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Computing well diagrams for vector fields on $\mathbb{R}^n$

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\begin{abstract}
Using topological degree theory, we present a fast algorithm for computing the well diagram, a quantitative property, of a vector field on Euclidean space.© 2012 Elsevier Ltd. All rights reserved.
\end{abstract}

1. Introduction

Persistent homology is a popular tool in the field of topological data analysis. It takes as input a filtration of a space, usually derived from a function $f : X \to \mathbb{R}$, and outputs a persistence diagram [1]. The newer idea of the well diagram is inspired by the persistence diagram [2]. Given a mapping $f : X \to Y$ and a subspace $A \subseteq Y$, the well diagram encodes the robustness of the homology of $f^{-1}(A)$ to perturbations of the mapping $f$. The well diagram is a finite representation of a zigzag of well groups [3]. Except for in a few special cases, there is as yet no general method for computing a well group [4,5].

In this work, we focus on computing the well diagram for a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$ where $A = \{0\}$. The well diagram is interesting because it is both a quantitative and a stable property of the zeros of a vector field. We show that the rank of a well group is completely determined by the topological degree of an appropriate mapping. This insight leads to a fast algorithm. In Section 2, we define the degree diagram and illustrate a simple algorithm for computing it. In Section 3, we show that the degree diagram is equivalent to a well diagram. The stability of the degree diagram is then implied by the stability of the well diagram. In Section 4, we present the well diagram for three example vector fields on $\mathbb{R}^2$. Unless otherwise stated, all homology groups are computed using integer coefficients.

2. Degree diagrams

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous proper mapping from an n-dimensional Euclidean space to itself. Define a function $f_0 : \mathbb{R}^n \to \mathbb{R}$ as the $l_2$-norm, $f_0(x) = \|f(x)\|_2$, of the image of each point in $\mathbb{R}^n$. A value $r > 0$ is a regular value of $f_0$ if the sublevel set $F_r = f_0^{-1}[0, r]$ is an n-manifold, possibly with boundary, and, for all sufficiently small $\varepsilon > 0$, $f_0^{-1}[r - \varepsilon, r + \varepsilon]$ retracts to $f_0^{-1}(r)$. If $r$ is not regular, then it is critical.

The Tameness Assumption. We assume that the function $f_0$ has a finite number of critical values and that the inverse $f^{-1}(0) = F_0$ is a finite collection of points.

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Orientation. The local homology group $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$, for a point $x \in \mathbb{R}^n$, is isomorphic to the integers $\mathbb{Z}$. An orientation $\mu_x$ at $x$ is the choice of a generator $\mu_x \in H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$. An orientation at $x$ extends to an orientation $\mu_y$, for any point $y \in \mathbb{R}^n$. This follows from the fact that if $C \subset \mathbb{R}^n$ is a closed path connected set containing both $x$ and $y$, then there are isomorphisms: $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$. An orientation of $\mathbb{R}^n$ is the choice of a generator $\mu_x$, for some point $x \in \mathbb{R}^n$. If $C \subset \mathbb{R}^n$ is a compact path connected $n$-manifold with a tubular neighborhood, then an orientation on $\mathbb{R}^n$ implies the choice of a generator $\mu(C)$ of $H_n(\mathbb{C}, \mathbb{B}_d) \cong \mathbb{Z}$.

Degrees. Let $r$ be a regular value of $f_0$, and let $C \subseteq \mathbb{F}_r$ be a path connected component of the sublevel set. The mapping $f$ restricted to $C$, written as $f|_C : (C, \mathbb{B}_d C) \rightarrow (B_r, \mathbb{B}_d B_r)$, takes $C$ to the closed ball $B_r$ of radius $r$ centered at the origin. The mapping $f$ restricted to the boundary, $B_d C$, of $C$, written as $f|_{B_d C} : \mathbb{B}_d C \rightarrow S^{n-1}$, takes $\mathbb{B}_d C$ to the boundary of $B_r$, which is the $(n-1)$-sphere $S^{n-1}$. The continuous mappings $f|_C$ and $f|_{B_d C}$ induce the homomorphisms $f_*|_C : H_n(\mathbb{B}_d C) \rightarrow H_n(B_r, \mathbb{B}_d B_r)$ and $f_*|_{B_d C} : H_{n-1}(\mathbb{B}_d C) \rightarrow H_{n-1}(S^{n-1})$. Recall that the degree of the mapping $f|_C$ is the unique integer $\deg(f|_C)$ such that $f_*|_C(\mu(C)) = \deg(f|_C) \cdot \mu(B_r)$ [6, p. 134].

The boundary homomorphisms between the chain groups induce the following two homomorphisms on the homology, where the second is an isomorphism: $\partial_C : H_n(\mathbb{B}_d C) \rightarrow H_{n-1}(\mathbb{B}_d C)$ and $\partial_S : H_n(B_r, \mathbb{B}_d B_r) \rightarrow H_{n-1}(S^{n-1})$. Denote by $v(\mathbb{B}_d C)$ the class $\partial_C(\mu(C))$ and by $v(S^{n-1})$ the class $\partial_S(\mu(B_r))$. The degree of $f|_{B_d C}$ is the unique integer $\deg(f|_{B_d C})$ such that $f_*|_{B_d C}(v(\mathbb{B}_d C)) = \deg(f|_{B_d C}) \cdot v(S^{n-1})$. Note that $\mathbb{B}_d C$ may not be connected.

