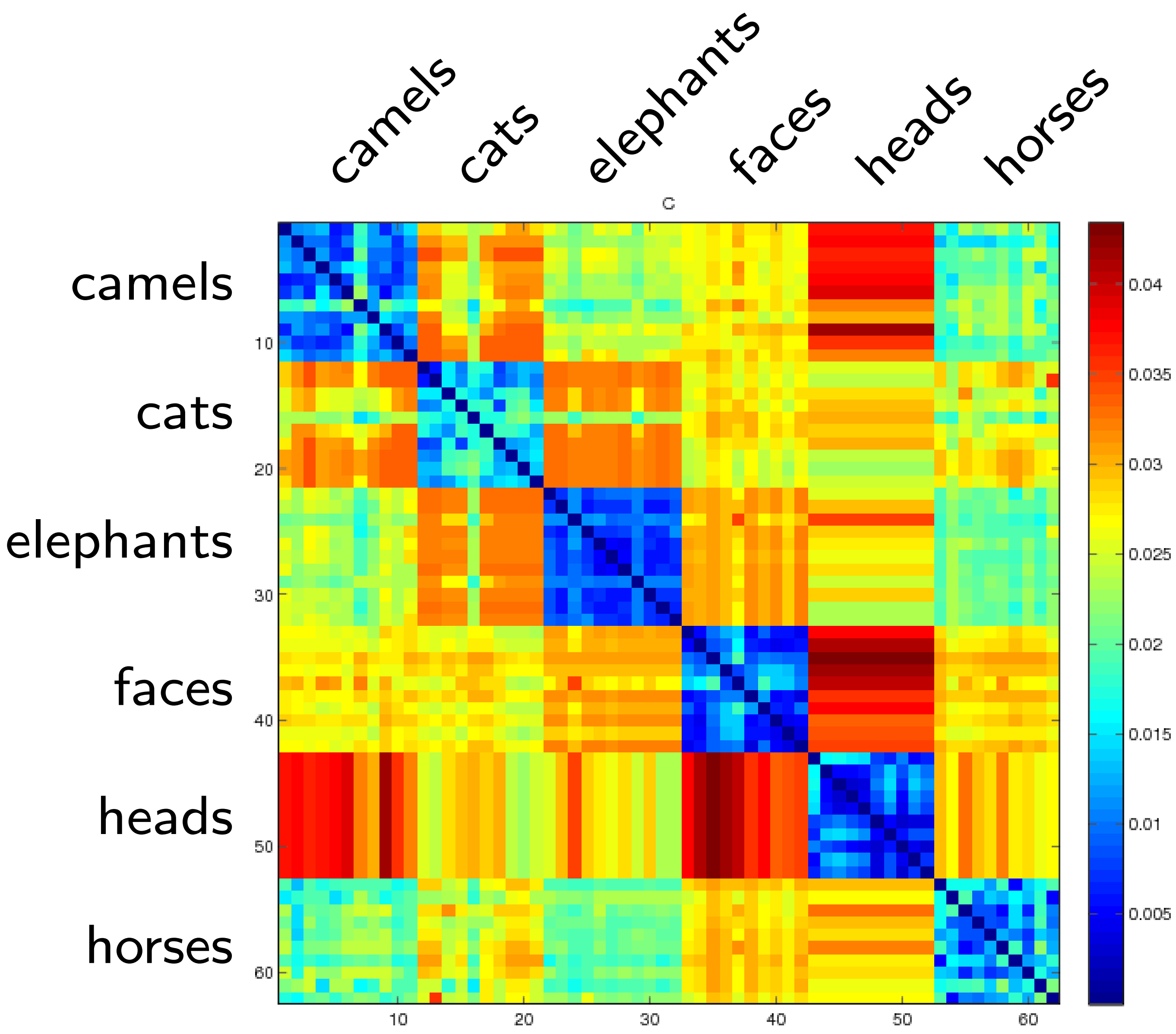

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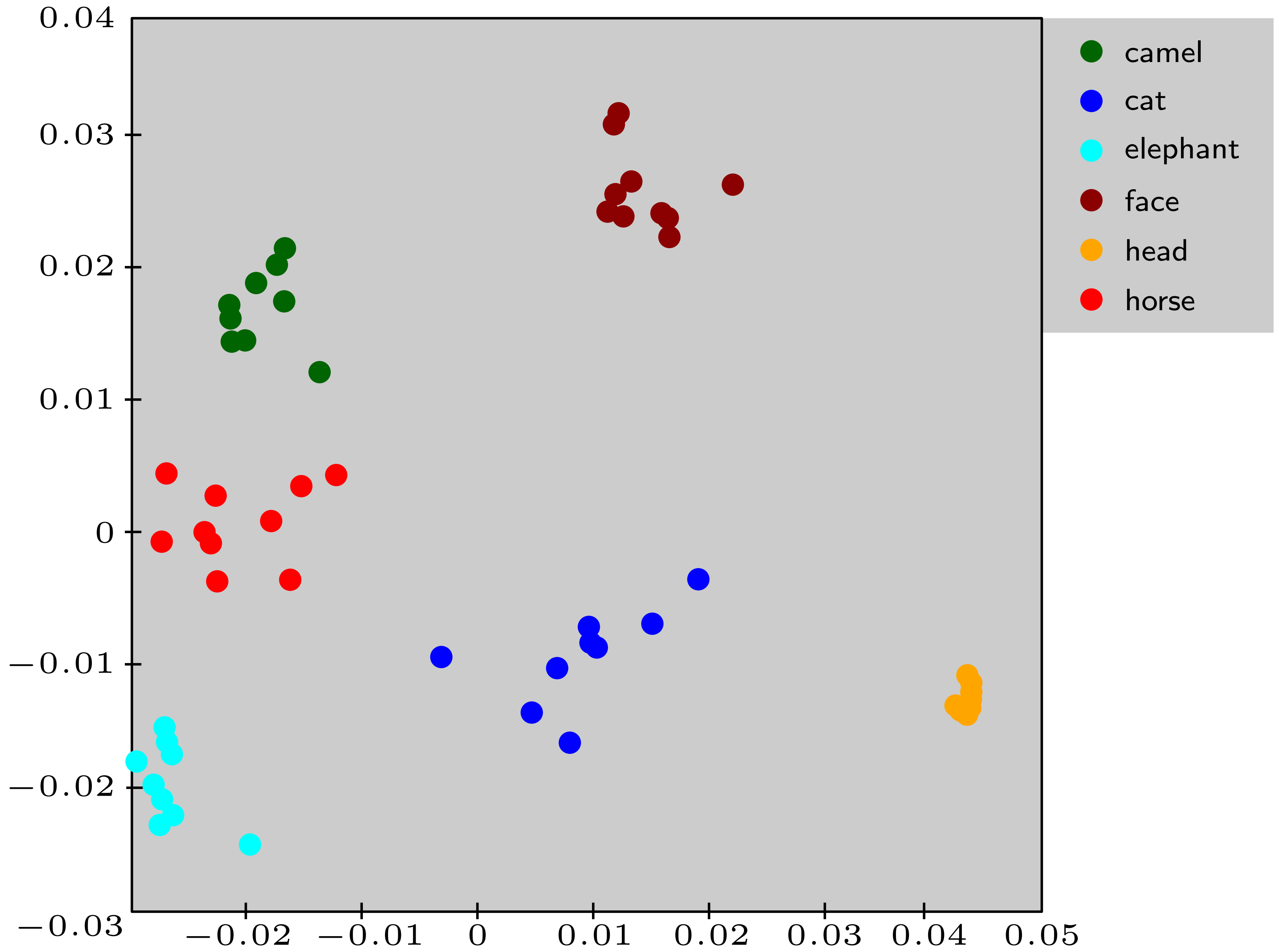

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Input: a compact metric space $(X, \mathrm{d}_X)$ and a Lipschitz function $f : X \to \mathbb{R}$

Signature: $\mathrm{Dg}\, f$

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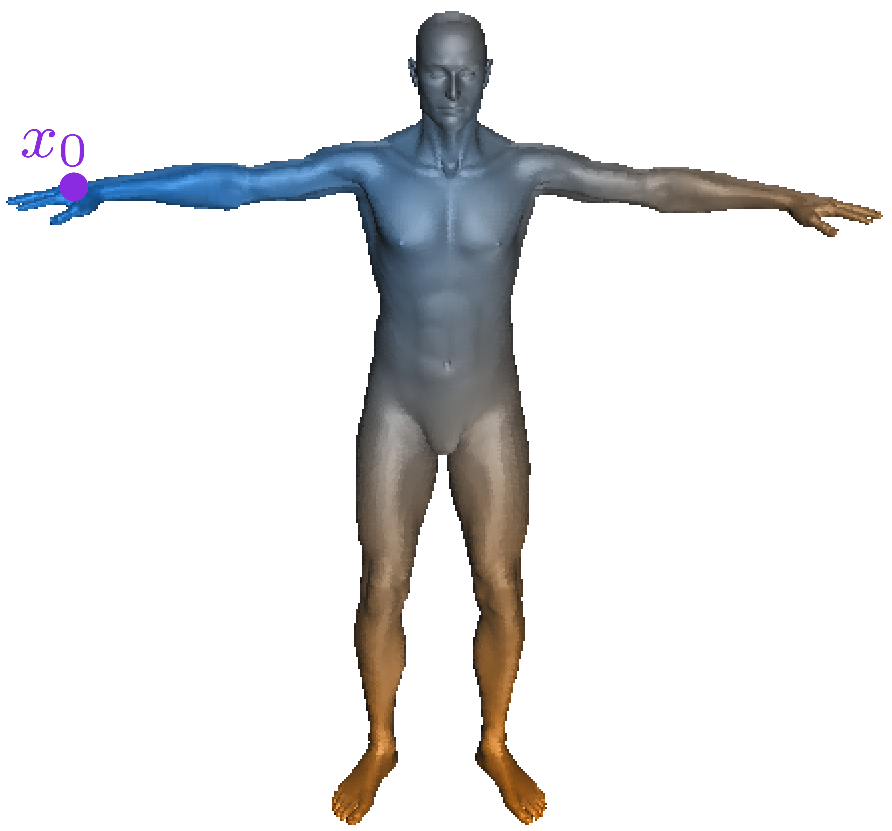
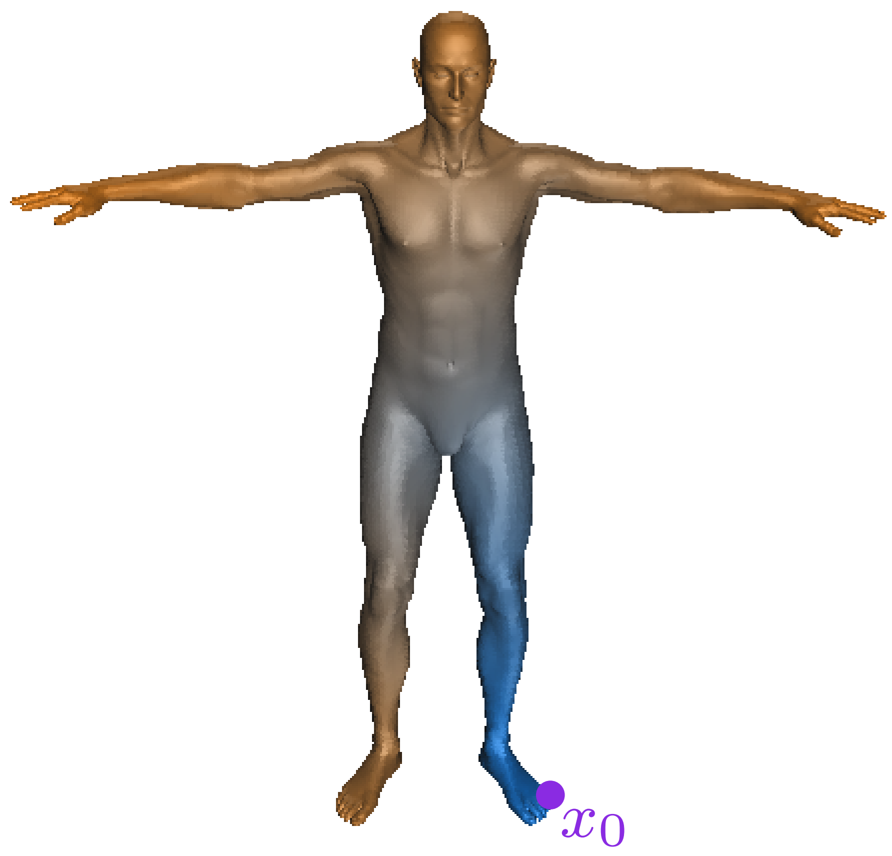
Signature: $\mathrm{Dg}\, f$

