International Journal of Computational Geometry & Applications

Ball-map : Homeomorphism between compatible surfaces

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Homeomorphisms between curves and between surfaces are fundamental to many applications of 3D modeling, graphics, and animation. They define how to map a texture from one object to another, how to morph between two shapes, and how to measure the discrepancy between shapes or the variability in a class of shapes. Previously proposed maps between two surfaces, S and S', suffer from two drawbacks: (1) it is difficult to formally define a relation between S and S' which guarantees that the map will be bijective and (2) mapping a point x of S to a point x' of S' and then mapping x' back to S does in general not yield x, making the map asymmetric. We propose a new map, called ball-map, that is symmetric. We define simple and precise conditions for the ball-map to be a homeomorphism. We show that these conditions apply when the minimum feature size of each surface exceeds their Hausdorff distance. The ball-map, $\mathrm{BM}_{S,S'}$, between two such manifolds, S and S', maps each point x of S to a point $x' = \mathrm{BM}_{S,S'}(x)$ of S'. $\mathrm{BM}_{S',S}$ is the inverse of $\mathrm{BM}_{S,S'}$, hence BM is symmetric. We also show that, when S and S' are C^k $(n-1)$-manifolds in $\mathbb{R}^n$, $\mathrm{BM}_{S,S'}$ is a C^{k-1} diffeomorphism and defines a C^{k-1} ambient isotopy that smoothly morphs between S to S'. In practice, the ball-map yields an excellent map for transferring parameterizations and textures between ball compatible curves or surfaces. Furthermore, it may be used to define a morph, during which each point x of S travels to the corresponding point x' of S' along a broken line that is normal to S at x and to S' at x'.

Keywords: homeomorphism; surface; Frechet

1. Introduction

Many problems in 3D graphics, animation, computer vision and data analysis require building a map between two curves or between two surfaces [13,23,9]. Such a map may be used to transfer a texture from a surface S to a simplified version S' of S [26] or to formulate the discrepancy between S and S', [6,20,24] better than it has been possible so far by using variations of the Hausdroff distance [18,19]. Finally, it provides a point-to-point association for computing 3D morphs [1,15].

In most cases, a bijective map (homeomorphism) is desired. Several maps have been used in the past. The closest-point map $C_{S',S} : S' \to S$ maps a point x' of S' to its closest point $x = C_{S',S}(x')$ of S. Its inverse, the orthomap [12] $O_{S,S'} : S \to S'$ maps a point x of S to the point x' of S' that is the first intersection of a ray starting from x along the normal to S at x oriented towards the interior of the symmetric difference between the regions bounded by S and S'. According to [8], the orthomap and the closest-point map from S to S' are homeomorphisms if the minimum feature sizes of S and S' both exceed $h/(2-\sqrt{2})$, where h is the Hausdorff distance between S and S'. We refer to this condition as the normal-compatibility condition between two surfaces in 3D or curves in 2D. Unfortunately, the orthomap and its inverse are not symmetric, and thus, in a sense, suboptimal [13]. In particular, if x' is the closest point on S' to a point x of S, x is not in general the closest point on S to x'. Hence, one anticipates the existence of a better map producing a shorter average distance between points x of S and their closest point image on S'.

In this paper, we propose a new map, which we named *ball-map*, between any two $(n-1)$-manifolds in $\mathbb{R}^n$. We show that under specific conditions, the two manifolds are, what we call, ball-compatible which means that the ball-map $\mathrm{BM}_{S,S'} : S \to S'$ from the manifold S onto the manifold S' is a homeomorphism and $\mathrm{BM}_{S,S'}$ is the inverse of $\mathrm{BM}_{S',S}$. Furthermore, we show that in this case the ball-map defines a smooth isotopy through which S may morph into S'. We say that two curves or surfaces are minimum feature compatible when the minimum feature sizes of S and S' both exceed h, where h is the Hausdorff distance between S and S'. We show that two minimum feature compatible curves or surfaces are ball-compatible. Note that this sufficient feature-respecting compatibility condition ensuring ball-compatibility is less restrictive than the corresponding condition for normal-compatibility, which requires a tighter ratio of $h/(2-\sqrt{2})$.

Minimum-feature compatibility is an optimal condition for the equality of the Hausdorff and Fréchet distances and provides a mild condition for surface isotopy. Consequently, we anticipate that the ball-map will be of value for comparing smooth surfaces and for formulating the error between a shape and its approximation that ensures topological compatibility. In solid modeling, the ball-map will make it possible to simplify the expression of the discrepancy between a CAD model of a nominal part and 3D measurements of manufactured products. In particular, it provides a generalized and constructive version of the theorem proven in [3] that uses metric conditions to *guarantee surface isotopy* .

Finally, it follows from the definition of the ball-map that it is a conformal invariant (i.e. invariant under Möbius transforms). Note that the closest-point projection is invariant under isometries only.

When the discrepancy between the two surfaces or the two curves exceeds the feature-respecting compatibility condition, the ball-map may not be a homeomorphism. In such cases, one may need to rely on more general, and less precise, mappings [9]. Still, even in such incompatible cases, the ball-map may be of use for automatically generating, an optimal map and morph between consecutive frames in a family that samples the morph between disparate curves or surfaces.

The remainder of this paper is organized as follows. Section 2 provides preliminary definitions and assumptions used throughout the rest of the paper. In section 3, we introduce the ball-compatibility and prove the continuity of the ball-map. Section 4 provides a necessary and sufficient condition for C^1 manifolds to be ball-compatible and then the sufficient feature-respecting compatibility condition formulated in terms of the Hausdorff distance. We show in section 5 that for a pair of surfaces satisfying the feature-respecting compatibility condition the Fréchet distance equals the Hausdorff distance. Section 6 relates the smoothness of the ball-map and of the median surface that it defines to the smoothness of the two ball compatible surfaces. We show in section 7 that the ball-map is not only a homeomorphism, but in fact an isotopy. The sufficient conditon for ball-map is then a metric based sufficient condition for two manifolds to be isotopic. Section 8 reviews the anticipated impact of the new results developed in this paper on practical applications. Finally section 9 concludes and mentions possible extensions and future work.

2. Background concepts and definitions

For any set X, $\overline{X}$ and X^c denote respectively the closure and the complement of X. For two sets X and Y, $X \cup Y$, $X \cap Y$ and $X \setminus Y$ denote respectively the union, intersection and set difference of X and Y.

2.1. *Compact connected $(n-1)$-manifolds embedded in $\mathbb{R}^n$*

We consider here a $(n-1)$-manifold S embedded in $\mathbb{R}^n$. Hence, each one of its points has a neighborhood homeomorphic to an $(n-1)$-dimensional disk [25]. For example, it may be a curve embedded in $\mathbb{R}^2$ or a surface embedded in $\mathbb{R}^3$. When the $(n-1)$-manifold S is orientable and smooth, a unit-length normal vector field to S defines a map $\vec{n} : x \to \vec{n}(x)$ known as the Gauss map.

A compact connected $(n-1)$-manifold S embedded in $\mathbb{R}^n$ is orientable and the complement of S in $\mathbb{R}^n$ has two connected components ([10] pp. 234). In other words, S decomposes $\mathbb{R}^n$ into three connected parts: S itself, S_i, the interior, and S_e, the exterior. S_e is the unbounded part of the complement of S. Note that S_i and S_e are open. Using Computer Aided Design terminology, in two dimensions S would be a closed curve separating the interior face S_i from the unbounded exterior face S_e.

