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Gromov-Hausdorff Approximation of Filament Structure Using Reeb-type Graph

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Abstract In many real-world applications data appear to be sampled around 1-dimensional filamentary structures that can be seen as topological metric graphs. In this paper we address the metric reconstruction problem of such filamentary structures from data sampled around them. We prove that they can be approximated, with respect to the Gromov-Hausdorff distance by well-chosen Reeb graphs (and some of their variants) and provide an efficient and easy to implement algorithm to compute such approximations in almost linear time. We illustrate the performances of our algorithm on a few data sets.

1 Introduction

Motivation. With the advance of sensor technology, computing power and the Internet, massive amounts of geometric data are being generated and collected in various areas of science, engineering and business. As they are becoming widely available, there is a real need to analyze and visualize these large scale geometric data to extract useful information out of them. In many cases this data is not embedded in Euclidean spaces and come as (finite) sets of points with pairwise distance information, i.e. (discrete) metric spaces. A large amount of research has been done on dimensionality reduction, manifold learning and geometric inference for data embedded in (possibly high dimensional) Euclidean spaces and assumed to be concentrated around low dimensional manifolds [11,13,5]. However, the assumption of data lying on a manifold may fail in many applications. In addition, the strategy of representing data by points in Euclidean space may introduce large metric distortions as the data may lie in highly curved spaces, instead of in flat Euclidean space raising many difficulties in the analysis of metric data. In the past decade, with the development of topological methods in data analysis, new theories such as topological persistence (see, for example, [9,14,22,31]) and new tools such as the Mapper algorithm [18] have given rise to new algorithms to extract and visualize geometric and topological information from metric data without the need of an embedding into an Euclidean space. In this paper we focus on a simple but important setting where the underlying geometric structure approximating the data can be seen as a branching filamentary structure i.e., more precisely, as a *metric graph* which is a topological graph endowed with a length assigned to each edge. Such structures appear naturally in various real-world data such as collections of GPS traces collected by vehicles on a road network, earthquakes distributions that concentrate around geological faults, distributions of galaxies in the universe, networks of blood vessels in anatomy or hydrographic networks in geography just to name a few. It is thus appealing to try to capture such filamentary structures and to approximate the data by metric graphs that will summarize the metric and allow convenient visualization.

Contribution. In this paper we address the metric reconstruction problem for filamentary structures. The input of our method and algorithm is a metric space (X, d_X) that is assumed to be close with respect to the so-called

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Gromov-Hausdorff distance d_{GH} to a much simpler, but unknown, metric graph $(G', d_{G'})$. Our algorithm outputs a metric graph (G, d_G) that is proven to be close to $(G', d_{G'})$ in both geometry and topology. Our approach relies on the notion of Reeb graph (and some variants of it introduced in Section 3) and our main theoretical results are stated in the following two theorems.

Theorem 3 [Recovery of Geometry]. *Let (X, d_X) be a compact connected geodesic space, let $r \in X$ be a fixed base point such that the metric Reeb graph (G, d_G) of the function $d = d_X(r, .) : X \to \mathbb{R}$ is a finite graph. If for a given $\epsilon > 0$ there exists a finite metric graph $(G', d_{G'})$ such that $d_{GH}(X, G') < \epsilon$ then we have*

$$d_{GH}(X, G) < 2(\beta_1(G) + 1)(17 + 8N_{E,G'}(8\epsilon))\epsilon$$

where $N_{E,G'}(8\epsilon)$ is the number of edges of G' of length at most 8ϵ and $\beta_1(G)$ is the first Betti number of G, i.e. the number of edges to remove from G to get a spanning tree. In particular if X is at distance less than ϵ from a metric graph with shortest edge larger than 8ϵ then $d_{GH}(X, G) < 34(\beta_1(G) + 1)\epsilon$.

Note that $\beta_1(G) \leq \beta_1(X)$ and thus $d_{GH}(X, G)$ is upper bounded by the quantities depending only on the input X.

Theorem 5 [Recovery of Topology]. *Let (X, d_X) be a compact connected path metric space and $(G', d_{G'})$ is a metric graph so that $d_{GH}(X, G') < \epsilon$. Let $r \in X$, $\alpha > 60\epsilon$ and $\mathcal{I} = \{[0, 2\alpha), (i\alpha, (i + 2)\alpha) | 1 \leq i \leq m\}$ covers the segment $[0, Diam(X)]$ such that the 2α-Reeb graph G associated to $\mathcal{I}$ and the function $d = d_X(r, .) : X \to \mathbb{R}$ is a finite graph. If no edges of G' are shorter than L and no loops of G' are shorter than $2L$ with $L \geq 32\alpha + 9\epsilon$, then we have G and G' are homotopy equivalent.*

To turn this result into a practical algorithm we address two issues:

(1) Raw data usually do not come as geodesic spaces. They are given as discrete sets of points (and thus not connected metric spaces) sampled from the underlying space (X, d_X). Moreover in many cases only distances between nearby points are known. A geodesic space (see Section 2 for a definition of geodesic space) can then be obtained from these raw data as a neighborhood graph where nearby points are connected by edges whose length is equal to their pairwise distance. The shortest path distance in this graph is then used as the metric. In our experiments we use this new metric as the input of our algorithm. The question of the approximation of the metric on X by the metric induced on the neighborhood graphs is out of the scope of this paper.

(2) Approximating the Reeb graph (G, d_G) from a neighborhood graph is usually not obvious. If we compute the Reeb graph of the distance function to a given point defined on the neighborhood graph we obtain the neighborhood graph itself and do not achieve our goal of representing the input data by a simple graph. See Table 1. It is then appealing to build a two dimensional complex having the neighborhood graph as 1-dimensional skeleton and use the algorithm of [26,32] to compute the Reeb graph of the distance to the root point. Unfortunately adding triangles to the neighborhood graph may widely change the metric between the data points on the resulting complex and significantly increase the complexity of the algorithm. We overcome this issue by introducing a variant of the Reeb graph, the α-Reeb graph, inspired from [18] and related to the recently introduced notion of graph induced complex [33], that is easier to compute than the Reeb graph but also comes with approximation guarantees (see Theorem 4). As a consequence our algorithm runs in almost linear time (see Section 6).

Related work. Approximation of data by 1-dimensional geometric structures has been considered by different communities. In statistics, several approaches have been proposed to address the problem of detection and extraction of filamentary structures in point cloud data. For example Arial-Castro et al [16] use multiscale anisotropic strips to detect linear structure while [21,29] and more recently [30] base their approach upon density gradient descents or medial axis techniques. These methods apply to data corrupted by outliers embedded in Euclidean spaces and focus on the inference of individual filaments without focus on the global geometric structure of the filaments network.

In computational geometry, the curve reconstruction problem from points sampled on a curve in an euclidean space has been extensively studied and several efficient algorithms have been proposed [2,4,7]. Unfortunately, these methods restrict to the case of simple embedded curves (without singularities or self-intersections) and hardly extend to the case of topological graphs. In a more intrinsic setting where data come as finite abstract metric spaces, [28] propose an algorithm that outputs a topologically correct (up to a homeomorphism) reconstruction of the approximated graph. However this algorithm requires some tedious parameters tuning and relies on quite restrictive sampling assumptions. When these conditions are not satisfied, the algorithm may fail and not even outputs a graph. Compared to the algorithm of [28], our algorithm not only comes with metric guarantees but also whatever the input data is, it always outputs a metric graph and does not require the user to choose any parameters. Closely related to our approach is the data skeletonization algorithm proposed in [27] that computes the Reeb graph of an approximation of the distance function to a root point on a 2-dimensional complex built on top of the

data whose size might be significantly larger than a neighboring graph. The algorithm of [27] also always output a graph but it does not come with metric guaranties. Recently, Bauer, Ge and Wang [34] define a metric based on the function for Reeb graph and show it is stable under Gromov-Hausdorff distance. The implementation of our algorithm relies on the Mapper algorithm [18], that provides a way to visualize data sets endowed with a real valued function as a graph, where the considered function is the distance to the chosen root point. However, unlike the general Mapper algorithm, our methods provides an upper bound on the Gromov-Hausdorff distance between the reconstructed graph and the underlying space from which the data points have been sampled.

In theoretical computer science, there is much of work on approximating metric spaces using trees [17, 19, 20] or distribution of trees [15, 12] where the trees are often constructed as spanning trees possibly with Steiner points. Our approach is different as our reconstructed graph or tree is a quotient space of the original metric space where the metric only gets contracted (see Proposition 2). Finally we remark that the recovery of filament structure is also studied in various applied settings, including road networks [23, 3], galaxies distributions [24].