Examples:

- distance to a base point $x_0 \in X$: $f_{x_0}(x) = \mathrm{d}_X(x, x_0)$ is 1-Lipschitz



# Function-Based Signatures

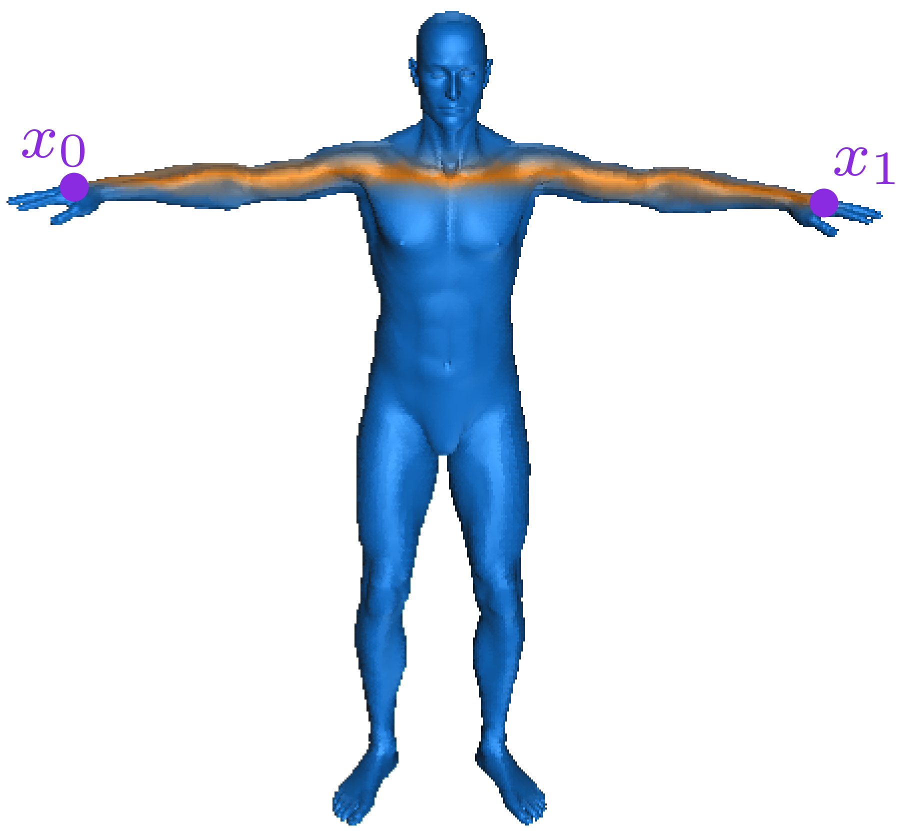
Input: a compact metric space $(X, \mathrm{d}_X)$ and a Lipschitz function $f : X \to \mathbb{R}$

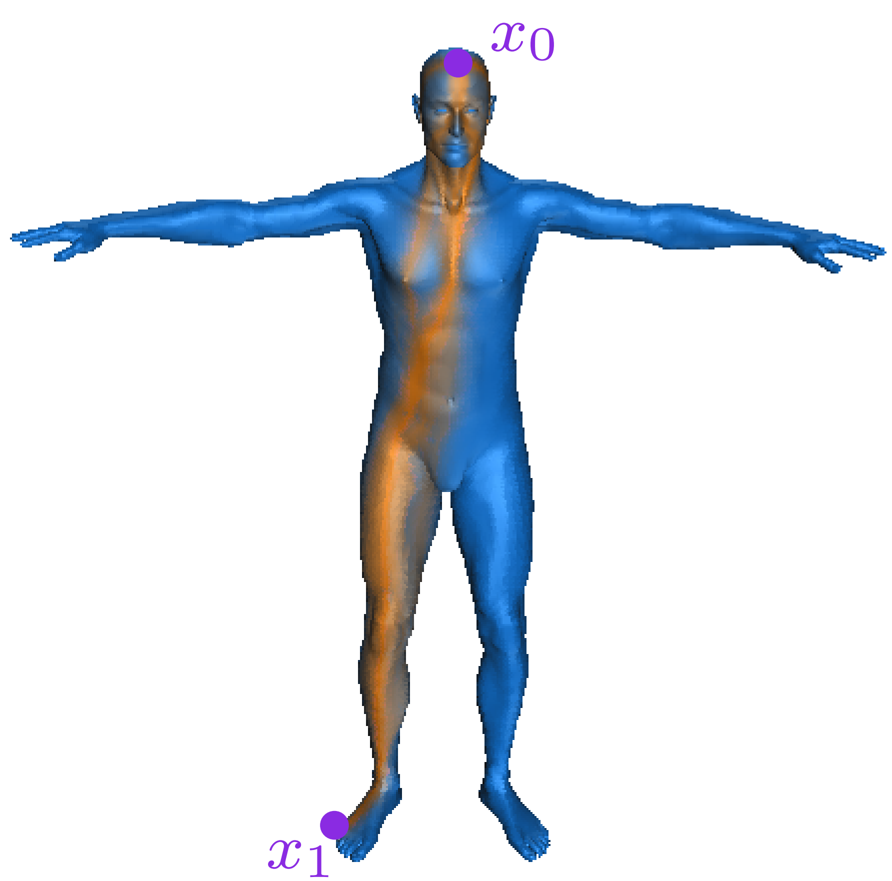
Signature: $\mathrm{Dg}\, f$

Examples:

- *fuzzy geodesic* [SCF'10] of a pair of base points $x_0, x_1 \in X$:

$$f_{x_0,x_1}(x) = \exp\left(-\frac{|\mathrm{d}_X(x,x_0)+\mathrm{d}_X(x,x_1)-\mathrm{d}_X(x_0,x_1)|}{\sigma}\right)$$ is $\frac{2}{\sigma}$-Lipschitz,



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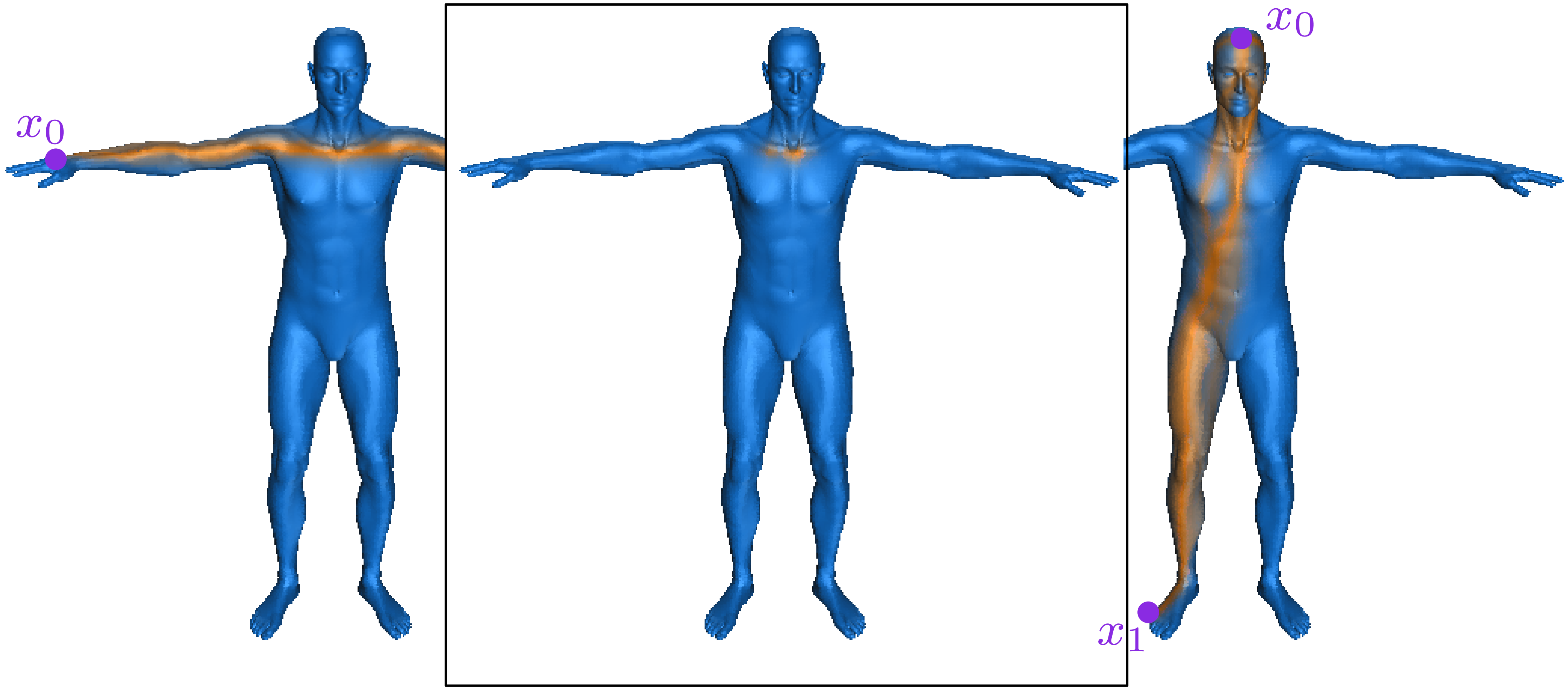
Input: a compact metric space $(X, \mathrm{d}_X)$ and a Lipschitz function $f : X \to \mathbb{R}$