In three dimensions, S would be a single shell surface without borders, although possibly with handles.
The *cut* $C(S)$ of S is the *medial axis* of S^c. It is the set of points (of S^c) that have at least two closest points on S. From [16], we know that $C(S)$ has exactly one connected component in S_i, that we denote $C_i(S)$ and may have one or more connected components in S_e that we denote $C_e(S)$. From the definition we have of course: $C(S) = C_i(S) \cup C_e(S)$.
The *local feature size* [17], or lfs, is defined for each point $x \in S$ as the distance to $C(S)$: $\mathrm{lfs}(x) = \inf_{y \in C(S)} d(x, y)$.
The *minimal feature size* or mfs is the infimum of the values of lfs on S:

$$\mathrm{mfs}(S) = \inf_{x \in S} \mathrm{lfs}(x) = \inf_{x \in S, y \in C(S)} d(x, y)$$

the concept of mfs has been introduced by Federer in the 50's and called *reach*.

2.2. *Mean feature size and smoothness*

The ball-map is not a homeomorphism when the curves or surfaces considered exhibit sharp features, i.e., points with unbounded curvature. To preclude these, we could simply require that each surface S for which the ball-map is defined be C^2, by which we mean that it has a *continuous* curvature. Unfortunately, the boundary of a solid in which the sharp edges have been rounded by smooth fillets or blends [21] will have bounded curvature, but its boundary may not be C^2. Since we wish to extend the results discussed here to such shapes, which are common in practices, we need a less constraining characterization of smoothness.
Note that, in the Computer Aided Geometric Design terminology, a distinction has been introduced between C^k and G^k ("G " for Geometric) continuity. However, this distinction does not apply here because we are using the classical terminology of differential geometry, where a C^k manifold is one that admits a local regular C^k parameterization which corresponds to the G^k continuity notion used by the CAGD community. (The term *regular* means here: "whose derivative has full rank everywhere".)
A natural approach to try and address this problem is to require that the surfaces be C^1. Unfortunately, this is not sufficient. Indeed, requiring only that the surfaces be C^1 not guarantee that the curvature is bounded. As an example, consider the curve defined by:

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \; : \; y^3 - x^4 = 0\} = \{(x, y) \in \mathbb{R}^2 \; : \; y = |x|^{\frac{4}{3}}\}$$

The curve $\mathcal{C}$ is C^1, but $\mathrm{mfs}(\mathcal{C}) = 0$, indeed, the map $x \mapsto |x|^{\frac{4}{3}}$ is differentiable and has derivative 0 for $x = 0$ but its second derivative tends toward infinity when $x \to 0$.
For compact manifolds, the condition $\mathrm{mfs}(S) > 0$ is equivalent to the $C^{1,1}$ property, which means that the surfaces are C^1 manifolds with a Lipschitz condition on the

Gauss map. The Lipschitz constant is then bounded by $\frac{1}{\mathrm{mfs}(S)}$ and the Lipschitz condition may therefore be expressed as:

$$\forall x, y \in S,\ \|\overrightarrow{n_S}(x) - \overrightarrow{n_S}(y)\| \leq \frac{1}{\mathrm{mfs}(S)}\|x - y\| \tag{1}$$

where $\overrightarrow{n_S}$ is the Gauss map of S.

2.3. *Comparing two $(n-1)$-manifolds embedded in* $\mathbb{R}^n$

We consider two compact, connected $(n-1)$-manifolds S and S' embedded in $\mathbb{R}^n$. The discrepancy between two such manifolds may be measured in a variety of ways. In this paper we consider both Hausdorff and Fréchet distances.

Definition 1 (Hausdorff distance). Let A and B be two compact subsets of $\mathbb{R}^n$. The Hausdorff distance between A and B is defined by

$$d_H(A, B) = \max\left(\sup_{x \in A} d(x, B),\ \sup_{y \in B} d(y, A)\right)$$

The Hausdorff distance $d_H(A, B)$ between two compact sets A and B may also be defined in terms of r-thickening. The r-thickening A^r of A is the union of all open balls of radius r and center on A. Note that A^r is the Minkowski sum of A with an open ball of radius r and center at the origin. The r-thickening operator was used as a tool for offsetting, rounding and filleting operations [22] and for shape simplification [14]. The Hausdorff distance, $d_H(A, B)$, between two sets A and B is the infimum of the radius r such that $A \subset B^r$ and $B \subset A^r$. The Hausdorff distance defines a distance on the space of compact subsets of $\mathbb{R}^n$: $d_H(A, B) = 0 \Rightarrow A = B$, $d_H(A, B) = d_H(B, A)$ and $d_H(A, C) \leq d_H(A, B) + d_H(B, C)$ (see [2]).

We recall that a *homeomorphism* is a continuous bijection, the inverse of which is also continuous. We say that two sets are homeomorphic if there exists a homeomorphism between them: in this case they are identical regarding intrinsic topological properties.

Hausdorff distance allows us to measure the discrepancy between any two compact sets, even when they are not homeomorphic. When one wants to measure the discrepancy between two homeomorphic compact sets A and B, the Hausdorff distance does not provide any information about how far one has to move points of A to B in order to realize a homeomorphism. In other words, two compact sets A and B may have a very small Hausdorff distance yet any homeomorphism between them will map at least some distant pairs (see figure 1). So instead of considering the Hausdorff distance, it may be more relevant for homeomorphic shapes to consider the Fréchet distance as a discrepancy measure.

Definition 2 (Fréchet distance). Let S and S' be two compact homeomorphic submanifolds of $\mathbb{R}^n$. Let $\mathcal{F} = \{f : S \to S' : f \textit{ is an homeomorphism}\}$ be the set of all homeomorphisms between S and S'. Given such a homeomorphism f, $\sup_{x \in S} d(x, f(x))$ is the maximum displacement of the points of S by f. The Fréchet

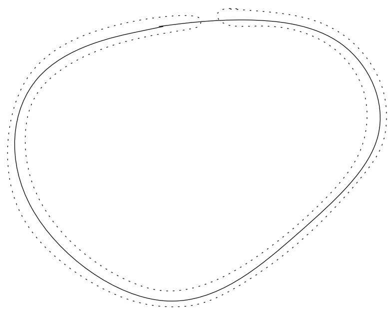

Fig. 1. The curves in solid and dotted line are close to each other in terms of the Hausdorff distance, while they are significantly different in terms of the Fréchet distance.

distance between S and S' is the infimum of this maximum displacement among all the homeomorphisms. It is defined by

$$d_F(S, S') = \inf_{f \in \mathcal{F}} \sup_{x \in S} d(x, f(x)).$$

It is a classical exercise to check that the Fréchet distance satisfies the properties defining a distance and that one always has

$$d_H(S, S') \leq d_F(S, S').$$

In general, the Fréchet distance is more difficult to compute than the Hausdorff distance since one has to find an infimum among all the homeomorphisms between S and S'. In section 5, we show that, under specific conditions (theorem 3), the Hausdorff and Fréchet distances are equal.
When a homeomorphism and its inverse are both C^k-smooth, the homeomorphism is a *C^k-diffeomorphism*. If a homeomorphism may be realized by a continuous deformation, it defines an *isotopy*:

Definition 3 (Isotopy and ambient isotopy). An *isotopy* between S and S' is a continuous map $F : S \times [0, 1] \to \mathbb{R}^n$ such that $F(., 0)$ is the identity of S, $F(S, 1) = S'$, and for each $t \in [0, 1]$, $F(., t)$ is a homeomorphism onto its image.
An *ambient isotopy* between S and S' is a continuous map $F : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ such that $F(., 0)$ is the identity of $\mathbb{R}^n$, $F(S, 1) = S'$, and for each $t \in [0, 1]$, $F(., t)$ is a homeomorphism of $\mathbb{R}^n$. If the map F is C^k-smooth, it is called a *C^k-smooth (ambient) isotopy*.