Part of the result (Theorem 3) shown in the paper also appears in [35]. The paper is organized as follows. The basic notions and definitions used throughout the paper are recalled in Section 2. The Reeb and α-Reeb graphs endowed with a natural metric are introduced in Section 3, and the approximation results in metric are stated and proven in Section 4, and the results of recovery of topology are stated and proven in Section 5. Our algorithm is described in Section 6 and experimental results are presented and discussed in Section 7.

2 Preliminaries

Recall that a metric space is a pair (X, d_X) where X is a set and $d_X : X \times X \to \mathbb{R}$ is a non negative map such that for any $x, y, z \in X$, $d_X(x, y) = 0$ if and only if $x = y$, $d_X(x, y) = d_X(y, x)$ and $d_X(x, z) \leq d_X(x, y) + d_X(y, z)$. Two compact spaces (X, d_X) and (Y, d_Y) are isometric if there exits a bijection $\phi : X \to Y$ that preserves the distances, namely: for any $x, x' \in X$, $d_Y(\phi(x), \phi(x')) = d_X(x, x')$. The set of isometry classes of compact metric spaces can be endowed with the Gromov-Hausdorff distance that can be defined using the following notion of correspondence ([6] Def. 7.3.17).

Definition 1 Let (X, d_X) and (Y, d_Y) be two compact metric spaces. Given $\epsilon > 0$, an *ϵ-correspondence* between (X, d_X) and (Y, d_Y) is a subset $C \subset X \times Y$ such that: i) for any $x \in X$ there exists $y \in Y$ such that $(x, y) \in C$; ii) for any $y \in Y$ there exists $x \in X$ such that $(x, y) \in C$; iii) for any $(x, y), (x', y') \in C$, $|d_X(x, x') - d_Y(y, y')| \leq \epsilon$.

Definition 2 The *Gromov-Hausdorff distance* between two compact metric spaces (X, d_X) and (Y, d_Y) is defined by

$$d_{GH}(X, Y) = \frac{1}{2} \inf\{\epsilon \geq 0 : \text{there exists an } \epsilon\text{-correspondence between } X \text{ and } Y\}$$

A metric space (X, d_X) is a *path metric space* if the distance between any pair of points is equal to the infimum of the lengths of the continuous curves joining them [1]. In the sequel of the paper we consider compact path metric spaces. It follows from the Hopf-Rinow theorem (see [10] p.9) that such spaces are *geodesic*, i.e. for any pair of point $x, x' \in X$ there exists a minimizing geodesic joining them.[2] A continuous path $\delta : I \to X$ where I is a real interval or the unit circle is said to be *simple* if it is not self intersecting, i.e. if δ is an injective map.

Recall that a *(finite) topological graph* $G = (V, E)$ is the geometric realization of a (finite) 1-dimensional simplicial complex with vertex set V and edge set E. If moreover each 1-simplex $e \in E$ is a metric edge, i.e. $e = [a, b] \subset \mathbb{R}$, then the graph G inherits from a metric d_G which is the unique one whose restriction to any $e = [a, b] \in E$ coincides with the standard metric on the real segment $[a, b]$. Then (G, d_G) is a *metric graph* (see [6], Section 3.2.2 for a more formal definition). Intuitively, a metric graph can be seen as a topological graph with a length assigned to each of its edges.

The *first Betti number* $\beta_1(G)$ of a finite topological graph G is the rank of the first homology group of G, or equivalently, the number of edges to remove from G to get a spanning tree.

[1] see [10] Chap.1 for the definition of the length of a continuous curve in a general metric space

[2] recall that a minimizing geodesic in X is any curve $\gamma : I \to X$, where I is a real interval, such that $d_X(\gamma(t), \gamma(t')) = |t - t'|$ for any $t, t' \in I$.

3 Reeb-type Graph

In this section, we describe a construction to build a Reeb-type graph for approximating the metric space. Let (X, d_X) be a compact geodesic space and let $r \in X$ be a fixed base point. Let $d : X \to \mathbb{R}$ be the distance function to r, i.e., $d(x) = d_X(r, x)$.

The Reeb graph. Define relation $x \sim y$ if and only if $d(x) = d(y)$ and x, y are in the same path connected component of $d^{-1}(d(x))$. This relation is an equivalence relation. The quotient space $G = X/\sim$ is called the *Reeb graph* of d and we denote by $\pi : X \to G$ the quotient map. Notice that π is continuous and as X is path connected, G is path connected. The function d induces a function $d_* : G \to \mathbb{R}_+$ that satisfies $d = d_* \circ \pi$. The relation defined by: for any $g, g' \in G$, $g \leq_G g'$ if and only if $d_*(g) \leq d_*(g')$ and there exist a continuous path γ in G connecting g to g' such that $d \circ \gamma$ is non decreasing, makes G a partially ordered set.

The α-Reeb graphs. Computing or approximating the Reeb graph of (X, d) from a finite set of point sampled on X is usually a difficult task. To overcome this issue we also consider a variant of the Reeb graph that shares very similar properties to the Reeb graph. Let $\alpha > 0$ and let $\mathcal{I} = \{I_i\}$ be a covering of the range of d by open intervals of length at most α. The transitive closure of the relation $x \sim_\alpha y$ if and only if $d(x) = d(y)$ and x, y are in the same path connected component of $d^{-1}(I_i)$ for some interval $I_i \in \mathcal{I}$ is an equivalence relation that is also denoted by $\sim_\alpha$. The quotient space $G_\alpha = X/\sim_\alpha$ is called the *α-Reeb graph*[3] of d and we denote by $\pi : X \to G_\alpha$ the quotient map. Notice that π is continuous and as X is path connected, G_α is path connected. The function d induces a function $d_* : G_\alpha \to \mathbb{R}_+$ that satisfies $d = d_* \circ \pi$. The relation defined by: for any $g, g' \in G_\alpha$, $g \leq_{G_\alpha} g'$ if and only if $d_*(g) \leq d_*(g')$ and there exist a continuous path γ in G_α connecting g to g' such that $d \circ \gamma$ is non decreasing, makes G_α a partially ordered set.

The α-Reeb graph is closely related to the graph constructed by the Mapper algorithm introduced in [18] making its computation much easier than the Reeb graph (see Section 6).

Notice that without making assumptions on X and d, in general G and G_α are not finite graphs. However when the number of path connected components of the level sets of d is finite and changes only a finite number of times then the Reeb graph turns out to be a finite directed acyclic graph. Similarly, when the covering of X by the connected components of $d^{-1}(I_i), I_i \in \mathcal{I}$ is finite, the α-Reeb graph also turns out to be a finite directed acyclic graph. This happens in most applications and for example when (X, d_X) is a finite simplicial complex or a compact semialgebraic (or more generally a compact subanalytic space) with d being semi-algebraic (or subanalytic).

All the results and proofs presented in Section 4 are exactly the same for the Reeb and the α-Reeb graphs. In the following paragraph and in Section 4.1, G denotes indifferently the Reeb graph or an α-Reeb graph for some $\alpha > 0$. We also always assume that X and d (and α and $\mathcal{I}$) are such that G is a finite graph.

A metric on Reeb and α-Reeb graphs. Let us define the set of vertices V of G as the union of the set of points of degree not equal to 2 with the set of local maxima of d_* over G, and the base point $\pi(r)$. The set of edges E of G is then the set of the connected components of the complement of V. Notice that $\pi(r)$ is the only local (and global) minimum of d_*: since X is path connected, for any $x \in X$ there exists a geodesic γ joining r to x along which d is increasing; d_* is thus also increasing along the continuous curve $\pi(\gamma)$, so $\pi(x)$ cannot be a local minimum of d_*. As a consequence d_* is monotonic along the edges of G. We can thus assign an orientation to each edge: if $e = [p, q] \in G$ is such that $d_*(p) < d_*(q)$ then the positive orientation of e is the one pointing from p to q. Finally, we assign a metric to G. Each edge $e \in E$ is homeomorphic to an interval to which we assign a length equal to the absolute difference of the function d_* at two endpoints. The distance between two points p, p' of e is then $|d_*(p) - d_*(p')|$. This makes G a metric graph (G, d_G) isometric to the quotient space of the union of the intervals isometric to the edges by identifying the endpoints if they correspond to the same vertex in G. Note that d_* is continuous in (G, d_G) and for any $p \in G$, $d_*(p) = d_G(\pi(r), p)$. Indeed this is a consequence of the following lemma.