Signature: $\mathrm{Dg}\, f$

Examples:

- *intersection configuration* [SCF'10] of a quadruple of base points:

$$f_{x_0,x_1,x_2,x_3}(x) = f_{x_0,x_1}(x) \cdot f_{x_2,x_3}(x) \text{ is } \tfrac{4}{\sigma}\text{-Lipschitz.}$$



# Function-Based Signatures (continued)

Desired stability result:

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact metric spaces equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$,

$$\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \in O(c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)).$$

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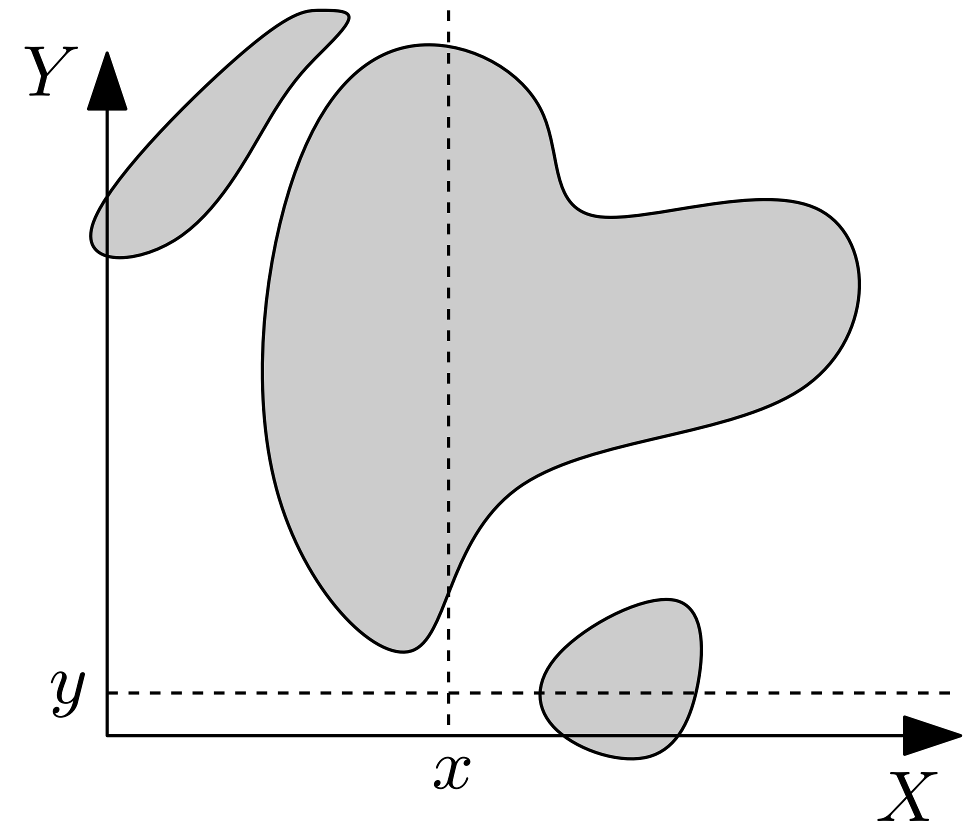
**Definitions:**

- correspondence



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**Definitions:**

- correspondence
- distortion

$$\mathrm{dist}_m(C) = \sup_{(x,y),(x',y') \in C} |\mathrm{d}_X(x, x') - \mathrm{d}_Y(y, y')|$$

$$\mathrm{dist}_f(C) = \sup_{(x,y) \in C} |f(x) - g(y)|$$

# Function-Based Signatures (continued)

Desired stability result:

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact metric spaces equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$,

$$\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \in O(c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)).$$

**Definitions:**

- correspondence
- distortion
- Gromov-Hausdorff distance $\quad \mathrm{d}_{\mathrm{GH}}(X, Y) = \frac{1}{2} \inf_{C \in \mathcal{C}(X,Y)} \mathrm{dist}_m(C)$

# Function-Based Signatures (continued)

Desired stability result:

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact metric spaces equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$,

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Note: this is a stability theorem for persistence diagrams

- improves over [CEH'05] (functions have different domains)

- improves over [dAFL'08] (domains are in different homeomorphism classes)

- relies on and is more specific than [CCGGO'09]

# Function-Based Signatures (continued)

Desired stability result:

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact metric spaces equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$,

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Note: this is a stability theorem for persistence diagrams

But it is not true in such generality:

- $\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) < \infty \Rightarrow (X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ are homologically equivalent
- $\mathrm{dist}_m(C)$ and $\mathrm{dist}_f(C)$ are finite regardless of homological types of $X, Y$

# Function-Based Signatures (continued)

Desired stability result:

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact metric spaces equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$,

$$\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \in O(c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)).$$

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- $\mathrm{dist}_m(C)$ and $\mathrm{dist}_f(C)$ are finite regardless of homological types of $X, Y$

$\to$ Restrict the focus to a class of *sufficiently regular* metric spaces

# Function-Based Signatures (continued)

Obtained stability result:

length spaces of curvature bounded above

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact ~~metric spaces~~ equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\operatorname{dist}_m(C) < \frac{1}{10} \min\{\varrho(X), \varrho(Y)\}$,

$$\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \in O(c \operatorname{dist}_m(C) + \operatorname{dist}_f(C)).$$

$$\leq 19c \operatorname{dist}_m(C) + \operatorname{dist}_f(C)$$

# Function-Based Signatures (continued)

Obtained stability result:

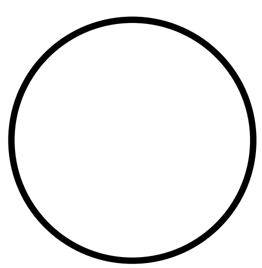
length spaces of curvature bounded above

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact ~~metric spaces~~ equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10} \min\{\varrho(X), \varrho(Y)\}$,

$$\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \; \sout{\in O(c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C))}.$$

$$\leq 19c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)$$

Prerequisite: $\mathrm{d}_\mathrm{GH}(X, Y) < \frac{1}{20} \min\{\varrho(X),\ \varrho(Y)\}$

$X =$ 

$Y =$ 

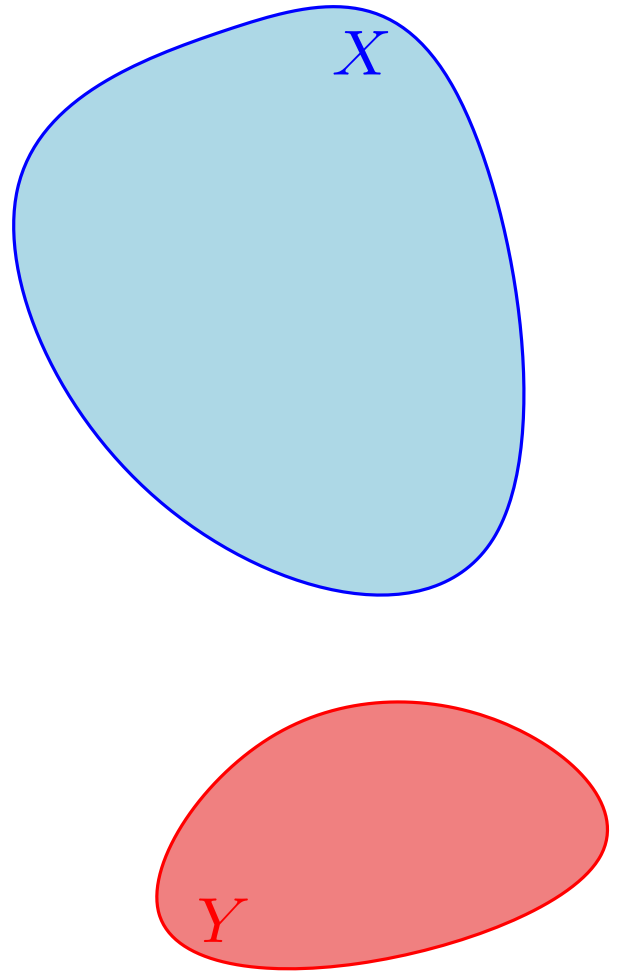
$$\mathrm{d}_\mathrm{GH}(X, Y) < \infty = \varrho(Y)$$

$$\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) = \infty$$

# Function-Based Signatures (continued)

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact Riemannian manifolds equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10}\min\{\varrho(X),\ \varrho(Y)\}$, $\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 19c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