Restricting an ambient isotopy between S and S' to $S \times [0, 1]$ thus yields an isotopy between them. If there exists an isotopy between S and S', then there is an ambient isotopy between them [11], so that both notions are equivalent in our case.

2.4. *Moat and median*

Given two $(n-1)$-manifolds S and S', we define their moat and median (see figure 2). The *moat* of S and S', $\mathrm{Moat}(S, S')$ is the union of S, S' and the symmetric

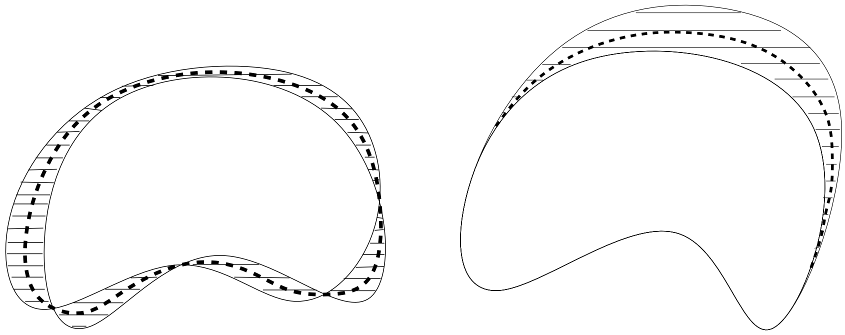

Fig. 2. On both examples the moat is depicted as a hatched area and the median curve as a dashed line. In the example on the right, the curves S and S' overlap partially.

difference of their interiors:

$$\text{Moat}(S, S') = S \cup S' \cup (S_i \cap S'_e) \cup (S'_i \cap S_e)$$

or, equivalently:

$$\text{Moat}(S, S') = \left((S_i \cap S'_i) \cup (S_e \cap S'_e)\right)^c$$

The *median* of S and S', $\text{Me}(S, S')$ is defined as the set of points in $\text{Moat}(S, S')$ which are equidistant from S and S':

$$\text{Me}(S, S') = \{x \in \text{Moat}(S, S') \quad | \quad d(x, S) = d(x, S')\}$$

Both $\text{Moat}(S, S')$ and $\text{Me}(S, S')$ are clearly compact sets. Notice that $S \cap S' \subset \text{Me}(S, S')$.
Alternatively, the median can be defined as the locus of centers of closed balls included in the moat that intersect (in fact touch) both S and S'.

3. Ball-compatibility and ball-map

We define now the main object of the paper: the ball-map.

Definition 4 (Ball-pair). Given two compact connected $(n-1)$-manifolds S and S' in $\mathbb{R}^n$, we say that $(x, x') \in S \times S'$ is a *ball-pair* if there is $c \in \text{Me}(S, S')$ such that $d(c, x) = d(c, S) = d(c, x') = d(c, S')$.

Obviously, one has the following alternate definition.

Definition 5 (Ball-pair, alternate definition). $(x, x') \in S \times S'$ is a ball-pair if and only if there is a closed ball $\mathbb{B} \subset \text{Moat}(S, S')$ such that $x \in \mathbb{B} \cap S$ and $x' \in \mathbb{B} \cap S'$.

It follows from this last definition that the ball-pairing is a conformal invariant, which means that any isometry or any inversion preserving the inner and outer connected components of $\mathbb{R}^n \setminus S$ and $\mathbb{R}^n \setminus S'$ will preserve the ball pairing.

Definition 6 (Ball-compatibility). If each point of S and each point of S' belongs to exactly one ball pair, the ball pairing defines a bijection, which is then called the *ball-map* and S and S' are said to be *ball-compatible*

Notices that the definition entails trivially that a surface S, even non-smooth, is ball-compatible with itself.

Definition 7 (Ball-map). If S and S' are ball-compatible, the bijection $\mathrm{BM}_{S,S'}$ and its inverse $\mathrm{BM}_{S',S}$ defined by the ball pairing are called *ball-maps*.

Note that, when S and S' are ball compatible, the projections $\pi_S : \mathrm{Me}(S, S') \to S$ and $\pi_{S'} : \mathrm{Me}(S, S') \to S'$ that associates to each point of $\mathrm{Me}(S, S')$ its unique closest point on S (resp. S') are also bijections and:

$$\mathrm{BM}_{S,S'} = \pi_{S'} \circ \pi_S^{-1} \tag{2}$$

$$\mathrm{BM}_{S',S} = \pi_S \circ \pi_{S'}^{-1} \tag{3}$$

When the manifolds are ball-compatible, the ball-map is a homeomorphism:

Lemma 1. *If S and S' are ball-compatible,* $\mathrm{BM}_{S,S'}$ *is a homeomorphism.*

Proof. It is clear that $\mathrm{BM}_{S,S'}$, π_S and $\pi_{S'}$ are bijections. Recall that a continuous bijection between compact sets is a homeomorphism [7]. From equation (2), it is then enough to prove that $\pi_S : \mathrm{Me}(S, S') \to S$ is continuous.
For that, we consider, in the condition of the Lemma, a sequence of points $(c_n)_{n\in\mathbb{N}}$ in $\mathrm{Me}(S, S')$ that converges to some $c \in \mathrm{Me}(S, S')$. Let us denote by $(a_n)_{n\in\mathbb{N}}$ the respective closest points on S: $a_n = \pi_S(c_n)$. Because S is compact, there is at least one point a such that a subsequence $(a_{n_i})_{i\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ converges toward a. S and S' being metric spaces, it is enough to prove that $a = \pi_S(c)$ to ensure the continuity of the map π_S. Because both the distance function and the distance to S are continuous, the sequence of distances $d(c_n, S)$ converges toward $d(c, S) = d(c, a)$, which entails that $a = \pi_S(c)$. □

Note that Lemma 1 does not require any smoothness condition on the surfaces S and S'. It follows immediately from Lemma 1 and from its proof that:

Corollary 1. *If S and S' are ball-compatible, the median* $\mathrm{Me}(S, S')$ *is a compact, connected $(n-1)$-manifold.*

4. Conditions for ball-compatibility

4.1. *Necessary and sufficient condition for C^1 manifolds*

If we restrict ourselves to C^1 manifolds, one has the following necessary and sufficient condition for ball-compatibility:

Theorem 1. *Let S and S' be compact, connected C^1 $(n-1)$-manifolds. S and S' are* ball-compatible *if and only if the following conditions hold:*

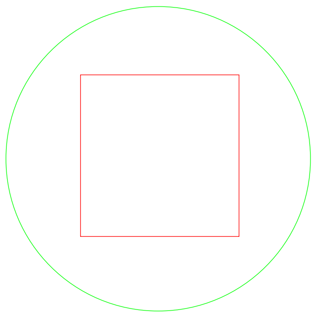

Fig. 3. The square and circle curves are not ball-compatible (the square corners belong to many ball-pairs) even though they match all conditions of theorem 1, except the C^1 condition: the square is not smooth.