Lemma 1 *If δ is a path joining two points $p, p' \in G$ such that $d_* \circ \delta$ is strictly increasing then δ is a shortest path between p and p' and $d_G(p, p') = d_*(p') - d_*(p)$.*

Proof As $d_* \circ \delta$ is strictly increasing, when δ enters an edge e by one of its end points, either it exits at the other end point or it stops at p' if $p' \in e$. Moreover δ cannot go through a given edge more than one time. As a consequence δ can be decomposed in a finite sequence of pieces $e_0 = [p, p_1], e_1 = [p_1, p_2], \cdots, e_{n-1} = [p_{n-1}, p_n], e_n = [p_n, p']$ where e_0 and e_n are the segments joining p and p' to one of the endpoint of the edges that contain them and

[3] strictly speaking we should call it the α-Reeb graph associated to the covering $\mathcal{I}$ but we assume in the sequel that some covering $\mathcal{I}$ has been chosen and we omit it in notations

$e_1, \cdots, e_{n-1}$ are edges. So, the length of δ is equal to $(d_*(p_1) - d_*(p)) + (d_*(p_2) - d_*(p_1)) + \cdots + (d_*(p') - d_*(p_n)) = d_*(p') - d_*(p)$ and $d_G(p, p') \leq d_*(p') - d_*(p)$.

Similarly any simple path joining p to p' can be decomposed in a finite sequence of pieces $e'_0 = [p, p'_1], e'_1 = [p'_1, p'_2], \cdots, e'_{k-1} = [p'_{k-1}, p'_k], e'_k = [p'_k, p']$ where e'_0 and e'_k are the segments joining p and p' to one of the endpoint of the edges that contain them, and $e'_1, \cdots, e'_{k-1}$ are edges. Now, as we do not know that d_* is increasing along this path, its length is thus equal to $|d_*(p'_1) - d_*(p)| + |d_*(p'_2) - d_*(p'_1)| + \cdots + |d_*(p') - d_*(p'_n)| \geq d_*(p') - d_*(p)$. So, $d_G(p, p') \geq d_*(p') - d_*(p)$.

4 Approximation of Metric

4.1 Bounding the Gromov-Hausdorff distance between X and G

The goal of this section is to provide an upper bound of the Gromov-Hausdorff distance between X and G that only depends on the first Betti number $\beta_1(G)$ of G and the maximal diameter M of the level sets of π. An upper bound of M is given in the next section.

Theorem 1 *$d_{GH}(X, G) < (\beta_1(G) + 1)M$ where $d_{GH}(X, G)$ is the Gromov-Hausdorff distance between X and G, $\beta_1(G)$ is the first Betti number of G and $M = \sup_{p \in G}\{diam(\pi^{-1}(p))\}$ is the supremum of the diameters of in the level sets of π.*

Remark that as $\beta_1(G) \leq \beta_1(X)$, from the above theorem, $d_{GH}(X, G)$ is upper bounded by the quantities depending only on the input X. The proof of Theorem 1 is deduced from two propositions comparing the distances between pairs of points $x, y \in X$ and their images $\pi(x), \pi(y) \in G$ whose proofs rely on the notion of merging vertex. A vertex $v \in V$ is called a *merging vertex* if it is the end point of at least two edges e_1 and e_2 that are pointing to it according to the orientation defined in Section 3. Geometrically this means that there are at least two distinct connected components of $\pi^{-1}(d_*^{-1}(d_*(v) - \epsilon))$ that accumulate to $\pi^{-1}(v)$ as $\epsilon > 0$ goes to 0. The set of merging vertices is denoted by V_m. We have

Lemma 2 *The cardinality of V_m is at most $\beta_1(G)$ where $\beta_1(G)$ is the rank of the first homology group of G.*

Proof The result follows from classical persistence homology theory [25]. First remark that, as $\pi(r)$ is the only local minimum of d_*, the sublevel sets of the function $d_* : G \to \mathbb{R}_+$ are all path connected. Indeed if $\pi(x), \pi(y) \in G$ are in the same sublevel set $d_*^{-1}([0, \alpha])$, $\alpha > 0$, then the images by π of the shortest paths in X connecting x to r and y to r are contained in $d_*^{-1}([0, \alpha])$ and their union is a continuous path joining $\pi(x)$ to $\pi(y)$. As a consequence, the 0-dimensional persistence of d_* is trivial. So as we increase the α value, no merging vertices serve as connecting two different connected components. Thus, each merging vertex in V_m creates at least a cycle that never dies as G is one dimensional and does not contain any 2-dimensional simplex. Thus $|V_m| \leq \beta_1(G)$.

The following lemma shows that a shortest path in G is the projection of a shortest path in X as long as it does not meet a merging vertex and allow to prove proposition 1 below.

Lemma 3 *Let $p, p' \in G$ and let $\delta : [d_*(p), d_*(p')] \to G$ be a strictly increasing path going from p to p' that does not contain any point of V_m in its interior. Then for any $x' \in \pi^{-1}(p') \bigcap cl(\pi^{-1}(\delta(d_*(p), d_*(p'))))$ where $cl(.)$ denotes the closure, there exists a shortest path γ connecting a point x of $\pi^{-1}(p)$ to x' such that $\pi(\gamma) = \delta$ and $d_X(x, x') = d(x') - d(x) = d_*(p') - d_*(p) = d_G(p, p')$.*

Proof First assume that p' is not a merging point. Let $\gamma_0 : [0, d(x')] \to X$ be any shortest path between r and x' and let γ be the restriction of γ_0 to $[d_*(p), d(x')] = [d_*(p), d_*(p')]$. If the infimum t_0 of the set $I = \{t \in [d_*(p), d_*(p')] : \pi(\gamma(t')) \in \delta,\ \forall t' \geq t\}$ is larger than $d_*(p)$, then there exists an increasing sequence (t_n) that converges to t_0 such that $\pi(\gamma(t_n)) \notin \delta$. As a consequence $\delta(t_0)$ is a merging point; a contradiction. So $t_0 = d_*(p)$ and $\gamma(d_*(p))$ intersects $\pi^{-1}(p)$ at a point x.

Now if p' is a merging point, as x' is chosen in the closure of $\pi^{-1}(\delta(d_*(p), d_*(p'))$, for any sufficiently large $n \in \mathbb{N}$ one can consider a sequence of points $x'_n \in \pi^{-1}(\delta(d_*(p') - 1/n))$ that converges to x' and apply the first case to get a sequence of shortest path γ_n from a point $x_n \in \pi^{-1}(p)$ and x'_n. Then applying Arzelà-Ascoli's theorem (see [1] 7.5) we can extract from γ_n a sequence of points converging to a shortest path γ between a point $x \in \pi^{-1}(p)$ and x'.

To conclude the proof, notice that from Lemma 1 we have $d_G(p, p') = d_*(p') - d_*(p) = d(x') - d(x)$. Since γ is the restriction of a shortest path from r to x we also have $d_X(x, x') = d(x') - d(x)$.

Notice that from Lemma 1, δ is a shortest path and the parametrization by the interval $[d_*(p), d_*(p')]$ can be chosen to be an isometric embedding.

Proposition 1 *For any $x, y \in X$ we have*

$$d_X(x,y) \leq d_G(\pi(x),\pi(y)) + 2(\beta_1(G)+1)M$$

where $M = \sup_{p\in G}\{diam(\pi^{-1}(p))\}$ and $\beta_1(G)$ is the first Betti number of G.

Proof Let δ be a shortest path between $\pi(x)$ and $\pi(y)$. Remark that except at the points $\pi(x)$ and $\pi(y)$ the local maxima of the restriction of d_* to δ are in V_m. Indeed as δ is a shortest path it has to be simple, so if $p \in \delta$ is a local maximum then p has to be a vertex and δ has to pass through two edges having p as end point and pointing to p according to the orientation defined in Section 3. So p is a merging point.
Since δ is simple and V_m is finite, δ can be decomposed in at most $|V_m|+1$ connected paths along the interior of which the restriction of d_* does not have any local maxima. So along each of these connected paths the restriction of d_* can have at most one local minimum. As a consequence, δ can be decomposed in a finite number of continuous paths $\delta_1, \delta_2, \cdots, \delta_k$ with $k \leq 2(|V_m|+1)$, such that the restriction of d_* to each of these path is strictly monotonic. For any $i \in \{1, \cdots, k\}$ let p_i and p_{i+1} the end points of δ_i with $p_1 = \pi(x)$ and $p_{k+1} = \pi(y)$. We can apply Lemma 3 to each δ_i to get a shortest path γ_i in X between a point $x_i \in \pi^{-1}(p_i)$ and a point in $y_{i+1} \in \pi^{-1}(p_{i+1})$ such that $\pi(\gamma_i) = \delta_i$ and $d_X(x_i, y_{i+1}) = d_G(p_i, p_{i+1})$. The sum of the lengths of the paths γ_i is equal to the sum of the lengths of the path δ_i which is itself equal to $d_G(\pi(x),\pi(y))$. Now for any $i \in \{1, \cdots, k\}$, since $\pi(x_i) = \pi(y_i)$ we have $d_X(x_i, y_i) \leq M$ and x_i and y_i can be connected by a path of length at most M (x_1 is connected to x and y_{k+1} is connected to y. Gluing these paths to the paths γ_i gives a continuous path from x to y whose length is at most $d_G(\pi(x),\pi(y)) + kM \leq d_G(\pi(x),\pi(y)) + 2(|V_m|+1)M$. Since from Lemma 2, $|V_m| \leq \beta_1(G)$, we finally get that $d_X(x,y) \leq d_G(\pi(x),\pi(y)) + 2(\beta(G)+1)M$.