**Proof:** reduction to *Scalar Fields Analysis from Point Cloud Data* [CGOS'09]:



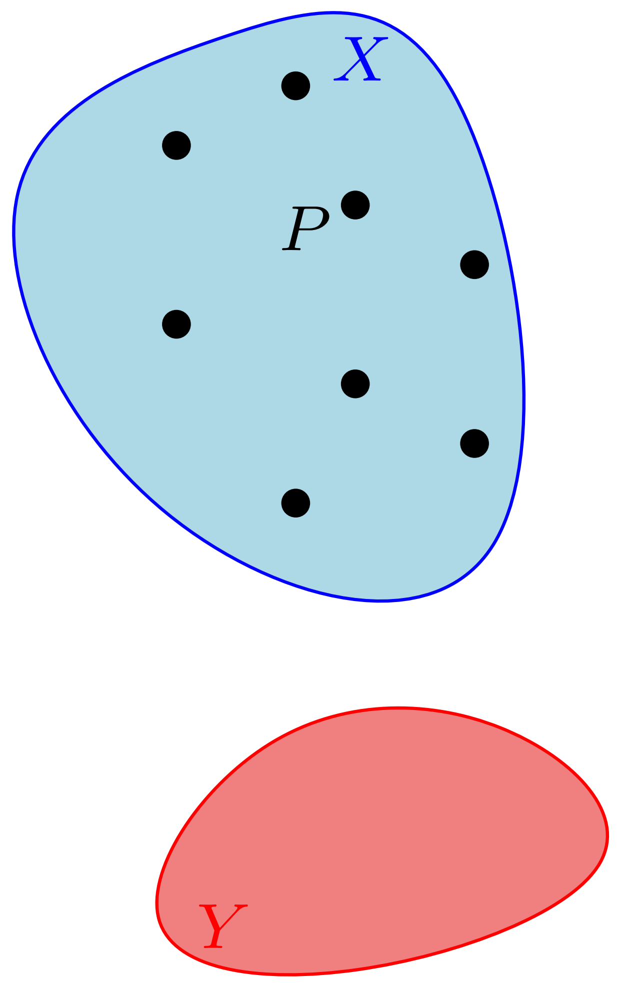
# Function-Based Signatures (continued)

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact Riemannian manifolds equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10}\min\{\varrho(X),\ \varrho(Y)\}$, $\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 19c\,\mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

**Proof:** reduction to *Scalar Fields Analysis from Point Cloud Data* [CGOS'09]:

Given any positive $\varepsilon < \frac{1}{10}\min\{\varrho(X),\ \varrho(Y)\} - \mathrm{dist}_m(C)$,

- take a finite $\varepsilon$-sample $P$ of $X$ ($P \subseteq X$)
- equip it with the induced metric $\mathrm{d}_P = \mathrm{d}_X|_{P\times P}$
- equip it with the restriction $h = f|_P$



# Function-Based Signatures (continued)

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact Riemannian manifolds equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10}\min\{\varrho(X),\ \varrho(Y)\}$, $\mathrm{d}_B^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 19c\,\mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

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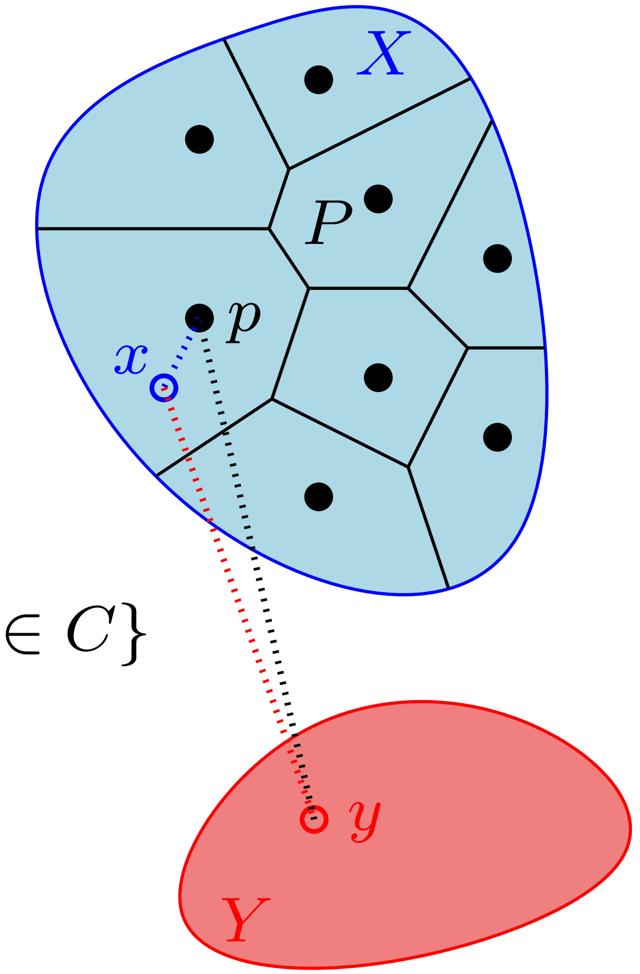
$C_{PX} = \{(p, x) \in P \times X : \mathrm{d}_X(x, p) = \min_{q\in P} \mathrm{d}_X(x, q)\}$

$C_{PY} = \{(p, y) \in P \times Y : \exists x \in X \text{ s.t. } (p, x) \in C_{PX} \text{ and } (x, q) \in C\}$



# Function-Based Signatures (continued)

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact Riemannian manifolds equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10} \min\{\varrho(X),\ \varrho(Y)\}$, $\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 19c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

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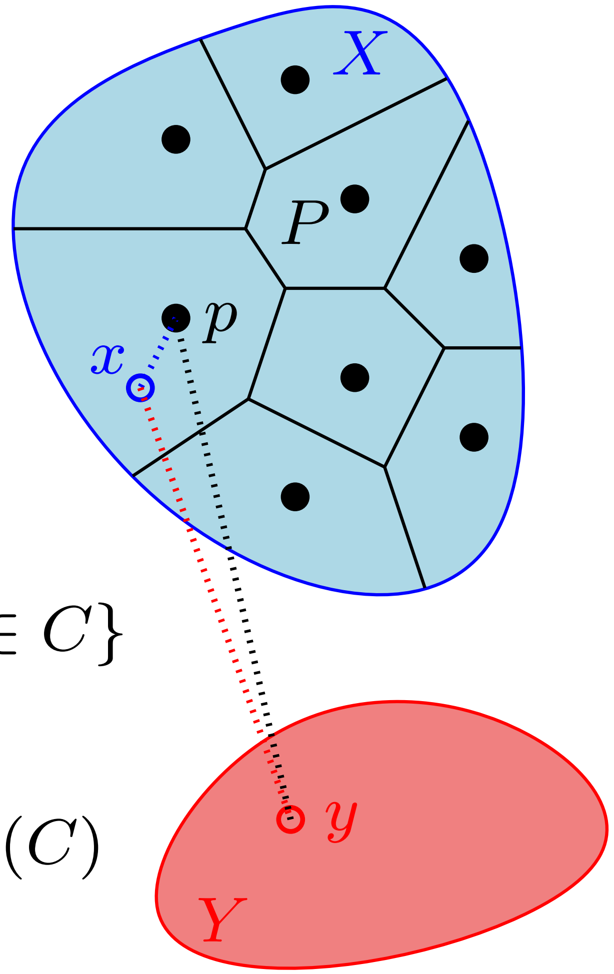
$\to$ $\mathrm{dist}_m(C_{PX}) \leq 2\varepsilon$ and $\mathrm{dist}_f(C_{PX}) = c\varepsilon$
$\mathrm{dist}_m(C_{PY}) \leq 2\varepsilon + \mathrm{dist}_m(C)$ and $\mathrm{dist}_f(C_{PY}) \leq c\varepsilon + \mathrm{dist}_f(C)$




$\to$ goal: approximate persistence diagram from GH-close finite metric space

# Function-Based Signatures (continued)

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact Riemannian manifolds equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10}\min\{\varrho(X),\ \varrho(Y)\}$, $\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 19c\,\mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

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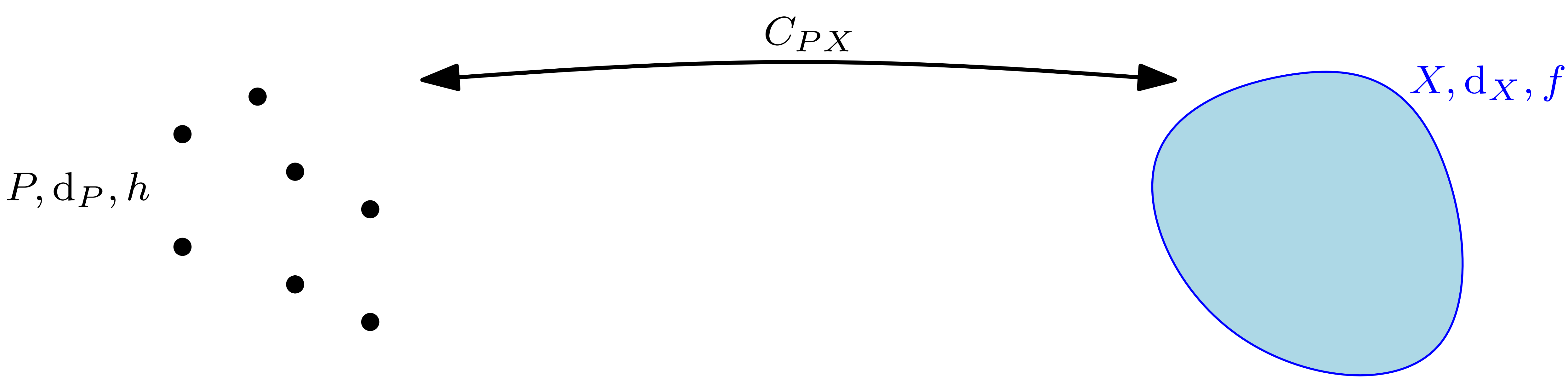

# Function-Based Signatures (continued)

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact Riemannian manifolds equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10}\min\{\varrho(X),\ \varrho(Y)\}$, $\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 19c\,\mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

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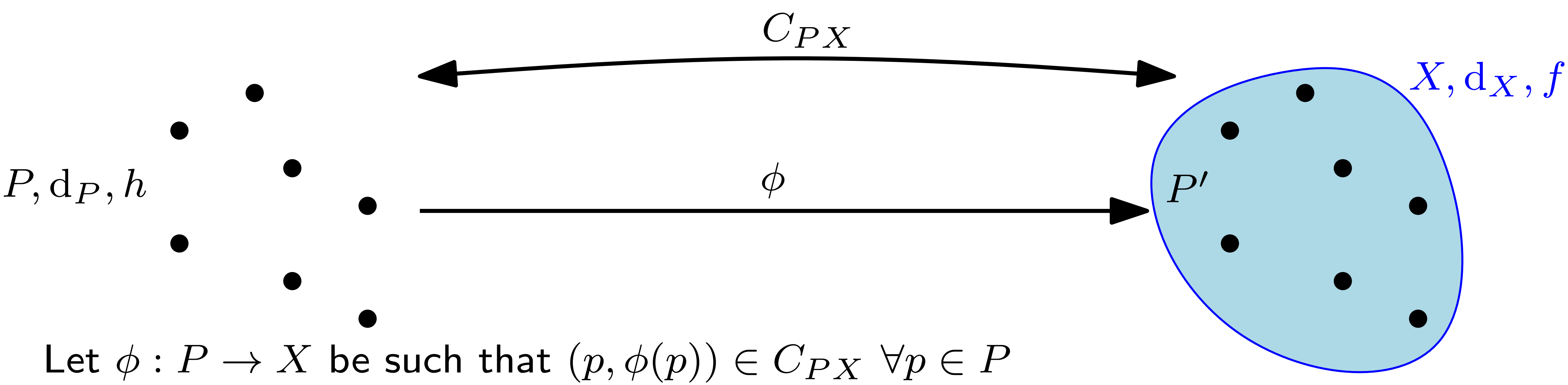