(i) $S_i \cap S'_i \neq \emptyset$
(ii) $\mathrm{Me}(S, S') \cap C(S) = \mathrm{Me}(S, S') \cap C(S') = \emptyset$

Note that, as seen on figure 3, the theorem does not hold if we drop the C^1 condition.

Proof of Theorem 1.
The proof of the "only if" part of theorem 1 is rather trivial: indeed, if condition (ii) of the theorem does not hold, for example if $\mathrm{Me}(S, S') \cap C(S) \neq \emptyset$, a point $c \in \mathrm{Me}(S, S') \cap C(S)$ will have several closest point on S and the surfaces can not be ball-compatible. If the condition (i) does not hold, one has $\mathrm{Moat}(S, S') = S \cup S' \cup S_i \cup S'_i$ and therefore $\mathrm{Me}(S, S') = S \cap S'$. Since $S \neq S'$, there exists $x \in S \cup S' \setminus S \cap S'$ which cannot be in a ball-pair.
For the "if" part we prove first, using the C^1 smoothness and the condition (ii), that a point $x \in S$ cannot belong to more than one ball-pair:

Lemma 2. *If S and S' are compact, connected C^1 $(n-1)$-manifolds such that $\mathrm{Me}(S, S') \cap C(S') = \emptyset$ then any point $x \in S$ belongs to at most one ball-pair.*

Symmetrically, if $\mathrm{Me}(S, S') \cap C(S) = \emptyset$, then any point $x \in S'$ belongs to at most one ball pair.

Proof. Under the conditions of the Lemma, let us consider $x \in S$. If $x \in S \cap S'$, x belongs only to the zero radius ball corresponding to the unique ball-pair (x, x). Now if for example $x \in S'_i$ (see figure 4), let us assume that it belongs to a ball-pair corresponding to a ball centered in c. Because $x \notin S'$ the ball has positive radius and, because S is C^1, c must lies on the line through x orthogonal to the plane tangent to S at x. Moreover, because the ball is included in the moat, its radius $[c, x]$ is in the moat and then in $\overline{S_e}$. But there may be only one ball through x centered on the same half line orthogonal to the plane tangent to S at x and maximal in the moat. Therefore, the ball is unique and the condition of the Lemma

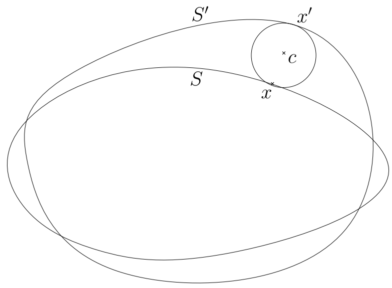

Fig. 4. A point $x \in S \cap S'_i$ and its "ball-companion" $x' \in S'$.

implies that its center $c \in \mathrm{Me}(S, S')$ is not in $C(S')$ so that there is only one $x' \in S'$ closest to c and x belongs to no other ball-pair. The case $x \in S'_e$ is similar. $\square$

We still have to prove that a point $x \in S$ belongs to at least one ball pair. For that, the main argument makes use of the fact that S_i and therefore (see [16]) $C_i(S)$ are connected and, for the outer part, that S_e is connected and unbounded, and therefore the connected components of $C_e(S)$, if any, are not bounded. This, together with condition (i) and (ii) of the theorem, shows that the median separates $C_i(S)$ and $C_e(S)$ (corollary 2). For that we consider the function $\psi : \mathbb{R}^n \to \mathbb{R}$:

$$\psi(x) = d\left(x, \overline{S_i} \cap \overline{S'_i}\right) - d\left(x, \overline{S_e} \cap \overline{S'_e}\right)$$

We let the reader check that:

$$\psi(x) = 0 \quad \Longleftrightarrow \quad x \in \mathrm{Me}(S, S') \tag{4}$$

Hence, we may now introduce the open sets Me_i and Me_e:

$$\mathrm{Me}_i(S, S') = \{x \mid \psi(x) < 0\}$$
$$\mathrm{Me}_e(S, S') = \{x \mid \psi(x) > 0\}$$

We have the following Lemma.

Lemma 3. *If condition (i) of theorem 1 holds, then:*

$$\mathrm{Me}_i(S, S') \cap C_i(S) \neq \emptyset$$
$$\mathrm{Me}_i(S, S') \cap C_i(S') \neq \emptyset$$

In other words, a non-empty subset of the cut of S (and the cut of S') lies inside $\mathrm{Me}_i(S, S')$.

Proof. We prove here that $\mathrm{Me}_i(S, S') \cap C_i(S) \neq \emptyset$.
Let us take $x \in S_i \cap S'_i$. Because $S_i \cap S'_i \subset \mathrm{Me}_i(S, S')$, if $x \in C_i(S)$ one has $x \in \mathrm{Me}_i(S, S') \cap C_i(S)$ and $\mathrm{Me}_i(S, S') \cap C_i(S) \neq \emptyset$. Let us suppose now that $x \notin C_i(S)$. There is a unique $y \in S$ which is closest to x: $d(x, y) = d(x, S)$ and $y \neq x$. The half line $[yx$ cuts the closure of the medial axis $\overline{C_i(S)}$ at a point $x_0 \in \overline{C_i(S)}$.

To see this, grows a ball centered at x and touching S at y by sliding its center on $[yx$ as long as it does not contain any other point of S and the boundary point of the set of such centers is x_0. Recall that $x \in S_i \cap S'_i$. One has

$$d(x_0,\, \overline{S_e} \cap \overline{S'_e}) \geq d(x_0, \overline{S_e}) = d(x_0,\, y) > d(x_0,\, x) \geq d(x_0,\, \overline{S_i} \cap \overline{S'_i})$$

$x_0 \in \mathrm{Me}_i(S, S')$ and $x_0 \in \overline{C_i(S)}$. But $\mathrm{Me}_i(S, S')$ is open and this entails

$$\mathrm{Me}_i(S, S') \cap C_i(S) \neq \emptyset$$ □

Lemma 3, implies the following.

Corollary 2. *If conditions (i) and (ii) of theorem 1 hold, then:*

$$C_i(S) \subset \mathrm{Me}_i(S, S')$$
$$C_e(S) \subset \mathrm{Me}_e(S, S')$$
$$C_i(S') \subset \mathrm{Me}_i(S, S')$$
$$C_e(S') \subset \mathrm{Me}_e(S, S')$$

Proof. Because S_i is connected, $C_i(S)$ is connected (see [16]). $\mathrm{Me}(S, S') \cap C_i(S) = \emptyset$ (condition (ii) of the theorem) and equation (4) entails that ψ does not vanish on $C_i(S)$. Lemma 3 says us that ψ takes negative values on $C_i(S)$. Since ψ is continuous and $C_i(S)$ is connected, ψ is negative on $C_i(S)$. Similarly, because S_e is connected, the connected components of $C_e(S)$, if there are any, are unbounded and necessarily lies in $\mathrm{Me}_e(S, S')$. Again $\mathrm{Me}(S, S') \cap C_e(S) = \emptyset$ allows us to conclude that $C_e(S) \subset \mathrm{Me}_e(S, S')$. The two other properties follow from similar proofs. □

So we prove now that $x \in S$ is the closest point of some point $c \in \mathrm{Me}(S, S')$. There are three possibilities for $x \in S$:

(1) $x \in \mathrm{Me}(S, S')$
(2) $x \in \mathrm{Me}_i(S, S')$
(3) $x \in \mathrm{Me}_e(S, S')$

In case (1), $x \in \mathrm{Me}(S, S')$ and, trivially, x is the closest point to $c = x$.
In case (2), $x \in S$ and $x \in \mathrm{Me}_i(S, S')$ entails $x \in S'_i$. The half-line starting at x and going outward S in the direction orthogonal to the plane tangent to S at x will cut the closed set $\mathrm{Me}(S, S')$ at a first point $c \in \mathrm{Me}(S, S')$. From Corollary 2, we have $C_e(S) \subset \mathrm{Me}_e(S, S')$ and therefore $\overline{\mathrm{Me}_i(S, S')} \cap C_e(S) = \emptyset$ and, the segment $[x, c]$ being in $\overline{\mathrm{Me}_i(S, S')}$ cannot meet the outer medial axis $C_e(S)$. $x \in \mathrm{Me}_i(S, S')$ entails that $d(x, C_e(S)) > 0$. Therefore, one can grow a small sphere (see figure 5) contained in $\overline{S_e}$, tangent to S at x until its center coincides with c. Consequently, x is the unique point of S closest to c. Case (3) is similar to case (2), using this time the relation $C_i(S) \subset \mathrm{Me}_i(S, S')$ of Corollary 2.

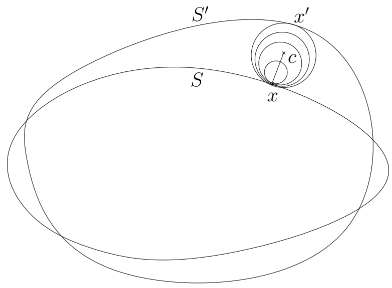

Fig. 5. One can grow a small sphere contained in $\overline{S_e}$, tangent to S at x until its center coincides with c.

4.2. *A sufficient condition relying on Hausdorff distance*

Given two surfaces S and S' sufficiently *close to each other* with respect to their minimum feature size, the *ball-map* is an homeomorphism between S and S'. More precisely:

Theorem 2. *Let S and S' be connected, compact $(n-1)$-manifolds in $\mathbb{R}^n$. If there exists $\epsilon > 0$ such that $\mathrm{mfs}(S) > \epsilon$, $\mathrm{mfs}(S') > \epsilon$, and $d_H(S, S') < \epsilon$, then the ball-pairing $\mathrm{BM}_{S,S'}$ defines a homeomorphism between S and S'.*

We use the term *feature-respecting compatibility* to describe the condition of theorem 2. We show below that under the condition of this theorem, S and S' meet the conditions of theorem 1.

Proof. We have seen in section 2.2 that $\mathrm{mfs}(S) > \epsilon$ (and $\mathrm{mfs}(S') > \epsilon$) entails that S (and S') are C^1 smooth.
One denotes by $S_e \downarrow^\epsilon$ the erosion of S_e: this is the set of points in S_e whose distance to S is greater than ϵ. Similarly, one denotes by $S_i \downarrow^\epsilon$ the set of points in S_i whose distance to S is greater than ϵ.

$$S_e \downarrow^\epsilon = \{x \in S_e : d(x, S) > \epsilon\}$$
$$S_i \downarrow^\epsilon = \{x \in S_i : d(x, S) > \epsilon\}$$

$S'_e \downarrow^\epsilon$ and $S'_i \downarrow^\epsilon$ are defined accordingly.
We have the following lemma:

Lemma 4. *Let S and S' be connected, compact $(n-1)$-manifolds in $\mathbb{R}^n$. If there exists $\epsilon > 0$ such that $\mathrm{mfs}(S) > \epsilon$, $\mathrm{mfs}(S') > \epsilon$, and $d_H(S, S') < \epsilon$ then*

$$S_i \downarrow^\epsilon \subset S'_i \text{ and } S_e \downarrow^\epsilon \subset S'_e$$
$$S'_i \downarrow^\epsilon \subset S_i \text{ and } S'_e \downarrow^\epsilon \subset S_e$$

Notice that the condition $\text{mfs}(S) > \epsilon$, $\text{mfs}(S') > \epsilon$ is crucial. Indeed in the example of figure 1 one has $d_H(S,S') < \epsilon$ for some small ϵ while the inclusions of the lemma does not hold.

Proof. proof of Lemma 4
The condition $d_H(S, S') < \epsilon$, together with the definition of erosion entails that the unions of the erosions $S_i \downarrow^\epsilon \cup S_e \downarrow^\epsilon$ has no intersection with the other surface S' and therefore:

$$S_i \downarrow^\epsilon \cup S_e \downarrow^\epsilon \subset S'_i \cup S'_e$$

Because both $S_i \downarrow^\epsilon$ and $S_e \downarrow^\epsilon$ are connected (see [4]), they are each either a subset of the connected component S'_i or a subset of the connected component S'_e. Because $S_e \downarrow^\epsilon$ is not bounded, it has to be a subset of S'_e. That is:

$$S_e \downarrow^\epsilon \subset S'_e$$

Assume now that $S_i \downarrow^\epsilon \subset S'_e$. Consequently

$$S_i \downarrow^\epsilon \cup S_e \downarrow^\epsilon \subset S'_e \tag{5}$$

We denote by $S^{+\epsilon}$ the set of points at distance less than ϵ from S:

$$S^{+\epsilon} = \{x \in \mathbb{R}^n : d(x, S) < \epsilon\}$$

By taking the complement and then the interior of both terms of inclusion (5), we obtain

$$S'_i \subset S^{+\epsilon} \tag{6}$$

We know (using a homotopy argument) that the medial axis $C_i(S')$ of S'_i is not empty [16]. Let us take $x' \in C_i(S')$. Because $\text{mfs}(S') > \epsilon$, let us take $\alpha > 0$ such that $\epsilon + \alpha < \text{mfs}(S')$.
The ball $\mathbf{B}_{x',\epsilon+\alpha}$ centered at x' with radius $\epsilon + \alpha$ is contained in S'_i:

$$\mathbf{B}_{x',\epsilon+\alpha} \subset S'_i$$

And, from the inclusion (6):

$$\mathbf{B}_{x',\epsilon+\alpha} \subset S^{+\epsilon} \tag{7}$$

Because $\text{mfs}(S) > \epsilon$, one has $x' \notin C(S)$. Let x be the unique point on S closest to x'. The line segment γ of length 2ϵ, centered at x supported by the line normal to S at x' is included in $S^{+\epsilon}$. Moreover, from $\text{mfs}(S) > \epsilon$, the two boundary points of this line segment belong to the boundary of $S^{+\epsilon}$. Note that $x' \in \gamma$. This is in contradiction with the inclusion (7). Consequently, the assumption $S_i \downarrow^\epsilon \subset S'_e$ cannot happen. Hence, because $S_i \downarrow^\epsilon$ is connected, we have $S_i \downarrow^\epsilon \subset S'_i$. □

An immediate consequence of lemma 4 is the following

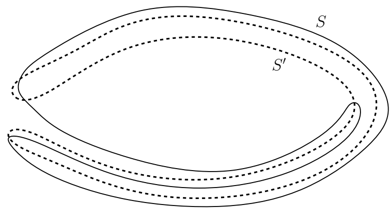

Fig. 6. The two curves S and S' are ball compatible. Still $d_F(S,S') \neq d_H(S,S')$, because the second (from the top) layer of S is close to the bottom part (third and forth layers) so the Hausdroff distance is small, while the Frechet distance is the height of the gap between the second layers of S and S'.