**Lemma 1.** $\deg(f|_C) = \deg(f|_{B_d C})$.

**Proof.** Consider the following diagram:

\[
\begin{array}{ccc}
H_n(C, \mathbb{B}_d C) & \xrightarrow{\partial_C} & H_{n-1}(\mathbb{B}_d C) \\
| f_*|_C \downarrow & & \downarrow | f_*|_{B_d C} \\
H_n(B_r, \mathbb{B}_d B_r) & \cong & H_{n-1}(S^{n-1}).
\end{array}
\]

The composition $f_*|_{B_d C} \circ \partial_C$ applied to $\mu(C)$ is $\deg(f|_{B_d C}) \cdot v(S^{n-1})$. The composition $f_*|_C \circ \partial_S$ applied to $\mu(C)$ is $\deg(f|_C) \cdot v(S^{n-1})$. The diagram commutes, and therefore the statement of the lemma holds. \qed

There is a third, and also equivalent, definition of a degree associated with $f$ and $C$. Let $\{x_1, \ldots, x_k\}$ be the collection of points $\mathbb{F}_r \cap C$. Choose a regular value $r$ of $f_0$ smaller than the smallest critical value. Then $\mathbb{F}_r \cap C$ has $k$ path connected components $\{D_1, \ldots, D_k\}$ such that $x_i \in D_i$. The mapping $f$ induces a homomorphism $f_*|_i : H_n(D_i, D_i - \{x_i\}) \rightarrow H_n(B_r, B_r - \{0\})$, for each $1 \leq i \leq k$. The local degree of $f$ at $x_i$ is the unique integer $\deg(f|_i)$ such that $f_*|_i(\mu(D_i)) = \deg(f|_i) \cdot \mu_0$.

**Lemma 2.** $\deg(f|_C) = \sum_{i=1}^k \deg(f|_i)$.

**Proof.** Consider the following diagram:

\[
\begin{array}{ccc}
H_n(C, \mathbb{B}_d C) & \xrightarrow{m} & H_n(C, C - \mathbb{F}_0) \\
| f_*|_C \downarrow & & \downarrow | f_*|_{\mathbb{F}_0} \\
H_n(B_r, \mathbb{B}_d B_r) & \cong & H_n(B_r, B_r - \{0\}).
\end{array}
\]

Both $m$ and $l$ are induced by inclusion, and $l$ is an isomorphism. The homomorphism $f_*|_{\mathbb{F}_0}$ is induced by the mapping $f$. The class $m(\mu(C))$ is the direct sum $\oplus_i \mu(B_r)$. We have $(f_*|_{\mathbb{F}_0} \circ m)(\mu(C)) = \sum_{i=1}^k \deg(f|_i) \cdot \mu_0$. Furthermore $(l \circ f_*|_C)(\mu(C)) = \deg(f|_C) \cdot \mu_0$. The diagram commutes, and therefore the statement of the lemma holds. \qed

The degree diagram. A degree diagram $DDgm(f_0)$ of $f_0$ is a multiset of points taken from the extended interval $[0, \infty]$. If $r$ is a regular value of $f_0$, then the number of points in $DDgm(f_0)$ with values greater than $r$ is defined as the number of components $C \subseteq \mathbb{F}_r$ such that $\deg(f|_C) \neq 0$. For reasons of stability, which will be explained in the next section, the point $0$ belongs to $DDgm(f_0)$ with infinite multiplicity. By the Tameness Assumption, $DDgm(f_0)$ has only finitely many positive valued points.

**Example.** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous mapping as shown in Fig. 1. There are four points in $\mathbb{F}_0$. From left to right, the local degrees of the four points are $+1, -1, +1,$ and $-1$. The tree in Fig. 1 shows how the components of the sublevel sets $\mathbb{F}_r$ evolve. By Lemma 2, the local degrees are sufficient to determine the degree of any component $C$ in $\mathbb{F}_r$, for $r$ regular. For all sufficiently small $\varepsilon > 0$, there are four components with nonzero degree in $\mathbb{F}_{r_1 - \varepsilon}$ and two components with nonzero degree in $\mathbb{F}_{r_1 + \varepsilon}$. Therefore, there are two points in the degree diagram $DDgm(f_0)$ with value $r_1$. Similarly, there are two points in $DDgm(f_0)$ with value $r_2$. The point $0$ in the diagram has infinite multiplicity.