Proposition 2 *The map $\pi : X \to G$ is 1-Lipschitz: for any $x, y \in X$ we have*

$$d_G(\pi(x),\pi(y)) \leq d_X(x,y).$$

Proof Let $x, y \in X$ and let $\gamma : I \to X$ be a shortest path from x to y in X where $I \subset \mathbb{R}$ is a closed interval. The path $\pi(\gamma)$ connects $\pi(x)$ and $\pi(y)$ in G.

We first claim that there exists a continuous path Γ contained in $\pi(\gamma)$ connecting $\pi(x)$ and $\pi(y)$ that intersects each vertex of G at most one time. The path Γ can be defined by iteration in the following way. Let $v_1, \cdots v_n \in V$ be the vertices of G that are contained in $\pi(\gamma) \setminus \{\pi(x), \pi(y)\}$ and let $\Gamma_0 = \pi(\gamma) : J_0 \to G, J_0 = I$. For $i = 1, \cdots n$ let $t_i^- = \inf\{t : \Gamma_{i-1}(t) = v_i\}$ and $t_i^+ = \sup\{t : \Gamma_{i-1}(t) = v_i\}$ and define Γ_i as the restriction of Γ_{i-1} to $J_i = J_{i-1} \setminus (t_i^-, t_i^+)$. The path Γ_i is a connected continuous path (although J_i is a disjoint union of intervals) that intersects the vertices $v_1, v_2, \cdots, v_i$ at most one time. We then define $\Gamma = \Gamma_n : J = J_n \to G$ where $J \subset I$ is a finite union of closed intervals. Notice that Γ is the image by π of the restriction of γ to J and that $\Gamma(t) \in \{v_1, \cdots v_n\}$ only if t is one of the endpoints of the closed intervals defining J.

Now, for each connected component $[t,t']$ of J, $\gamma((t,t'))$ is contained in $\pi^{-1}(e)$ where e is the edge of G containing $\Gamma([t,t'])$. As a consequence,

$$\begin{aligned} d_G(\pi(\gamma)(t),\pi(\gamma)(t')) &= |d_*(\pi(\gamma)(t)) - d_*(\pi(\gamma)(t'))| \\ &= |d(\gamma(t)) - d(\gamma(t'))|. \end{aligned}$$

Recalling that $d(\gamma(t)) = d_X(r,\gamma(t))$ and $d(\gamma(t')) = d_X(r,\gamma(t'))$ and using the triangle inequality we get that $|d(\gamma(t)) - d(\gamma(t'))| \leq d_X(\gamma(t),\gamma(t'))$. To conclude the proof, since γ is a geodesic path we just need to sum up the previous inequality over all connected components of J:

$$\begin{aligned} d_X(x,y) &\geq \sum_{[t,t']\in cc(J)} d_X(\gamma(t),\gamma(t')) \\ &\geq \sum_{[t,t']\in cc(J)} d_G(\pi(\gamma)(t),\pi(\gamma)(t')) \geq d_G(\pi(x),\pi(y)) \end{aligned}$$

where $cc(J)$ is the set of connected components of J.

The proof of Theorem 1 now easily follows from Propositions 1 and 2.

Proof (of Theorem 1) Consider the set $C = \{(x,\pi(x)) : x \in X\} \subset X \times G$. As π is surjective this is a correspondence between X and G. It follows from Propositions 1 and 2 that for any $(x,\pi(x)), (y,\pi(y)) \in C$,

$$|d_X(x,y) - d_G(\pi(x),\pi(y))| \leq 2(\beta_1(G)+1)M$$

So C is a $2(\beta_1(G)+1)M$-correspondence and $d_{GH}(X,G) \leq (\beta_1(G)+1)M$.

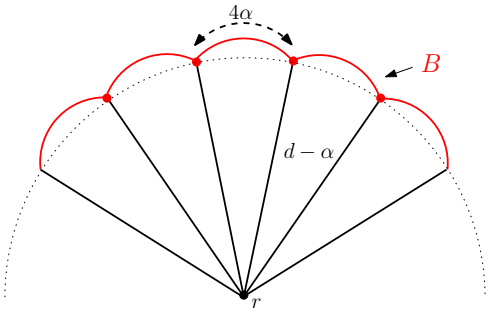


Fig. 1 Tightness of the bound in Lemma 3: there are 3 edges of length at most 4α and the diameter of B is equal to 20α. The range of the distances from r to the points on the red curve is $[d-\alpha, d+\alpha]$.

4.2 Bounding the diameter M

The two following lemmas allow to bound M, the diameter of the level sets of π.

Lemma 4 *Let (G, d_G) be a connected finite metric graph and let $r \in G$. We denote by $d_r = d_G(r, .) : G \to [0, +\infty)$ the distance to r. For any edge $e \subset G$, the restriction of d_r to e is either strictly monotonic or has only one local maximum. Moreover the length $l = l(e)$ of e is upper bounded by two times the difference between the maximum and the minimum of d_r restricted to e.*

Proof Let l be the length of E and let $t \mapsto e(t)$, $t \in [0, l]$, be an arc length parametrization of E. Since E is an edge of G, for $t \in [0, l]$ any shortest geodesic γ_t joining r to $e(t)$ must contain either $x_1 = e(0)$ or $x_2 = e(l)$. If it contains x_1 then for any $t' < t$ the restriction of γ_t between r and $e(t')$ is a shortest geodesic containing x_1 and if it contains x_2 then for any $t' > t$ the restriction of γ_t between r and $e(t')$ is a shortest geodesic containing x_2. Moreover in both cases, the function d_r is strictly monotonic along γ. As a consequence, the set $I_1 = \{t \in [0, l] : \text{a shortest geodesic joining } r \text{ to } e(t) \text{ contains } x_1\}$ is a closed interval containing 0. Similarly the set $I_2 = \{t \in [0, l] : \text{a shortest geodesic joining } r \text{ to } e(t) \text{ contains } x_2\}$ is a closed interval containing l and $[0, l] = I_1 \cup I_2$. Moreover d_r is strictly monotonic on $e(I_1)$ and on $e(I_2)$. As a consequence $I_1 \cap I_2$ is reduced to a single point t_0 that has to be the unique local maximum of d_r restricted to E.

The second part of the lemma follows easily from the previous proof: the minimum of d_r restricted to E is attained either at x_1 or x_2 and $d_r(e(t_0)) = d_r(x_1) + t_0 = d_r(x_2) + l - t_0$ is the maximum of d_r restricted to E. We thus obtain that $2t_0 = l + (d_r(x_2) - d_r(x_1))$. As a consequence if $d_r(x_1) \leq d_r(x_2)$ then $l/2 \leq t_0 = d_r(e(t_0)) - d_r(x_1)$; similarly if $d_r(x_1) \geq d_r(x_2)$ then $l/2 \leq l - t_0 = d_r(e(t_0)) - d_r(x_2)$.