Let $\phi : P \to X$ be such that $(p, \phi(p)) \in C_{PX}\ \forall p \in P$

# Function-Based Signatures (continued)

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact Riemannian manifolds equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10}\min\{\varrho(X),\ \varrho(Y)\}$, $\mathrm{d}_B^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 19c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

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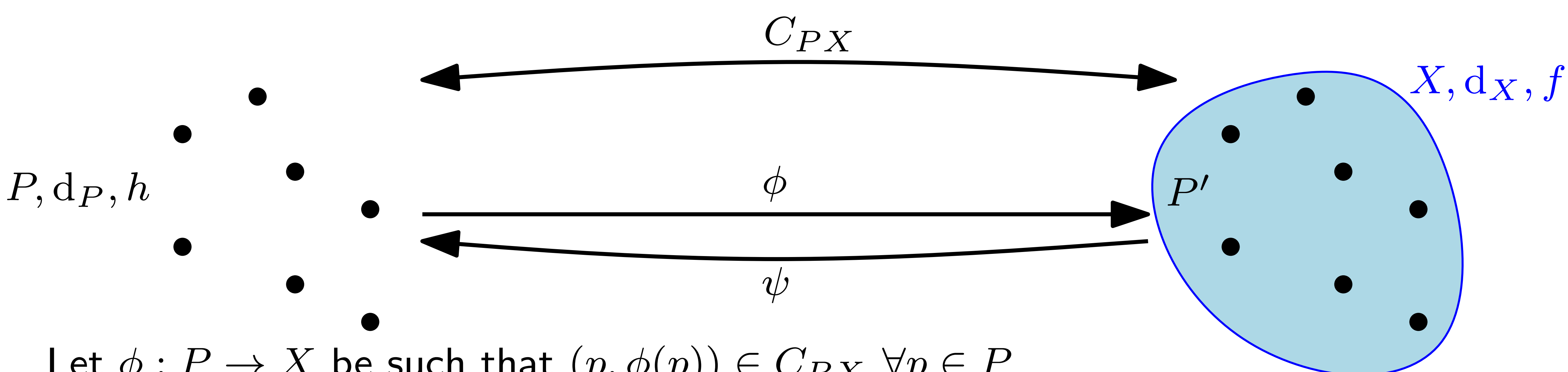


Let $\phi : P \to X$ be such that $(p, \phi(p)) \in C_{PX}$ $\forall p \in P$

Assume wlog that $\phi$ is **injective** and let $\psi : X \to P$ be a left inverse

# Function-Based Signatures (continued)

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact Riemannian manifolds equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10}\min\{\varrho(X),\ \varrho(Y)\}$, $\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 19c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

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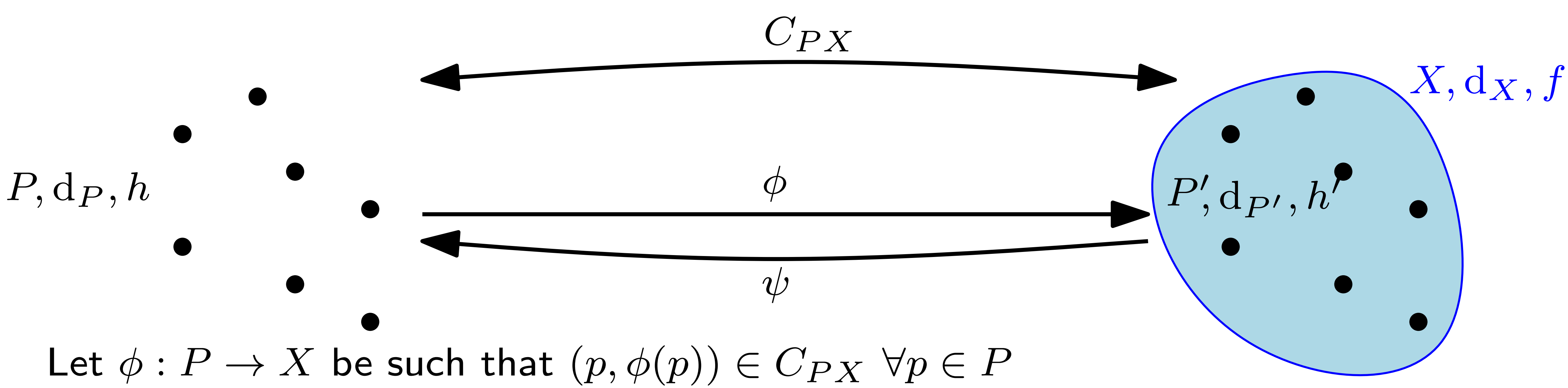


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Equip $P' = \phi(P)$ with $\mathrm{d}_{P'} = \mathrm{d}_P(\psi(\cdot), \psi(\cdot))$ and $h' = h \circ \psi$

$\to$ $\mathrm{d}_\mathrm{H}(P', X) \leq \mathrm{dist}_m(C)$
$\|h' - f|_{P'}\|_\infty \leq \mathrm{dist}_f(C)$ and $\|\mathrm{d}_{P'} - \mathrm{d}_X|_{P' \times P'}\|_\infty \leq \mathrm{dist}_m(C)$

# Function-Based Signatures (continued)

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact Riemannian manifolds equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X,Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10} \min\{\varrho(X),\ \varrho(Y)\}$, $\mathrm{d}_B^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 19c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

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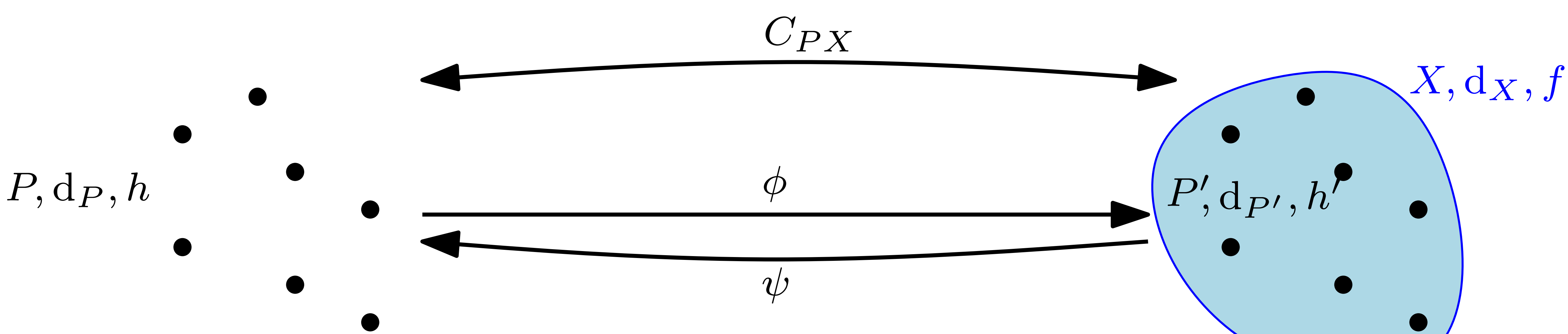


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Assume wlog that $\phi$ is **injective** and let $\psi : X \to P$ be a left inverse

Equip $P' = \phi(P)$ with $\mathrm{d}_{P'} = \mathrm{d}_P(\psi(\cdot), \psi(\cdot))$ and $h' = h \circ \psi$

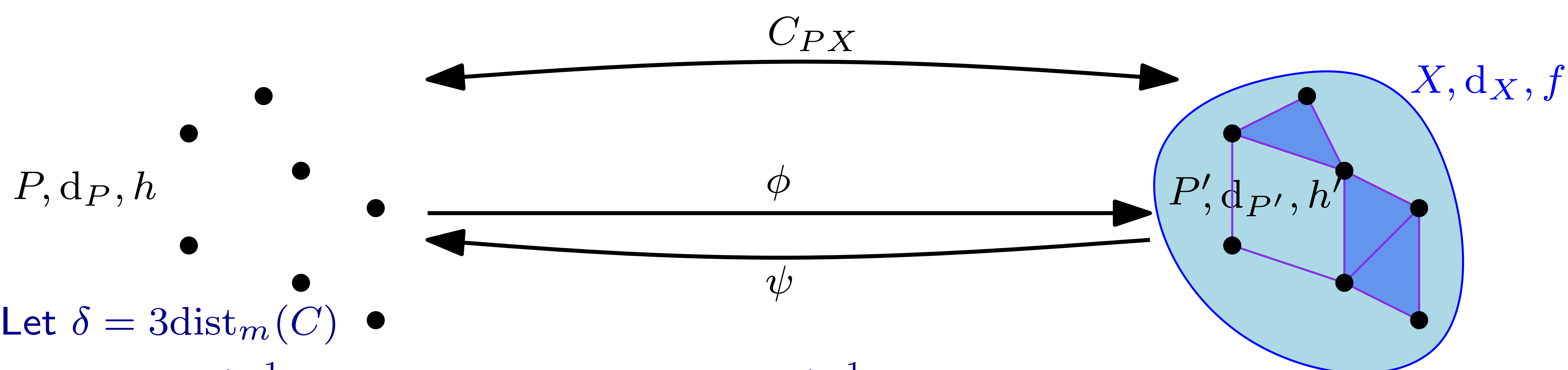
$\to$ $\mathrm{d}_H(P', X) \leq \mathrm{dist}_m(C)$ ← scenario considered in [CGOS'09]

$\|h' - f|_{P'}\|_\infty \leq \mathrm{dist}_f(C)$ and $\|\mathrm{d}_{P'} - \mathrm{d}_X|_{P' \times P'}\|_\infty \leq \mathrm{dist}_m(C)$ $\square$