3. Stability

Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a second proper mapping, and assume that $g_0$ satisfies the Tameness Assumption. Define the distance, $Dist(f, g)$, between the two mappings as $Dist(f, g) = \sup_{x \in \mathbb{R}^n} ||f(x) - g(x)||$. There is a notion of a distance
between the two diagrams DDgm(f₀) and DDgm(g₀). Order the positive points in both diagrams and add zeros so that both sequences have the same length. We have 0 ≤ u₁ ≤ u₂ ≤ · · · ≤ uᵣ₀ and 0 ≤ v₁ ≤ v₂ ≤ · · · ≤ vᵣ₁. Define the distance, Dist(DDgm(f₀), DDgm(g₀)), between two degree diagrams as Dist(DDgm(f₀), DDgm(g₀)) = max₁≤ᵣ≤ₚ |uᵢ − vᵢ|.

**Theorem 1.** Dist(DDgm(f₀), DDgm(g₀)) ≤ Dist(f₀, g₀).

**Theorem 1** says that the degree diagram is stable under perturbations of the mapping f. In the rest of this section, we show that the degree diagram is equivalent to the well diagram as introduced in [2]. The stability of the well diagram implies Theorem 1.

**Well diagrams.** Let h : Rⁿ → Rⁿ be a mapping such that Dist(f₀, h) < r. Then h⁻¹(0) ⊆ Fᵣ induces a homomorphism hₜ : h⁻¹(0); Z/2Z → hₜ(Fᵣ; Z/2Z). The well group, Uₚ(r; Z/2Z), is the subgroup of hₜ(Fᵣ; Z/2Z) such that

\[ Uₚ(r; Z/2Z) = \bigcap_{\forall h.s.t. Dist(h, f₀) < r} \text{im} \, hₜ. \]

Assuming that f₀ satisfies the Tameness Assumption, the rank of Uₚ(0; Z/2Z) is the number of fixed points of f. Furthermore, the rank of the well group decreases monotonically with increasing r. The well diagram, WDgm(f₀), is a multiset of points taken from [0, ∞). A point r ∈ (0, ∞) belongs to WDgm(f₀) with multiplicity k if the rank of the well group Uₚ(r) changes by k at r. The point ∞ is counted with multiplicity k if for all sufficiently large values of r, the rank of Uₚ(r) is k. The point 0 is counted with infinite multiplicity. The distance between two well diagrams is defined in the same way as the distance between two degree diagrams. In [2], it was shown that Dist(WDgm(f₀), WDgm(g₀)) ≤ Dist(f₀, g₀). The following lemma implies Theorem 1.

**Lemma 3.** For a regular value r of f₀, the rank of the well group Uₚ(r; Z/2Z) is the number of connected components C ⊆ Fᵣ, such that deg(f₀↾C) ≠ 0.

**Proof.** Consider the following diagram for any h such that Dist(f₀, h) ≤ r:

\[
\begin{array}{ccc}
Hₚ(C, Bd C) & \xrightarrow{i} & Hₚ(C, C - h⁻¹(0)) \\
|f₀|₀ & \downarrow & |hₜ|₀ \\
Hₚ(B₀, Bd B₀) & \xrightarrow{j₀} & Hₚ(B₀, B₀ - \{0\}).
\end{array}
\]

The homomorphisms i₀ and j₀ are induced by the inclusion of spaces. The mapping hₜ|₀ is induced by h. If deg(f₀↾C) ≠ 0, then, by commutativity, h⁻¹(0) ∩ C cannot be empty. Therefore the rank of Uₚ(r; Z/2Z) is at least the number of components C such that deg(f₀↾C) ≠ 0.

If deg(f₀↾C) = deg(f₀↾B₀) = 0, then by the Hopf Extension Theorem, there is an extension g : C → Sⁿ⁻¹ such that g restricted to Bd C equals f restricted to Bd C [7]. Define a mapping h : Rⁿ → Rⁿ as h(x) = 0.5-f + 0.5g. By the definition of a regular value, there is no point in the interior of C that maps to the boundary of B₀, and therefore Dist(f₀, h) < r. Furthermore, the preimage h⁻¹(0) ∩ C is empty. In other words, the component C does not contribute to the well group Uₚ(r; Z/2Z).

**4. Experiments**

To get a better feeling for the well diagram, we compute the well diagram for a random vector field on three subspaces of the plane: a disk, an annulus, and a thickened figure eight. The vector field on the boundary of each domain points inward with a magnitude of 3, and the magnitude of each vector in the interior is at most 3. See Fig. 2 for pictures of each of the three vector fields. A regular value r, the number of components of the sublevel set Fᵣ, with nonzero degree, is the number of points in the diagram with value at least r. In other words, the rank of the well group Uₚ(r; Z/2Z) is the number of points in the diagram with value at least r. We stop growing the sublevel set at radius 3. In Fig. 2(a) and (e), the rank of the well group Uₚ(3; Z/2Z) is 1, a hint of the Euler characteristic of Fₚ.
5. Conclusion

We have shown that the seemingly intractable definition of the well group is, for the case of vector fields on $\mathbb{R}^n$, easily understood using degree theory. This observation leads to a fast construction of the well diagram, a new, stable, and quantitative property of a vector field.

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