Proposition 3 *Let (G, d_G) be a connected finite metric graph and let $r \in G$. For $\alpha > 0$ we denote by $N_E(\alpha)$ the number of edges of G of length at most α. For any $d > 0$ and any connected component B of the set $B_{d,\alpha} = \{x \in G : d - \alpha \leq d_G(r, x) \leq d + \alpha\}$ we have*

$$diam(B) \leq 4(2 + N_E(4\alpha))\alpha$$

Proof Let $x, y \in B$ and let $t \mapsto \gamma(t) \in B$ be a continuous path joining x to y in B. Let E be an edge of G that does not contain x or y and with end points x_1, x_2 such that γ intersects the interior of E. Then $\gamma^{-1}(E)$ is a disjoint union of closed intervals of the form $I = [t, t']$ where $\gamma(t)$ and $\gamma(t')$ belong to the set $\{x_1, x_2\}$. If $\gamma(t) = \gamma(t')$ we can remove the part of γ between t and t' and still get a continuous path between x and y. So without loss of generality we can assume that if γ intersects the interior of E, then E is contained in γ. Using the same argument as previously we can also assume that if γ goes across E, it only does it one time, i.e. $\gamma^{-1}(E)$ is reduced to only one interval. As a consequence, γ can be decomposed in a sequence $[x, v_0], E_1, E_2, \cdot, E_k, [v_k, y]$ where $[x, v_0]$ and $[v_k, y]$ are pieces of edges containing x and y respectively and $E_1 = [v_0, v_1], E_2 = [v_1, v_2]\cdot, E_k = [v_{k-1}, v_k]$ are pairwise distinct edges of G contained in B. It follows from Lemma 4 that the lengths of the edges $E_1, \cdots E_k$ and of $[x, v_0]$ and $[v_k, y]$ are upper bounded by 4α. As a consequence the length of γ is upper bounded by $4(k + 2)\alpha$ which is itself upper bounded by $4(N_E(4\alpha) + 2)\alpha$ since the edges $E_1, \cdots E_k$ are pairwise distinct. It follows that $d_G(x, y) \leq 4(N_E(4\alpha) + 2)\alpha$.

The example of Figure 1 shows that the bound of Lemma 3 is tight.

Lemma 5 *Let X and Y be compact geodesic metric spaces and $C \subset X \times Y$ be an $\epsilon'-$correspondence between them. Assume $(x_0, y_0) \in C(X, Y)$, we define functions $d_{x_0}(.) = d_X(x_0, .)$ and $d_{y_0}(.) = d_Y(y_0, .)$ in X and Y*

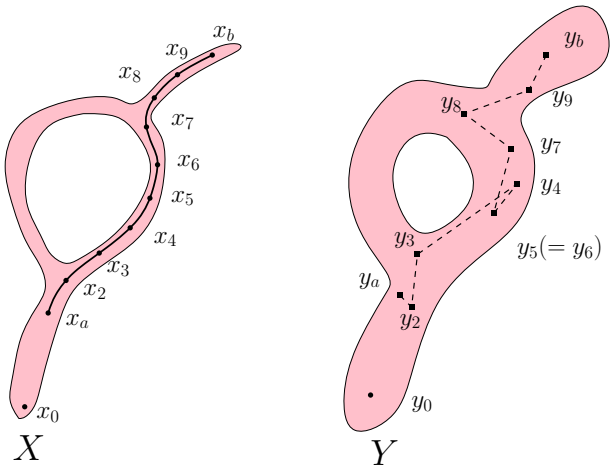


Fig. 2 The path correspondence between two metric spaces X and Y.

respectively. Then for any path γ_x in X connecting $x_a, x_b \in X$, we can find a path γ_y in Y such that its end points, y_a, y_b, are corresponding to x_a, x_b. And further more:

$$[min_{y\in\gamma_y} d_{y_0}(y), max_{y\in\gamma_y} d_{y_0}(y)] \subset [min_{x\in\gamma_x} d_{x_0}(x) - 2\epsilon', max_{x\in\gamma_x} d_{x_0}(x) + 2\epsilon']$$

Proof Let $\epsilon > \epsilon' > 0$ and u, l be the maximum and minimum of d_{x_0} restricted to γ_x. Since C is an ϵ'-correspondence for any $x \in \gamma_x$ there exists a point $(x, y) \in C$ such that $d_{x_0}(x) - \epsilon' \leq d_{y_0}(y) \leq d_{x_0}(x) + \epsilon'$. As illustrated in Figure 2, the set of points y obtained in this way is not necessarily a continuous path from y_a to y_b. However one can consider a finite sequence $x_1 = x_a, x_2, \cdots, x_n = x_b$ of points in γ_x such that for any $i = 1, \cdots n-1$ we have $d_X(x_i, x_{i+1}) < \epsilon - \epsilon'$. If $(x_i, y_i) \in C$ then we have $d_Y(y_i, y_{i+1}) < \epsilon - \epsilon' + \epsilon' = \epsilon$. As a consequence, since $l - \epsilon < l - \epsilon' < d_{y_0}(y_i) < u + \epsilon' < u + \epsilon$ the shortest geodesic connecting y_i to y_{i+1} in G remains in the set $d_{y_0}^{-1}([l-2\epsilon, u+2\epsilon])$ and connecting these geodesics for all $i = 1, \cdots, n-1$ we get a continuous path from y_a to y_b in $d_r^{-1}([l-2\epsilon, u+2\epsilon])$. Now decreasing ϵ to ϵ', we finish the construction.

As a corollary, we have the following theorem.

Theorem 2 *Let (G, d_G) be a connected finite metric graph and let (X, d_X) be a compact geodesic metric space such that $d_{GH}(X, G) < \epsilon$ for some $\epsilon > 0$. Let $x_0 \in X$ be a fixed point and let $d_{x_0} = d_X(x_0, .) : X \to [0, +\infty)$ be the distance function to x_0. Then for $d \geq \alpha \geq 0$ the diameter of any connected component L of $d_{x_0}^{-1}([d-\alpha, d+\alpha])$ satisfies*

$$diam(L) \leq 4(2 + N_E(4(\alpha + 2\epsilon)))(\alpha + 2\epsilon) + \epsilon$$

where $N_E(4(\alpha + 2\epsilon))$ is the number of edges of G of length at most $4(\alpha + 2\epsilon)$. In particular if $\alpha = 0$ and 8ϵ is smaller that the length of the shortest edge of G then $diam(L) < 17\epsilon$.

Proof Let $\epsilon' > 0$ be such that $d_{GH}(X, G) < \epsilon' < \epsilon$. Let $C \subset X \times G$ be an ϵ'-correspondence between X and G and $(x_0, r) \in C$. we denote by $d_r = d_G(r, .) : G \to [0, +\infty)$ the distance function to r in G. Let $x_a, x_b \in L$ and let $(x_a, y_a), (x_b, y_b) \in C$. There exists a continuous path $\gamma \subseteq L$ joining x_a to x_b. Following lemma 5, we get a continuous path from y_a to y_b in $d_r^{-1}([d - \alpha - 2\epsilon', d + \alpha + 2\epsilon'])$. It then follows from Proposition 3 that $d_G(y_a, y_b) \leq 4(2 + N_E(4(\alpha + 2\epsilon)))(\alpha + 2\epsilon)$ and since C is an ϵ'-correspondence (and so an ϵ-correspondence), $d_X(x_a, x_b) < 4(2 + N_E(4(\alpha + 2\epsilon)))(\alpha + 2\epsilon) + \epsilon$.

From Theorems 2 and 1 we obtain the following results for the Reeb and α-Reeb graphs.

Theorem 3 *Let (X, d_X) be a compact connected path metric space, let $r \in X$ be a fixed base point such that the metric Reeb graph (G, d_G) of the function $d = d_X(r, .) : X \to \mathbb{R}$ is a finite graph. If for a given $\epsilon > 0$ there exists a finite metric graph $(G', d_{G'})$ such that $d_{GH}(X, G') < \epsilon$ then we have*

$$d_{GH}(X, G) < (\beta_1(G) + 1)(17 + 8N_{E,G'}(8\epsilon))\epsilon$$

where $N_{E,G'}(8\epsilon)$ is the number of edges of G' of length at most 8ϵ. In particular if X is at distance less than ϵ from a metric graph with shortest edge length larger than 8ϵ then $d_{GH}(X, G) < 17(\beta_1(G) + 1)\epsilon$.

Theorem 4 *Let (X, d_X) be a compact connected path metric space. Let $r \in X$, $\alpha > 0$ and $\mathcal{I}$ be a finite covering of the segment $[0, Diam(X)]$ by open intervals of length at most α such that the α-Reeb graph G_α associated to $\mathcal{I}$ and the function $d = d_X(r, .) : X \to \mathbb{R}$ is a finite graph. If for a given $\epsilon > 0$ there exists a finite metric graph $(G', d_{G'})$ such that $d_{GH}(X, G') < \epsilon$ then we have*

$$d_{GH}(X, G_\alpha) < (\beta_1(G_\alpha) + 1)(4(2 + N_{E,G'}(4(\alpha + 2\epsilon)))(\alpha + 2\epsilon) + \epsilon$$

where $N_{E,G'}(4(\alpha + 2\epsilon))$ is the number of edges of G' of length at most $4(\alpha + 2\epsilon)$. In particular if X is at distance less than ϵ from a metric graph with shortest edge length larger than $4(\alpha + 2\epsilon)$ then $d_{GH}(X, G_\alpha) < (\beta_1(G_\alpha) + 1)(8\alpha + 17\epsilon)$.