Corollary 3. *Let S and S' be connected, compact $(n-1)$-manifolds in $\mathbb{R}^n$ with $\epsilon > 0$ such that $\mathrm{mfs}(S) > \epsilon$, $\mathrm{mfs}(S') > \epsilon$, and $d_H(S,S') < \epsilon$. Then:*

$$C_i(S) \subset S'_i \text{ and } C_e(S) \subset S'_e$$
$$C_i(S') \subset S_i \text{ and } C_e(S') \subset S_e$$

In other words, S separates $C_i(S')$ from $C_e(S')$ and, symetrically, S' separates $C_i(S)$ from $C_e(S)$.

Corollary 3 implies that, if $\mathrm{mfs}(S) > \epsilon$, $\mathrm{mfs}(S') > \epsilon$ and $d_H(S,S') < \epsilon$, then one has $C_i(S) \subset S_i \cap S'_i$ and $C_e(S) \subset S_e \cap S'_e$. Therefore $\mathrm{Me}(S,S') \cap C(S) = \emptyset$. One shows similarly that $\mathrm{Me}(S,S') \cap C(S') = \emptyset$ and hence theorem 1 applies.

5. Bounding Fréchet distance

From the definition of the Fréchet distance (definition 2), taking an infimum on the set of homeomorphisms, we conclude that its computation for arbitrary (homeomorphic) pair of surfaces is expensive. In contrast, the definition of the Hausdorff distance whose nature is more geometric, makes it affordable.
Under the condition of ball-compatibility d_H can be arbitrary small with d_F arbitrarily large as suggested by figure 6.
However, we can show that, in the conditions of theorem 2, the Hausdorff and Fréchet distances are equal.

Theorem 3. *Let S and S' be connected, compact $(n-1)$-manifolds in $\mathbb{R}^n$ with $\epsilon > 0$ such that $\mathrm{mfs}(S) > \epsilon$, $\mathrm{mfs}(S') > \epsilon$, and $d_H(S,S') < \epsilon$. Then:*

$$d_F(S,S') = d_H(S,S')$$

Proof. The inequality $d_H(S,S') \leq d_F(S,S')$ holds in general and follows from the definitions of d_H and d_F. Because S is compact, there is a ball-pair $(x,x') \in S \times S'$, such that:

$$d(x,x') = \sup_{y \in S} d(y, \mathrm{BM}_{S,S'}(y)) \geq d_F(S,S')$$

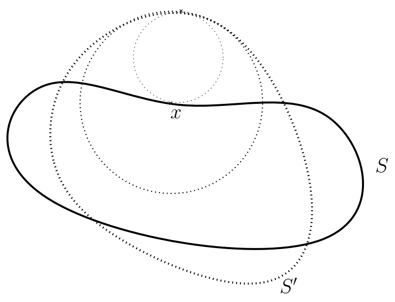

Fig. 7. The ball centered at x and tangent to S' at x' contains no point of S' other than x'.

As seen in section 2.2 under the hypothesis of the theorem, both S and S' are C^1. Therefore, in order for the distance $d(x, x')$ to be maximal among ball-pairs, the segment $[x, x']$ must be orthogonal to S at x and to S' at x'.

On the other hand, it follows from Lemma 4 that $\text{Moat}(S, S') \subset S'^{\epsilon}$. But the segment $[x, x']$ being in $\text{Moat}(S, S') \subset S'^{\epsilon}$ and orthogonal to S' at x', it follows that $d(x, x') < \epsilon < \text{mfs}(S')$. Therefore, the ball of radius $d(x, x')$ tangent to S' at x' centered at x contains no point of S' other than x' (see figure 7). But this implies that x' is the closest point to x on S': $d(x, S') = d(x, x')$, and then:

$$d_F(S, S') \leq d(x, x') = d(x, S') \leq d_H(S, S') \qquad \square$$

6. Smoothness

6.1. *The smoothness of the ball-map*

In the condition of theorem 2, the smoothness of $\text{BM}_{S,S'}$ is related to the smoothness of S and S':

Theorem 4 (2 improved with smoothness of $\text{BM}_{S,S'}$). *In the conditions of theorem 2, $\text{BM}_{S,S'}$ is in fact Lipschitz and, if S and S' are C^k manifolds, with $k \geq 2$, then $\text{BM}_{S,S'}$ is a C^{k-1} diffeomorphism.*

Recall that we denote by $S^{+\epsilon}$ the set of points at distance less than ϵ from S:

$$S^{+\epsilon} = \{x \in \mathbb{R}^n : d(x, S) < \epsilon\}$$

We start with a corollary of lemma 4. If one compares the definition of the moat with lemma 4, one gets:

Corollary 4. *Let S and S' be connected, compact $(n-1)$-manifolds in $\mathbb{R}^n$ with $\epsilon > 0$ such that $\text{mfs}(S) > \epsilon$, $\text{mfs}(S') > \epsilon$, and $d_H(S, S') < \epsilon$. Then:*

$$\text{Moat}(S, S') \subset S^{+\epsilon}$$
$$\text{Moat}(S, S') \subset (S')^{+\epsilon}$$

Corollary 4 helps in the proof below.

Proof of theorem 4.
When the surfaces are not C^2 and mfs$(S) > \epsilon$, mfs$(S') > \epsilon$, equation (1) in section 2.2 entails that $\overrightarrow{n_S}$ and $\overrightarrow{n_S}$ are $\frac{1}{\epsilon}$-Lipchitz. If moreover S is a C^k manifold with $k \geq 2$ $x \mapsto \overrightarrow{n_S}(x)$ is a C^{k-1} map.
Let us consider the map $\varphi : S \times \mathbb{R} \to \mathbb{R}^n$ defined by:

$$\varphi(x,t) = x + t \cdot \overrightarrow{n_S}(x)$$

$\varphi' : S' \times \mathbb{R} \to \mathbb{R}^n$ is defined similarly. Notice that if $x' = \mathrm{BM}_{S,S'}(x)$, then, there exists $t \in (-\epsilon, \epsilon)$ such that:

$$\varphi(x,t) = \varphi'(x', -t)$$

Indeed, if $x \in S \cap S'$, the equality holds for $t = 0$.
If $x \in S \setminus S \cap S'$, then either the ball map is defined by a ball in $\overline{S_i \cap S'_e}$, in which case its center y satisfies $y = \varphi(x,t)$ for some $t < 0$, or the ball-map is defined through a ball in $\overline{S_e \cap S'_i}$, in which case its center y satisfies $y = \varphi(x,t)$ for some $t > 0$.
Define the map $F : S \times S' \times \mathbb{R} \to \mathbb{R}^n$ as

$$F(x, x', t) = \varphi(x,t) - \varphi'(x', -t)$$

Locally, the implicit equation $F(x, x', t) = 0$ defines the ball-map:

$$x' = \mathrm{BM}_{S,S'}(x) \Rightarrow \exists t :\ F(x, x', t) = 0 \text{ and } \varphi(x,t) \in \mathrm{Moat}(S, S')$$