# Function-Based Signatures (continued)

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact Riemannian manifolds equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10}\min\{\varrho(X),\ \varrho(Y)\}$, $\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \le 19c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

**Proof:** reduction to *Scalar Fields Analysis from Point Cloud Data* [CGOS'09]:




Let $\delta = 3\mathrm{dist}_m(C)$

Build $\{\mathrm{R}_\delta(h'^{-1}((-\infty, \alpha]))\}_{\alpha\in\mathbb{R}}$ and $\{\mathrm{R}_{3\delta}(h'^{-1}((-\infty, \alpha]))\}_{\alpha\in\mathbb{R}}$

$\to$ $\mathrm{d}_{\mathrm{H}}(P', X) \le \mathrm{dist}_m(C)$
$\|h' - f|_{P'}\|_\infty \le \mathrm{dist}_f(C)$ and $\|\mathrm{d}_{P'} - \mathrm{d}_X|_{P'\times P'}\|_\infty \le \mathrm{dist}_m(C)$

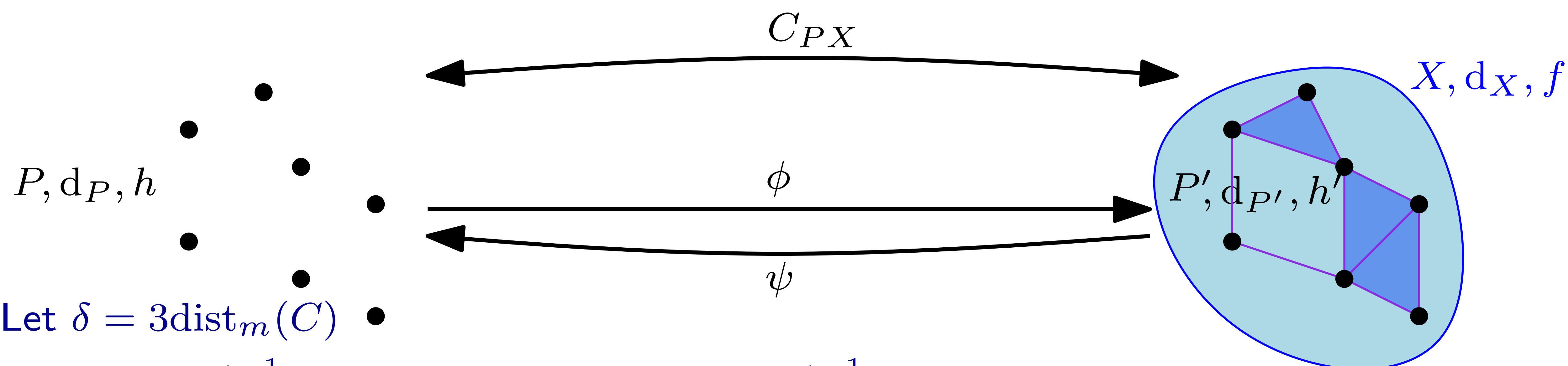
scenario considered in [CGOS'09]

□

# Function-Based Signatures (continued)

**Theorem (Stability):** Let $(X, \mathrm{d}_X)$ and $(Y, \mathrm{d}_Y)$ be two compact Riemannian manifolds equipped with $c$-Lipschitz functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then, for any correspondence $C \in \mathcal{C}(X, Y)$ such that $\mathrm{dist}_m(C) < \frac{1}{10} \min\{\varrho(X),\, \varrho(Y)\}$, $\mathrm{d}_B^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 19c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)$.

**Proof:** reduction to *Scalar Fields Analysis from Point Cloud Data* [CGOS'09]:




Let $\delta = 3\mathrm{dist}_m(C)$

Build $\{\mathrm{R}_\delta(h'^{-1}((-\infty, \alpha]))\}_{\alpha \in \mathbb{R}}$ and $\{\mathrm{R}_{3\delta}(h'^{-1}((-\infty, \alpha]))\}_{\alpha \in \mathbb{R}}$

**Thm:** $\mathrm{d}_B^\infty \left(\mathrm{Dg}\, f,\ \mathrm{Dg}\, \{\mathrm{Im}\, H_*(\mathrm{R}_\delta(h'^{-1}((-\infty, \alpha]))) \to H_*(\mathrm{R}_{3\delta}(h'^{-1}((-\infty, \alpha])))\}_{\alpha \in \mathbb{R}}\right)$
$\leq 10c\, \mathrm{dist}_m(C) + \mathrm{dist}_f(C)$

$\to$ $\mathrm{d}_H(P', X) \leq \mathrm{dist}_m(C)$ — scenario considered in [CGOS'09]

$\|h' - f|_{P'}\|_\infty \leq \mathrm{dist}_f(C)$ and $\|\mathrm{d}_{P'} - \mathrm{d}_X|_{P' \times P'}\|_\infty \leq \mathrm{dist}_m(C)$ $\square$

# Function-Based Signatures (continued)

Computing (approximating) the signatures in practice:

- when a triangulation of the manifold $X$ is given:

  - replace $f$ by its PL interpolation $\hat{f}$ over the triangulation

  - compute $\mathrm{Dg}\,\hat{f}$

  - $\mathrm{d}_\mathrm{B}^\infty(\mathrm{Dg}\,f,\, \mathrm{Dg}\,\hat{f})$ is controlled by the stability theorem for PDs [CEH'05]

# Function-Based Signatures (continued)

Computing (approximating) the signatures in practice:

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  - compute $\mathrm{Dg}\,\hat{f}$

  - $\mathrm{d}_{\mathrm{B}}^{\infty}(\mathrm{Dg}\,f,\,\mathrm{Dg}\,\hat{f})$ is controlled by the stability theorem for PDs [CEH'05]

- when a finite approximation $(P, \mathrm{d}_P, g)$ of $(X, \mathrm{d}_X, f)$ is given:

  - choose a neighborhood parameter $\delta > 0$

  - build the filtrations $\{\mathrm{R}_\delta(g^{-1}((-\infty,\alpha]))\}_{\alpha\in\mathbb{R}}$ and $\{\mathrm{R}_{3\delta}(g^{-1}((-\infty,\alpha]))\}_{\alpha\in\mathbb{R}}$

  - compute the PD of the image persistence module induced by inclusions:

$$\{\mathrm{Im}\,H_*(\mathrm{R}_\delta(g^{-1}((-\infty,\alpha])) \to H_*(\mathrm{R}_{3\delta}(g^{-1}((-\infty,\alpha]))\}_{\alpha\in\mathbb{R}}$$

  - bottleneck distance to $\mathrm{Dg}\,f$ is controlled by the results of [CGOS'09]

# Function-Based Signatures (continued)

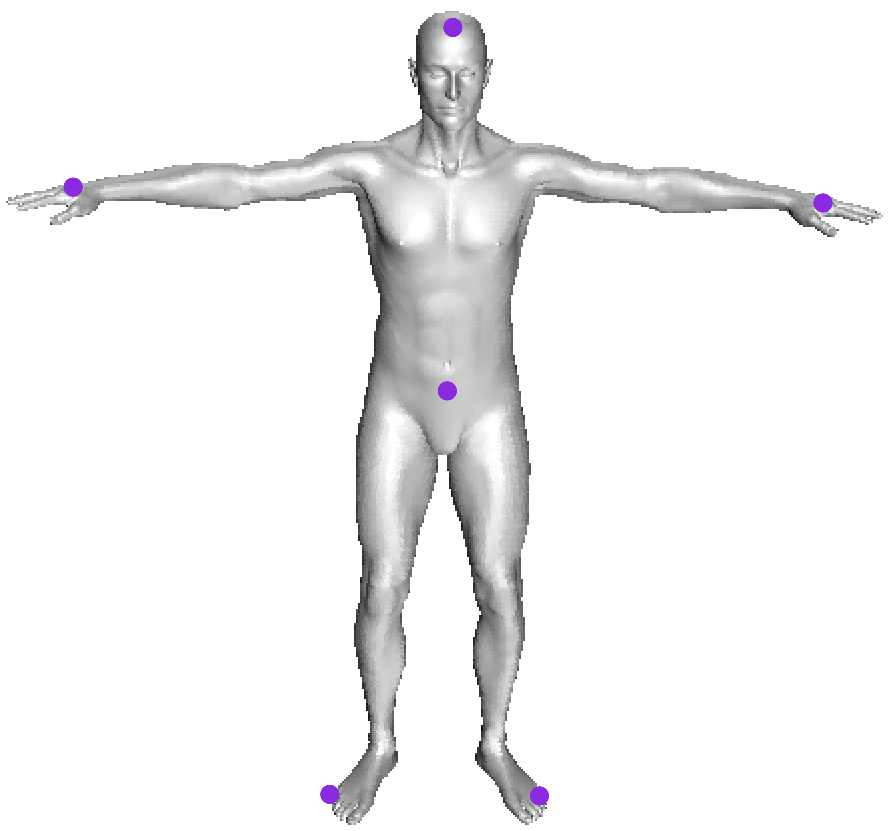
Experimental results:

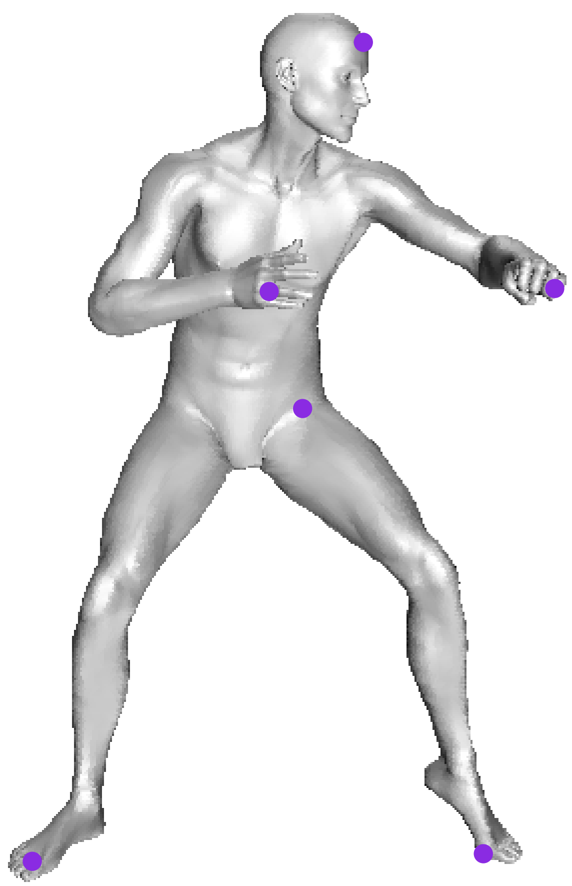
- input: shapes from the TOSCA database, in *mesh* form