5 Recovery of Topology

In this section, we show the following theorem which asserts that the α-Reeb graph G of (X, d) recovers some topology of X.

Theorem 5 *Let (X, d_X) be a compact connected path metric space and $(G', d_{G'})$ is a metric graph so that $d_{GH}(X, G') < \epsilon$. Let $r \in X$, $\alpha > 60\epsilon$ and $\mathcal{I} = \{[0, 2\alpha), (i\alpha, (i + 2)\alpha) | 1 \le i \le m\}$ covers the segment $[0, Diam(X)]$ such that the 2α-Reeb graph G associated to $\mathcal{I}$ and the function $d = d_X(r, .) : X \to \mathbb{R}$ is a finite graph. If no edges of G' are shorter than L and no loops('lengths) of G' are shorter than $2L$ with $L \ge 32\alpha + 9\epsilon$, then we have G and G' are homotopy equivalent.*

Our strategy of proving Theorem 5 is to construct some open covers for X and G' and relate the α-Reeb graph G and the graph G' to the nerves of the open covers. Specifically, we construct an initial open cover $\mathcal{V}_0$ of X whose nerve $N(\mathcal{V}_0)$ is homotopy equivalent to G. Then we obtain a new open cover $\tilde{\mathcal{V}}$ of X by merging certain elements in $\mathcal{V}_0$ while preserving the homotopy type of the nerve of the open cover, i.e., $N(\mathcal{V}_0)$ and $N(\tilde{\mathcal{V}})$ are homotopy equivalent. Based on the open cover $\tilde{\mathcal{V}}$, we construct an open cover $\tilde{\mathcal{U}}$ for G' whose nerve $N(\tilde{\mathcal{U}})$ is isomorphic to $N(\tilde{\mathcal{V}})$ as graphs and at the same time is homotopy equivalent to G'. In the following, we describe the constructions of the above open covers for X and G' and show the above claimed relations between them.

Since $d_{GH}(X, G') < \epsilon$, there exists an ϵ-correspondence between the two spaces, denoted $C(X, G')$. For any subset $V \subset X$, denote $C(V) = \{g' : (x, g') \in C(X, G'), x \in V\}$, and similarly for any subset $U \subset G'$, denote $C(U) = \{x : (x, g') \in C(X, G'), g' \in U\}$. We call $C(V)$ and $C(U)$ are the correspondence set of V and U respectively under $C(X, G')$. Recall that $r \in X$ is the root point. Choose a point $g_r \in C(r)$ and define a distance function $b : G' \to \mathbb{R}$ by $b(g) = d_{G'}(g_r, g)$. Let $N = \{g_{n_1}, g_{n_2}, \cdots, g_{n_p}\}$ be the vertices of G', i.e., N is the set of vertices whose degree is not equal to two. From the hypotheses of the above theorem, the distance between any pair of vertices g_{n_i}, g_{n_j} with $i \neq j$ is larger than L. For convenience, we also add into the vertices of G' the remaining local maximal/minimal points of the distance function b, which we denote using $M = \{g_{m_1}, \cdots, g_{m_q}\}$. Note any newly added vertex $g_{m_i} \in M$ is of degree two. We call the graph G' before adding the vertices in M the original G', and the edges in the original G' the original edges of G'. An original edge of G' contains at most one vertex in M and thus can be split into at most two edges in G'.

5.1 Construction of open cover for X

We start with the following open cover of X. For each $I_k \in \mathcal{I}$, denote $V_k = d^{-1}(I_k)$. V_k may have several connected components, which can be listed in an arbitrary order. Denote V_k^l the l-th connected component of V_k. Then $\mathcal{V}_0 = \{V_k^l\}_{k,l}$ is an open cover of X. Since at most two elements in $\mathcal{I}$ are overlapped, the nerve of $\mathcal{V}_0$, denoted $N(\mathcal{V}_0)$, is a graph. The following lemma states that any loop in the nerve $N(\mathcal{V}_0)$ is large, which is useful for the proof of Theorem 5. We say an open set $V_{k_1}^{l_1} \in \mathcal{V}_0$ is lower than the open set $V_{k_2}^{l_2} \in \mathcal{V}_0$ if $k_1 < k_2$ and is higher than $V_{k_2}^{l_2}$ if $k_2 > k_1$.

Lemma 6 *Let V_k^l and V_j^i are the lowest vertex and the highest vertex of a loop respectively in the nerve $N(\mathcal{V}_0)$. Then under the hypotheses of Theorem 5, we have $j - k \ge 15$.*

Proof First notice that $j > k$. Let $x_1 \in V_k^l \cap d^{-1}(k\alpha, (k+1)\alpha)$ and $x_2 \in V_j^i \cap d^{-1}((j+1)\alpha, (j+2)\alpha)$. From the hypotheses of the lemma, there are two different paths β_1, β_2 connecting x_1 to x_2 so that $\beta_1 \cap d^{-1}((k+1)\alpha, (j+1)\alpha)$ and $\beta_2 \cap d^{-1}((k+1)\alpha, (j+1)\alpha)$ are in the different connected components of $d^{-1}((k+1)\alpha, (j+1)\alpha)$. Choose $g_i \in G'$ from $C(x_i)$ for $i = 1, 2$. Following Lemma 5, the path β_i in X for $i = 1, 2$ traces out a simple path γ_i in G' connecting g_1 to g_2 so that γ_i lies in $b^{-1}(k\alpha - \epsilon, (j + 2)\alpha + \epsilon)$. One can verify that γ_1 and γ_2 are two different

paths due to the fact that β_1 and β_2 pass through different connected components of $d^{-1}((k+1)\alpha,(j+1)\alpha)$ and thus form a loop in G', denoted γ. We have $b(\gamma) \subset (k\alpha-\epsilon,(j+2)\alpha+\epsilon)$.

We claim the range of the function b restricted to any loop, in particular β, covers an interval with the length at least $\frac{L}{2}$. If the claim holds, then we have $(j-k+2)\alpha+2\epsilon \geq \frac{L}{2}$, which implies $j-k \geq 15$ from the hypothesis $L \geq 32\alpha+9\epsilon$ of Theorem 5. Indeed, if β contains at least two vertices in N, then it is obvious that the range of the function b restricted to γ covers an interval the length at least $\frac{L}{2}$ as any original edge of G' is longer than L. Now consider the case where γ contains one vertex in N, say g_a. If γ does not contain g_r, then there is exactly one local maximum on γ, say g_b. If γ contains g_r, let $g_b = g_r$. The removal of g_a and g_b cuts γ into two pieces. Along either piece, the function b has at most one local maximum. As the length of γ is longer than $2L$. We have $b(\gamma)$ covers an interval with length longer than $L/2$. Finally, if β contains no vertex in N, then G' is a single loop γ and the claim obviously holds.

In the following, we modify this open cover by merging while preserving the homotopy type of its nerve. The main purpose of the merging operation is make it easy to relate the open cover of X to the open cover of G' as constructed in Section 5.2. The merging operation is done in two steps.

For any vertex $g \in M \cup N$ of G', we construct a connected open set $V(g)$ as the union of a subset of the open cover $\mathcal{V}_0$ as follows. If $b(g) \geq \frac{\alpha}{2}$, then there exists a unique positive integer k' s.t. $k'\frac{\alpha}{2} \leq b(g) < (k'+1)\frac{\alpha}{2}$. Let $k = \lfloor \frac{k'+1}{2} \rfloor - 1 \geq 0$, and one can verify that $(k+\frac{1}{2})\alpha \leq b(g) \leq (k+\frac{3}{2})\alpha$. Therefore for all $x \in C(g)$, $d(x) \in [(k+\frac{1}{2})\alpha-\epsilon,(k+\frac{3}{2})\alpha+\epsilon] \subset I_k$. Moreover $C(g)$ is contained in $V_k^l \subset V_k$ for some l. Indeed, if not, assume $x_1,x_2 \in C(g)$ with $x_i \in V_k^i$ for $i \in \{1,2\}$. By the definition of V_k^i, the geodesic connecting x_1 and x_2 must pass through a point x_0 outside of V_k, which means $d_X(x_i,x_0) \geq |d(x_i)-d(x_0)| \geq \frac{\alpha}{2}-\epsilon$. Then $d_X(x_1,x_2) \geq \alpha-2\epsilon$ which contradicts to the fact that $d_X(x_1,x_2) \leq d_{G'}(g,g)+\epsilon \leq \epsilon$. Now we construct the open set $V(g)$ as the union of the elements in the open cover $\mathcal{V}_0$ having non-empty intersection with V_k^l, i.e.,

$$V(g) = \bigcup_{V \in \mathcal{V}_0 \text{ and } V \cap V_k^l \neq \emptyset} V.$$

In the case where $b(g) < \frac{\alpha}{2}$, we construct the open set $V(g) = V_0 \cup V_1 = d^{-1}([0,3\alpha))$. Note in both cases, $V(g)$ is a connected open set of X. We abuse the notation and also denote $V(g)$ the subset of $\mathcal{V}_0$ whose union is the open set $V(g)$. What $V(g)$ represents will be clear from the context. For convenience, we call V_k^l containing $C(g)$ the center of $V(g)$. Note that it is possible that $V(g) = V(g')$ for two different vertices g, g'.