We check below that the implicit functions theorem applies and it follows that the relation $F(x, x', t) = 0$ defines locally a C^{k-1} smooth one-to-one mapping: $x \mapsto (x'(x), t(x))$ such that $F(x, x'(x), t(x)) = 0$. In order to express the derivative of the map φ' at $x' \in S'$ by a matrix, one chooses a convenient local coordinate system for S' and R^n. It is a classic result that, when mfs$(S) > \epsilon$, $\varphi : S \times (-\epsilon, \epsilon) \to S^{+\epsilon}$ is one-to-one, but we give the proof here for completeness.
Let us take an orthonormal frame centered at x', with last vector being the unit normal to S' at $x' \in S'$ pointing outward. This defines a frame in R^n. Notice that, with the origin x', the first $n-1$ vectors of the frame define an orthogonal frame for the plane tangent to S' at x'.
The orthogonal projection of S' on its tangent plane at x' defines a local coordinate system for S', using the same $(n-1)$ orthonormal frame of the tangent plane defined above. Using these maps for S' and $\mathbb{R}^n$, it is possible to write the derivative of φ' with respect to x' and t in the form of a matrix. In the expression below, the first column and row correspond to the $n-1$ tangential directions and the second column and row to the normal direction to S' at x'. One denotes by $\mathbf{1}$ the $(n-1) \times (n-1)$ unit matrix. One has

$$\begin{pmatrix} \frac{\partial \varphi'}{\partial x'} & \frac{\partial \varphi'}{\partial t} \end{pmatrix} = \begin{pmatrix} \mathbf{1} + t \cdot \frac{\partial \overrightarrow{n_{S'}}(x')}{\partial x'} & 0 \\ 0 & 1 \end{pmatrix}$$

Notice that, with the chosen maps, $\frac{\partial \overrightarrow{n_{S'}}(x')}{\partial x'}$ is precisely the derivative of the Gauss map of S' at x'. If $F(x, x', t) = 0$, one has either $t = 0$ or $\varphi'(x', t) \in \mathrm{Moat}(S, S')$. Thus, from corollary 4, one has $t < \epsilon$. Using equation (1), we obtain:

$$\|t \cdot \frac{\partial \overrightarrow{n_{S'}}(x')}{\partial x'}\| < 1 \tag{8}$$

It follows that the $(n-1) \times (n-1)$ determinant $|\mathbf{1} + t \cdot \frac{\partial \overrightarrow{n_{S'}}(x')}{\partial x'}|$ does not vanish:

$$\left|\mathbf{1} + t \cdot \tfrac{\partial \overrightarrow{n_{S'}}(x')}{\partial x'}\right| \neq 0$$

Consequently:

$$\left(\tfrac{\partial F}{\partial x'} \; \tfrac{\partial F}{\partial t} \right)$$

has a non-zero determinant and therefore full rank, which allows to apply the implicit function theorem: there are C^{k-1} maps $x \mapsto x'(x)$ and $x \mapsto t(x)$ defined in a neighborhood of x such that $F(x, x'(x), t(x)) = 0$. Therefore, $\mathrm{BM}_{S,S'}$ and $\mathrm{BM}_{S,S'}$ are C^{k-1}, which proves the theorem when the surfaces are at least C^2.
When the surfaces are not C^2, but only with strictly positive mfs, one cannot apply the usual implicit function theorem. The local inversion can be built explicitly: using a weak version of equation (8), it is possible to inverse the map φ' explicitly, as the fix point of an iterative inversion algorithm. Alternatively, one can use a Lipschitz variant of the implicit function theorem (see Clarke [5]). □

6.2. *Smoothness of the median surface*

As stated in theorem 4, the map $\mathrm{BM}_{S,S'}$ loses one order of continuity with respect to S and S': if S and S' are C^k, then the ball-map is C^{k-1} only. This behavior is not a surprise if we recall the central role of the Gauss maps $\overrightarrow{n}$ and $\overrightarrow{n'}$, which are of course C^{k-1}. Consequently, $\mathrm{Me}(S, S')$ is then a C^{k-1} manifold. We prove that $\mathrm{Me}(S, S')$ is in fact not only C^{k-1} but C^k.

Theorem 5. *Let S and S' be connected, compact $(n-1)$-manifolds in $\mathbb{R}^n$ with $\epsilon > 0$ such that $\mathrm{mfs}(S) > \epsilon$, $\mathrm{mfs}(S') > \epsilon$, and $d_H(S, S') < \epsilon$. Then, the Gauss map $\overrightarrow{n_{\mathrm{Me}(S,S')}}$ of the median surface $\mathrm{Me}(S, S')$ is ϵ-Lipschitz. Moreover, if S and S' are C^k manifolds, then $\mathrm{Me}(S, S')$ is a C^k manifold.*

The proof of the theorem is based upon the characterization of $\mathrm{Me}(S, S')$ with distance functions to S and S'. For that, one uses the following Lemma.

Lemma 5. *Let S and S' be connected, compact $(n-1)$-manifolds in $\mathbb{R}^n$ with $\epsilon > 0$ such that $\mathrm{mfs}(S) > \epsilon$, $\mathrm{mfs}(S') > \epsilon$, and $d_H(S, S') < \epsilon$. Let $x \in S$, $x' \in S'$ and $c \in \mathrm{Me}(S, S')$ be such that x (resp. x') is the point of S (resp. S') closest to c. Let $\Pi_{x,S}$, $\Pi_{x',S'}$ and $\Pi_{c,\mathrm{Me}(S,S')}$ be the respective tangent planes at x, x' and c on the respective surfaces S, S' and $\mathrm{Me}(S, S')$.*
Then:

(1) if $x \neq x'$, $\Pi_{c,\mathrm{Me}(S,S')}$ is the bisector of the points x and x'
(2) $\Pi_{c,\mathrm{Me}(S,S')}$ is the bisector of the planes $\Pi_{x,S}$ and $\Pi_{x',S'}$

Proof. The fact that, if $x \neq x'$, the bisector of the points x and x' happen to be the bisector of the planes $\Pi_{x,S}$ and $\Pi_{x',S'}$ results from the observation that the triangle xcx' is isosceles and that the planes $\Pi_{x,S}$ and $\Pi_{x',S'}$ are orthogonal respectively to cx and cx'. It is then enough to prove (2).
The distance function to S, $\mathbf{d}_S : \mathbb{R}^n \to \mathbb{R}^+$ is 1-Lipschitz and is differentiable at any point that does not belong to the closure of the cut $\overline{C}$. In particular, it is differentiable at $c \in \mathrm{Moat}(S, S')$. Without loss of generality, assume that $c \in S_i \cap S'_e$. In this case, if one denotes by $\nabla \mathbf{d}_S$ the gradient of $\mathbf{d}_S$, one has:

$$\nabla \mathbf{d}_S(c) = -\overrightarrow{n}(x)$$

and, similarly:

$$\nabla \mathbf{d}_{S'}(c) = \overrightarrow{n'}(x')$$

Therefore, the map $c \mapsto \mathbf{d}_{S'}(c) - \mathbf{d}_S(c)$ is differentiable and has gradient:

$$\nabla \left(\mathbf{d}_{S'} - \mathbf{d}_S\right)(c) = \overrightarrow{n}(x) + \overrightarrow{n'}(x')$$