- select a few base points by hand on each shape

- approximate geodesic distances to base points using the 1-skeleton graph

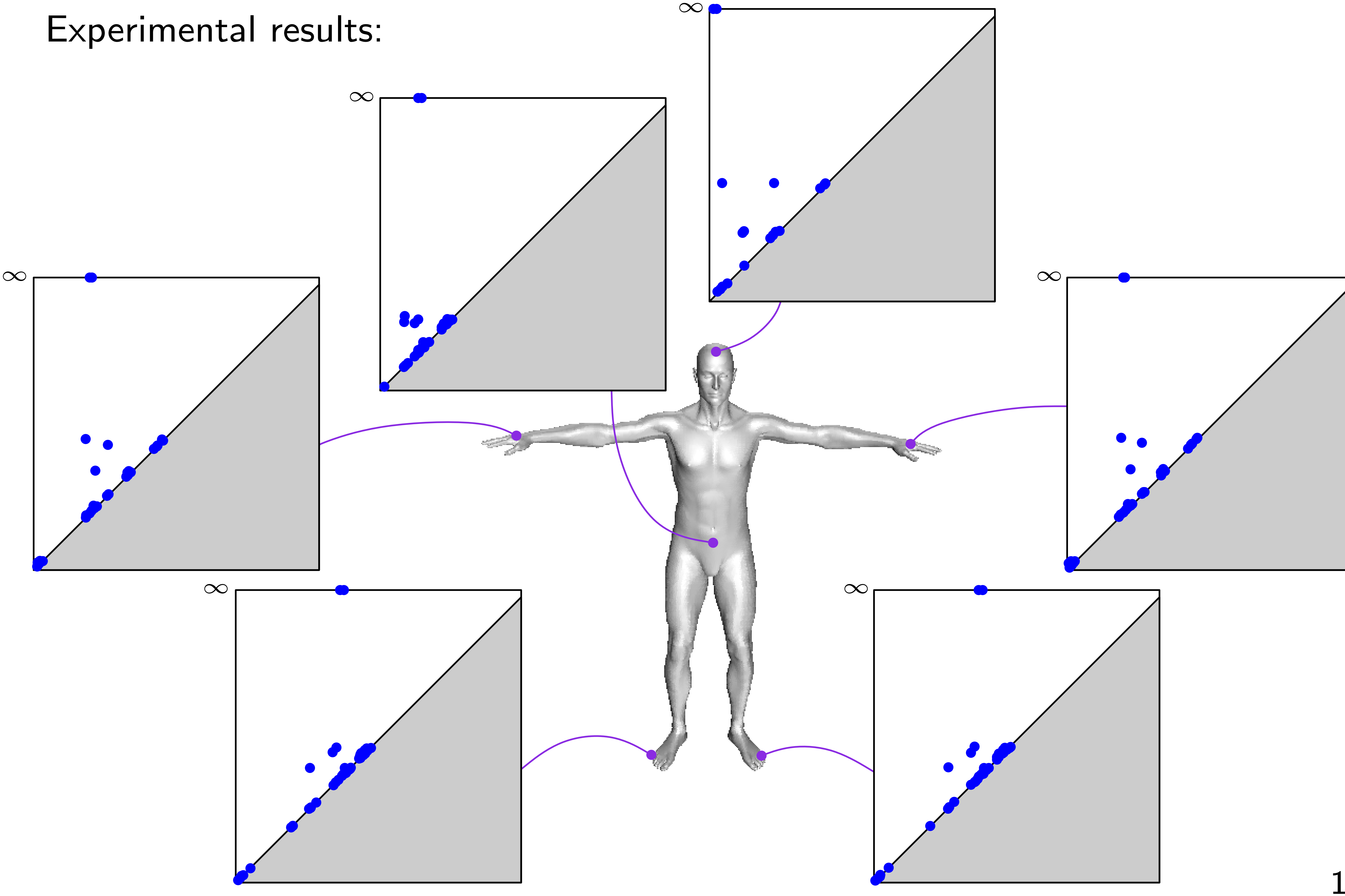
- use the PDs of the PL interpolations over the meshes as signatures





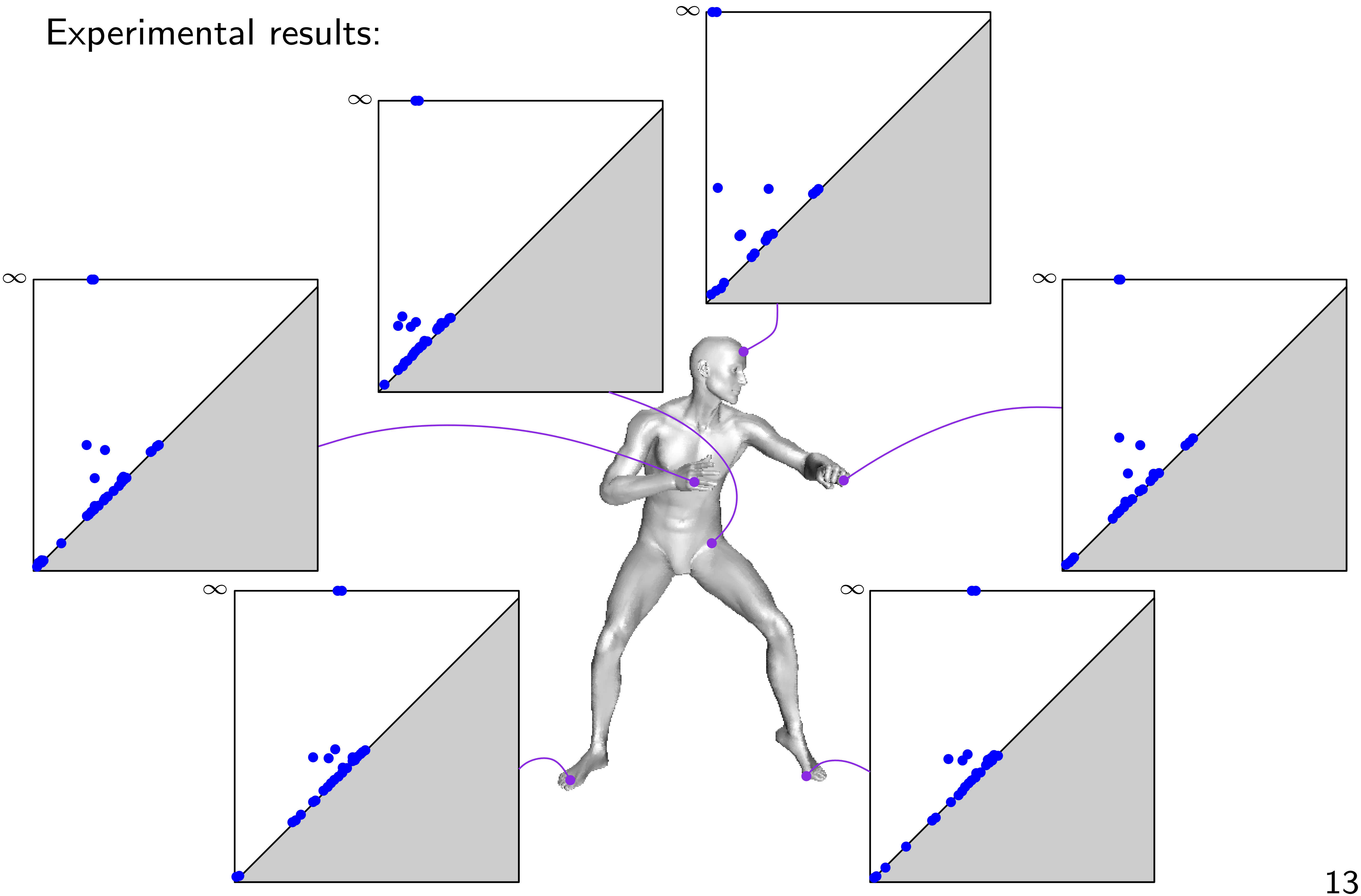
# Function-Based Signatures (continued)

Experimental results:



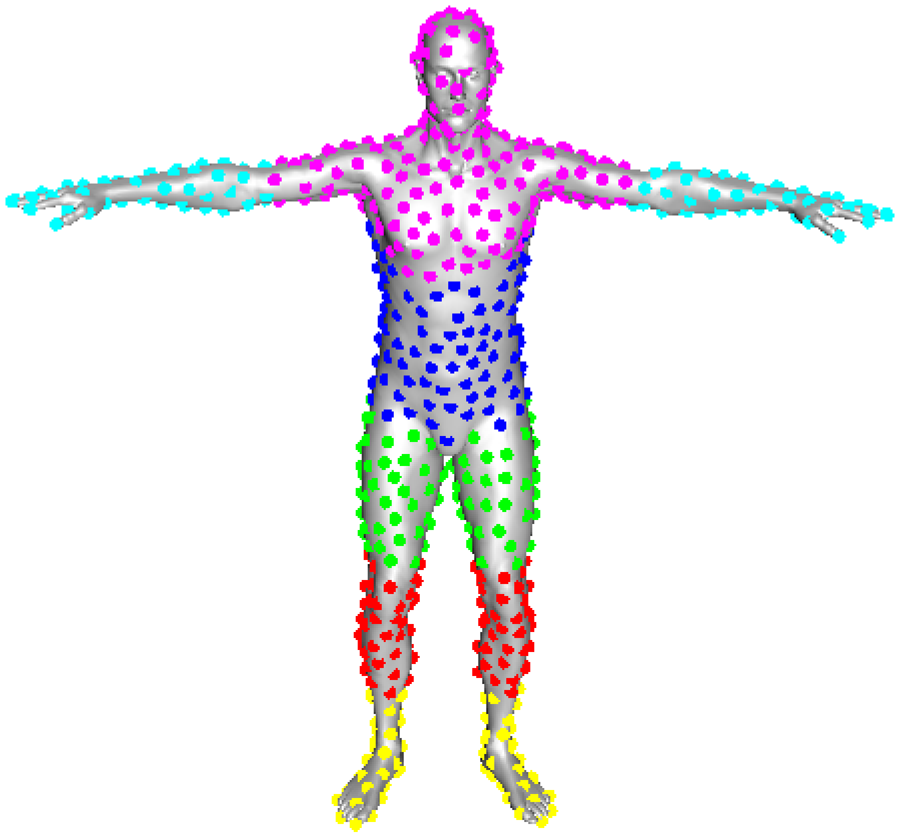
# Function-Based Signatures (continued)