Now we obtain an intermediate open cover of X

$$\mathcal{V} = \{V(g) : g \in M \cup N\} \bigcup \{V \in \mathcal{V}_0 : V \notin V(g), \forall g \in M \cup N\}$$

Note as a set, $\mathcal{V}$ does not have duplicated elements, i.e., if $V(g) = V(g')$ for $g \neq g'$, then $\mathcal{V}$ only contains one copy of $V(g)$. We call an open set $V(g) \in \mathcal{V}$ for any $g \in M \cup N$ critical and the remaining ones regular. The following two lemmas describe the properties of the critical open sets and the regular open sets.

Lemma 7 *Under the hypotheses of Theorem 5, we have for any vertex $g \in M \cup N$,*
i) $d(V(g)) \subset [s\alpha,(s+4)\alpha]$ for some integer $s \geq 0$, and
ii) for any point $x \in \bigcup_{V \in \mathcal{V}_0 \setminus V(g)} V$ and any $g_x \in C(x) \subset G'$, $d_{G'}(g,g_x) \geq \frac{\alpha}{2}-2\epsilon$.

Proof The claim (i) is obvious from the construction of $V(g)$. We now prove claim (ii). In the case where $b(g) < \frac{\alpha}{2}$, for any $x \in \bigcup_{V \in \mathcal{V}_0 \setminus V(g)} V$, we have $d(x) > 3\alpha$ and $b(g_x) > 3\alpha-\epsilon$. Thus $d_{G'}(g,g_x) \geq |b(g_x)-b(g)| > 3\alpha-\epsilon-\frac{\alpha}{2} > \frac{\alpha}{2}-2\epsilon$. Now consider the case where $b(g) \geq \frac{\alpha}{2}$. Assume V_k^l is the center of $V(g)$. If $d(x) \notin I_k$, then $d_X(x,y) \geq \frac{\alpha}{2}-\epsilon$ for any point $y \in C(g)$ from the construction of $V(g)$, which implies $d_{G'}(g_x,g) \geq \frac{\alpha}{2}-2\epsilon$. Otherwise $d(x) \in I_k$. Then x is not in V_k^l and the geodesic from x to any point $y \in C(g)$ must pass $x_0 \notin V_k$. This implies that $d_X(x,y) > d_X(x_0,y) \geq \frac{\alpha}{2}-\epsilon$ and $d_{G'}(g_x,g) \geq d_X(x,y)-\epsilon \geq \frac{\alpha}{2}-2\epsilon$. This proves the lemma.

Lemma 8 *For any regular open set $V \in \mathcal{V}$, V is also an open set in $\mathcal{V}_0$. Moreover, under the hypotheses of Theorem 5, it is of degree two in the nerve of $N(\mathcal{V}_0)$ with one neighboring vertex higher than V and one neighboring vertex lower than V.*

Proof We prove the lemma by contradiction. Assume $V \in \mathcal{V}_0 \setminus \bigcup_{g \in M \cup N} V(g)$ has two neighboring vertices, say V_a, V_b, which are lower than V. Without loss of generality, assume $d(V) \subset I_j$ and $d(V_a)$ and $d(V_b)$ are subsets of I_{j-1}. Let $x_a \in V_a$ and $x_b \in V_b$ such that $(j-1)\alpha < d(x_a), d(x_b) < j\alpha$. As V_a and V_b both have non-empty intersection with V, there exist a path in $d^{-1}((j-1)\alpha,(j+2)\alpha)$. Now let $l = inf\{s$: there exists a path connecting x_a, x_b in $d^{-1}((j-1)\alpha,s] \bigcap (V \cup V_a \cup V_b)\}$. We have $(j+1)\alpha \leq l < (j+2)\alpha$ as V_a, V_b are disconnected.

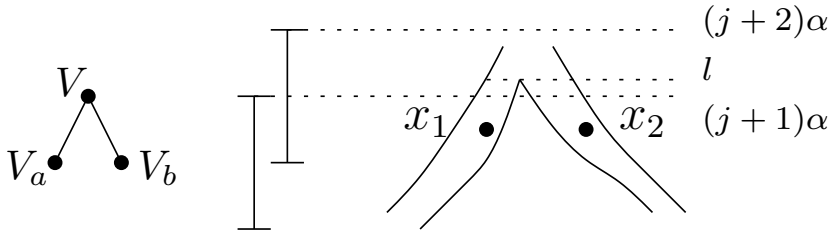


Fig. 3 V with two lower neighborhoods.

We can choose two points $x_1, x_2 \in V \cup V_a \cup V_b$ from a path connecting x_a and x_b such that $d(x_1) = d(x_2) = l - 2\epsilon$, and x_1, x_2 are disconnected in $d^{-1}([l-2\epsilon, l)) \bigcap (V \cup V_a \cup V_b)$ but are connected by a path, say β, in $d^{-1}([l-2\epsilon, l]) \bigcap (V \cup V_a \cup V_b)$. Obviously $d_X(x_1, x_2) \geq 2(l - (l - 2\epsilon)) = 4\epsilon$. Let $g_i \in C(x_i)$ for $i = 1, 2$. Then $b(g_i) \in [l - 3\epsilon, l - \epsilon]$ for $i = 1, 2$ and $d_{G'}(g_1, g_2) \geq d_X(x_1, x_2) - \epsilon \geq 3\epsilon$. Following Lemma 5, the path β traces out a simple path in G' denoted γ connecting g_1 and g_2 which lies in $b^{-1}(l - 4\epsilon, l + 2\epsilon)$. We claim γ must contain a vertex g_c in $M \cup N$. If not,

$$\begin{aligned} d_{G'}(g_1, g_2) &= |b(g_1) - b(g_2)| \\ &\leq |b(g_1) - d(x_1) + d(x_2) - b(g_2)| \\ &\leq |(b(g_1) - b(g_r)) - (d(x_1) - d(r))| + |(d(x_2) - d(r)) - (b(g_2) - b(g_r))| \\ &= |d_{G'}(g_1, g_r) - d_X(x_1, r)| + |d_{G'}(g_2, g_r) - d_X(x_2, r)| \\ &\leq 2\epsilon \end{aligned}$$

This contradicts to the fact that $d_{G'}(g_1, g_2) \geq 3\epsilon$. From the construction of γ, there exists a point $g \in \gamma$ so that $d_{G'}(g, g_c) \leq \epsilon$ and there exists a point $x \in \beta \cap C(g)$. Then for any point $x_c \in C(g_c)$, we have $d_X(x, x_c) \leq 2\epsilon$. For any vertex $x_0 \in V \cap d^{-1}(l)$, since x_0 and x are connected in $d^{-1}([l-2\epsilon, l]) \bigcap (V \cup V_a \cup V_b)$, $d_X(x_0, x) \leq 25\epsilon$ from Theorem 2, and therefore $d_X(x_0, x_c) \leq 27\epsilon$. However, since $x_0 \in V$ which is regular, $x \notin V(g_c)$. From Lemma 7, we have $d_X(x_0, x_c) \geq \frac{\alpha}{2} - 2\epsilon > 27\epsilon$. This is a contradiction. Therefore V can not have more than one neighboring vertices that are lower than V.

Using a similar argument we can also prove that V can not have more than one neighboring vertices that are higher than V.

We now perform a second step of merging. Two critical open set $V(g_1)$ and $V(g_2)$ in $N(\mathcal{V})$ are said to be close if there is a simple path γ in the nerve $N(\mathcal{V}_0)$ connecting the center $V_{k_1}^{l_1}$ of $V(g_1)$ and the center $V_{k_2}^{l_2}$ of $V(g_2)$ so that γ consists of at most 4 edges. If there is a regular open set along the above path, we say this regular open set connects the critical open sets $V(g_1)$ and $V(g_2)$. We have the following properties for two close critical open sets.