We claim that this gradient cannot be 0:

$$\overrightarrow{n}(x) + \overrightarrow{n'}(x') \neq 0, \tag{9}$$

Indeed, if $x = x'$, the relation (9) holds (if not, there exists a $\lambda > 0$ such that $x + \lambda \overrightarrow{n}(x) \in \mathrm{Moat}(S, S')$ which leads to a contradiction). Otherwise, if $x \neq x'$, one has $\|\overrightarrow{cx}\| = \|\overrightarrow{cx'}\| = t > 0$, $\overrightarrow{cx} = t\overrightarrow{n}(x)$, and $\overrightarrow{cx'} = -t\overrightarrow{n'}(x')$. Therefore:

$$\overrightarrow{n}(x) + \overrightarrow{n'}(x') = \frac{1}{t}\,\overrightarrow{x'x}$$

which proves (9).
Then, it follows from the implicit function theorem, that the surface $\mathrm{Me}(S, S')$ has a tangent plane at c that is normal to the vector $\overrightarrow{n}(x) + \overrightarrow{n'}(x')$. □

The previous Lemma allows us to prove the smoothness of C:

Proof. Proof of Theorem 5
In the proof of Lemma 5 above, we show that the implicit function theorem applies to the map $c \mapsto \mathbf{d}_{S'}(c) - \mathbf{d}_S(c)$. If S and S' are C^k manifolds, this map is C^k and, by the implicit function theorem, $\mathrm{Me}(S, S')$ is a C^k manifold. □

7. Ball-map and isotopy

In this section we introduce an isotopy from S to S' that "morph" any point $x \in S$ to $\mathrm{BM}_{S,S'}(x) \in S'$. Let us define the map $\pi_S : \mathrm{Me}(S,S') \to S$ as the projection on S which associates to $y \in \mathrm{Me}(S,S')$ its closest point on S : $\forall z \in S, d(y,z) \geq d(y, \pi_S(y))$. The map $\pi_{S'}$ is defined similarly. It is possible to "parametrize" $\mathrm{Moat}(S,S')$ by $\mathrm{Me}(S,S') \times [-1,1]$:

$$\mathcal{I}(y,t) = \begin{cases} (1+t)y - t\pi_S(y) & \text{if } t \in [-1,0], \\ (1-t)y + t\pi_{S'}(y) & \text{if } t \in [0,1] \ . \end{cases} \tag{10}$$

The map $\mathcal{I}$ is then an isotopy that "morphs" S onto S', called *broken line morph.* Notice that under the condition $\mathrm{mfs}(S) > \epsilon$, $\mathrm{mfs}(S') > \epsilon$, and $d_H(S,S') < \epsilon$, one can in fact relax the requirement for S and S' to be connected, because $S^{+\epsilon}$ has exactly one connected component for each connected component of S and each connected component of $S^{+\epsilon}$ contains exactly one connected component of S' and theorem 2 can be applied independently to each pair of respective connected components of S and S'. We have therefore:

Theorem 6. *Let S and S' be two compact $(n-1)$-manifolds in $\mathbb{R}^n$ with $\epsilon > 0$ such that $\mathrm{mfs}(S) > \epsilon$, $\mathrm{mfs}(S') > \epsilon$, and $d_H(S,S') < \epsilon$. The broken line morph associated to the ball-pairing $\mathrm{BM}_{S,S'}$ is an isotopy from S to S'.*

The existence of an isotopy from S to S' under the condition of this theorem when $n = 3$ has first been proven in [3]. Theorem 6 extends the result in arbitrary dimensions and provides an explicit isotopy. This theorem has a consequence on the determination of the isotopy type of a surface from a Hausdorff approximation, for example a set of sampled points. Any given compact set, possibly finite, whose Hausdorff distance to a compact set X is less than $\frac{\epsilon}{2}$ is called an $(\frac{\epsilon}{2})$-Hausdorff approximation of X. Note that if a $(\frac{\epsilon}{2})$-Hausdorff approximation of S with $\mathrm{mfs}(S) > \epsilon$ is given, then the topology and even the isotopy class of the surface is completely determined. Indeed if $\tilde{S}$ is any surface satisfying $\mathrm{mfs}(\tilde{S}) > \epsilon$ and $d_H(X,\tilde{S}) < \frac{\epsilon}{2}$, one has $d_H(S,\tilde{S}) \leq d_H(X,S) + d_H(X,\tilde{S})$ and, according to theorem 6, $\tilde{S}$ is isotopic to S.

8. Anticipated applications

We anticipate several possible applications of the ball-map. For example, the ball-map may be used to transfer parameterizations and texture maps [23] between two ball-compatible surfaces (Fig 8). Note that it is distortion-free when mapping a planar portion of S to a planar portion of S'.

It may also be used to compare two curves or two surfaces. Arguably, measuring the maximum, average, or mean square of the distances between all points x of S and their ball-image $\mathrm{BM}_{S,S'}(x)$ on S' may be more useful than measuring the Hausdorff distance between S and S'.

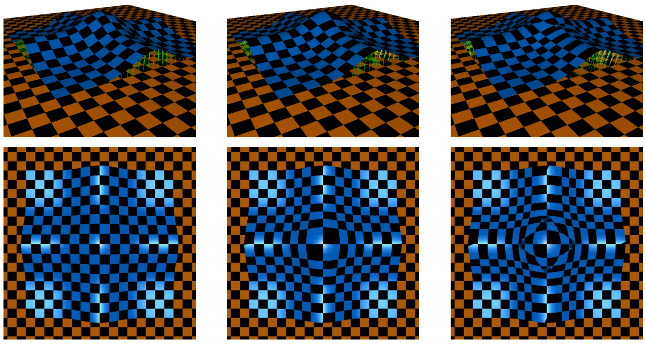

Fig. 8. The checkerboard texture of the horizontal plane S is mapped onto the curved surface S' in three ways. Left: a point x' of S' inherits the color of the closest point x of S. Center, a point x' of S' inherits the color of the point $\mathrm{BM}_{S',S}(x')$ on S. Right, a point x of S transfers its texture to the closest point x' on S'. The bottom row shows a top view.

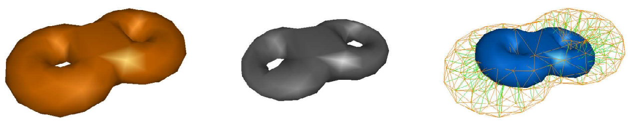

Fig. 9. The double-torus S (left) is morphed into the smaller double-torus S' (right), positioned partly inside S_i. Broken-lines, x-to-$c(x)$-to-x', defined by the ball-map are shown in green (right) along with a wireframe of S. The mid-time frame of the morph is shown (center).

The isotopies induced by the ball-map may be used to define a morph between S and S'.

9. Conclusion

We introduce the concept of ball-compatible manifolds and show that, when the Hausdorff distance between S and S' is strictly smaller than $\mathrm{mfs}(S)$ and $\mathrm{mfs}(S')$, S and S' are ball-compatible. We introduce the ball-map between two ball-compatible manifolds and show that it is a homeomorphism (theorem 1). Note that this is a weaker constraint than the one proposed in [8] ensuring that the orthomap is a bijection. For example, if the Hausdorff distance h is 0.8mfs, the ball-map is bijective, but the orthomap may not be. The ball-map may be used to transfer parameterizations, textures, and other properties and annotations between two curves or between two surfaces.

We also prove several smoothness results. In particular, when S and S' are C^k manifolds, then their ball-map is a C^{k-1} diffeomorphism and their median $\mathrm{Me}(S, S')$

is a C^k manifold. We also show that a morph defined by the ball-map is a C^{k-1} isotopy.

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