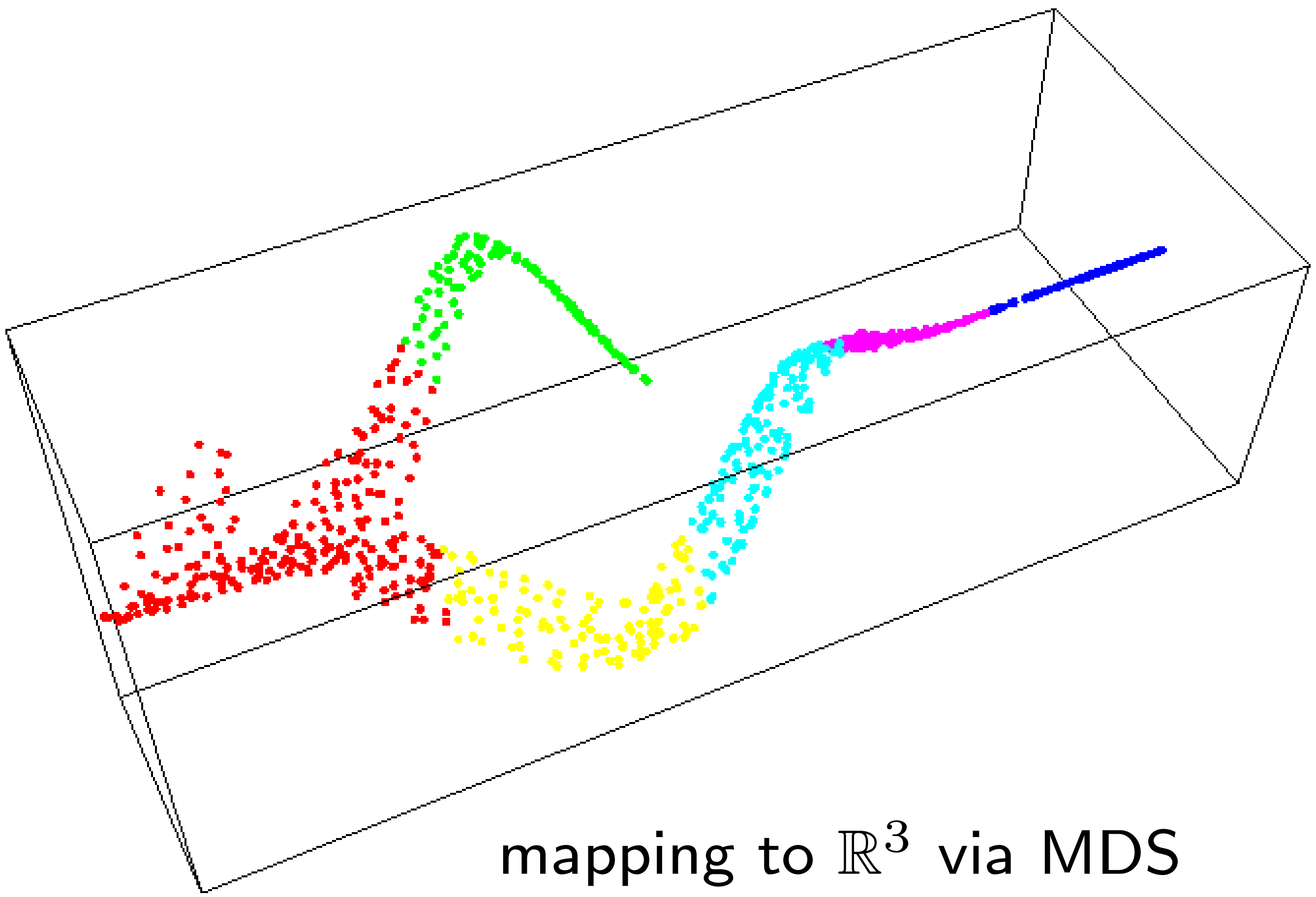
Experimental results:



# Function-Based Signatures (continued)

Experimental results:



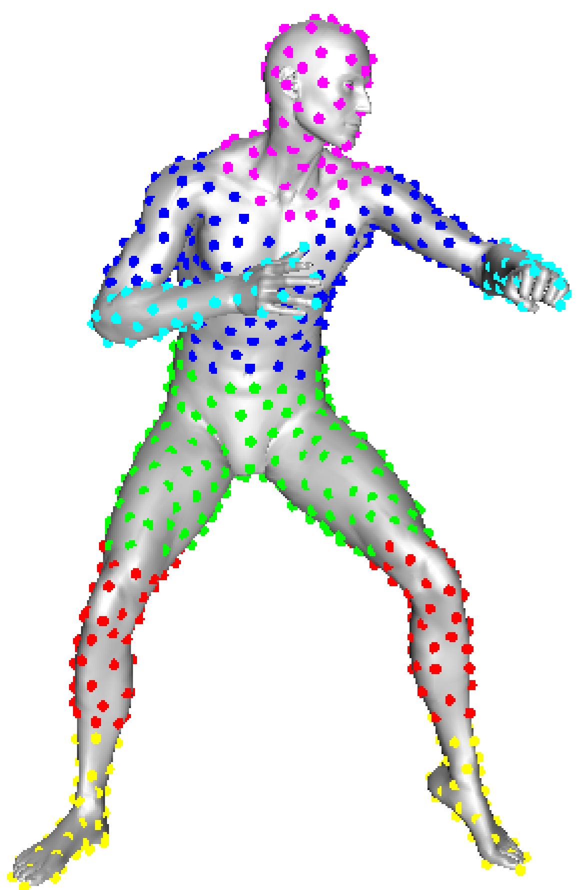


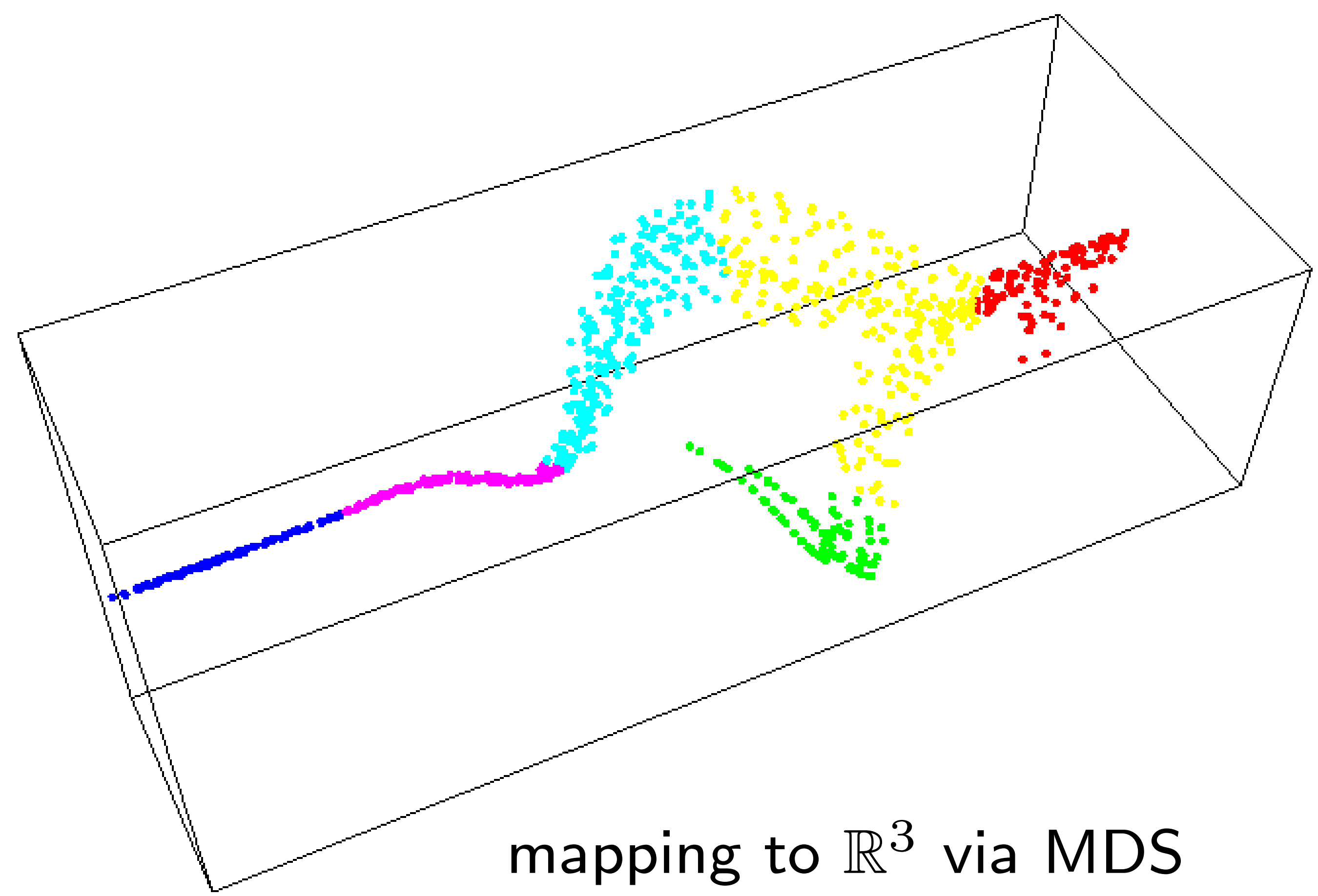
mapping to $\mathbb{R}^3$ via MDS

$k$-means in $\mathbb{R}^3$

# Function-Based Signatures (continued)

Experimental results:





mapping to $\mathbb{R}^3$ via MDS

$k$-means in $\mathbb{R}^3$

# Conclusion

- Two families of signatures: Rips-based and function-based
- Flexibility via choice of metric and/or function

$2$-parameters filtration $\to$ relate the $2$ parameters by a linear relation, *e.g.* $\alpha$

$\to$ $2$-parameter family of filtrations

**Q** Consider bi-filtration directly?