Lemma 9 *Under the hypotheses of Theorem 5, we have*

(i) for any two vertices $g_{n_1}, g_{n_2} \in N$, $V(g_{n_1})$ and $V(g_{n_2})$ can not be close;
(ii) for any $g_m \in M$, there exists at most one $g_n \in N$ such that $V(g_m)$ and $V(g_n)$ are close;
(iii) if $V(g_{m_1})$ and $V(g_{m_2})$ are close for any two vertices $g_{m_1}, g_{m_2} \in M$, then there must exist a vertex $g_n \in N$ such that at least one of $V(g_{m_1})$ and $V(g_{m_2})$ is close to $V(g_n)$. Moreover, there is a path in $N(\mathcal{V}_0)$ of at most 5 edges connecting the center of $V(g_n)$ to the center of $V(g_{m_i})$ for any $i = 1, 2$.

Proof Let V_j^p, V_k^q are the centers of $V(g_1)$ and $V(g_2)$ respectively for $g_1, g_2 \in N \cup M$. If $V(g_1)$ and $V(g_2)$ are close, then $|k - j| \leq 4$. Assume $j \leq k$. Then for any $x_1 \in C(g_1)$ and $x_2 \in C(g_2)$, there is a path in $d^{-1}((j\alpha, (k+2)\alpha))$ connecting x_1 and x_2. Note that $k + 2 - j \leq 6$. We claim that $V(g_1)$ and $V(g_2)$ are not close provided that $d_{G'}(g_1, g_2) > 12\alpha + 9\epsilon$. Indeed, since $d_{G'}(g_1, g_2) > 12\alpha + 9\epsilon$, the range of the function b restricted to any path connecting g_1 and g_2 in G' covers an interval of the length at least $6\alpha + 4.5\epsilon$. This implies that the range of the function d restricted to any path in X connecting x_1 and x_2 covers an interval of the length at least $6\alpha + 0.5\epsilon$. This means that $V(g_1)$ and $V(g_2)$ can not be close. Since $d_{G'}(g_{n_1}, g_{n_2}) \geq L > 12\alpha + 9\alpha$, $V(g_{n_1})$ and $V(g_{n_2})$ are not close. This proves (i).

Assume $V(g_m)$ is close to both $V(g_{n_1})$ and $V(g_{n_2})$ with $g_{n_1}, g_{n_2} \in N$. We have $d_{G'}(g_{n_1}, g_{n_2}) \leq d_{G'}(g_m, g_{n_1}) + d_{G'}(g_m, g_{n_2}) \leq 24\alpha + 18\epsilon < L$, which means $g_{n_1} = g_{n_2}$. This proves (ii).

We now prove (iii). Since at most one vertex in M is added into an original edge of G', any path in G' connecting g_{m_1} and g_{m_2} passes through at least one vertex from N. Furthermore, let γ be a geodesic in G' connecting g_{m_1} and g_{m_2}. If γ passes more than one vertices in N, $d_{G'}(g_{m_1}, g_{m_2}) \geq L > 12\alpha + 9\epsilon$, which contradicts to the fact that $V(g_{m_1})$ and $V(g_{m_2})$ are close. Therefore γ contains exactly one vertex in N. Denote this vertex by g_n.

Let $V_{k_1}^{l_1}$ and $V_{k_2}^{l_2}$ be the centers of $V(g_{m_1})$ and $V(g_{m_2})$ respectively, and δ be the simple path from $V_{k_1}^{l_1}$ to $V_{k_2}^{l_2}$ in $N(\mathcal{V}_0)$ so that δ consists of at most 4 edges, or equivalently at most five elements in $\mathcal{V}_0$.

Recall $V(g_n)$ consists of a subset of $\mathcal{V}_0$. We claim that δ must pass through an element in $V(g_n)$.

If the claim holds, it is easy to verify that at least one of $V(g_{m_1})$ and $V(g_{m_2})$ is close to $V(g_n)$. In addition, if we let V_k^l be the center of $V(g_n)$, then there is a path in $N(\mathcal{V}_0)$ with as most 5 edges connecting $V_{k_i}^{l_i}$ and V_k^l for any $i = 1, 2$. This proves (iii).

It remains to show the above claim. We prove by contradiction. If we let

$$V(\delta) = \{V \in \mathcal{V}_0 : V \text{ is on the path of } \delta\},$$

then $V(g_n)$ as a subset of $\mathcal{V}_0$ does not intersect with $V(\delta)$. We have $C(g_{m_1})$ and $C(g_{m_2})$ are contained in $V_{k_1}^{l_1}$ and $V_{k_2}^{l_2}$ respectively. For any $x_1 \in C(g_{m_1})$ and any $x_2 \in C(g_{m_2})$, there is a path β in X connecting x_1 and x_2 so that β is contained in $\bigcup_{V\in V(\delta)} V$. From Lemma 7, for any $x \in \beta$ and any $g_x \in C(x)$, $d_{G'}(g_x, g_n) \geq \frac{\alpha}{2} - 2\epsilon$. From the construction in the proof of Theorem 2, the path β can trace out a simple path γ' in G' connecting g_{m_1} and g_{m_2} so that for any point $g \in \gamma'$, $d_{G'}(g, g_n) \geq \frac{\alpha}{2} - 3\epsilon$. This means that γ and γ' form a loop in G'. Since β is contained in $\bigcup_{V\in V(\delta)} V$, $d(\beta)$ is contained in an interval with the length of 6α, which implies that $b(\gamma')$ is contained in an interval with the length of $6\alpha + 4\epsilon$. Thus the length of γ' is at most $2(6\alpha + 4\epsilon)$ as it passes through at most one vertex in N. Since γ is a geodesic in G' connecting g_{m_1} and g_{m_2}, the length of the above loop is at most $4(6\alpha + 4\epsilon)$, which contradicts to the hypotheses of Theorem 5. This proves the above claim.

We say $g_a, g_b \in M \cup N$ are equivalent, denoted $g_a \sim_c g_b$, if there exists a finite sequence $g_a = g_1, g_2, ..., g_k = g_b$ such that $V(g_i)$ and $V(g_{i+1})$ are close for any $i = 1, \cdots, k-1$. This is an equivalence relation. From Lemma 9 (iii), if an equivalence class contains at least two vertices in $M \cup N$, it must contain a vertex in N. We have the following lemma

Lemma 10 *Under the hypotheses of Theorem 5, an equivalence class contains at most one vertex from N.*

Proof If not, assume $g_{n_1} \neq g_{n_2}$ and $g_{n_1} \sim_c g_{n_2}$. Let $g_{n_1} = g_1, g_2, ..., g_k = g_{n_2}$ be a sequence so that $V(g_i)$ and $V(g_{i+1})$ are close for any $i = 1, \cdots, k-1$. WLOG, we can further assume $g_i \in M$ for $i = 2, \cdots, k-1$.

We first show that $k > 5$. Assume not. let $V_{k_i}^{l^i}$ be the center of $V(g_i)$ for $i = 1, \cdots, k$. From Lemma 9 (iii), there is a path in $N(\mathcal{V}_0)$ with at most $2 \times 5 = 10$ edges connecting $V_{k_1}^{l^1}$ to $V_{k_5}^{l^5}$. Thus for any $x_1 \in C(g_{n_1})$ and any $x_2 \in C(g_{n_2})$, there is a path β in X connecting x_1 and x_2 so that $d(\beta)$ is contained in an interval with the length at most 12α. The path β traces out a path γ in G' connecting g_{n_1} and g_{n_2} so that $b(\gamma)$ is contained in an interval with the length at most $12\alpha + 4\epsilon$, which implies $d_{G'}(g_{n_1}, g_{n_2}) \leq 2(12\alpha + 4\epsilon)$. This contradicts to the hypothesis concerning the lengths of the edges in G'.

Now we assume $k > 5$. Since $V(g_3)$ and $V(g_4)$ are close and $g_3, g_4 \in M$, from Lemma 9 (iii), there exists a $g_n \in N$ so that $V(g_n)$ is close to at least one of $V(g_3)$ and $V(g_4)$. Assume $V(g_n)$ is close to $V(g_3)$. If $V(g_n) \neq V(g_{n_1})$, we obtain a sequence of $g_1' = g_{n_1}, g_2' = g_2, g_3' = g_3, g_4' = g_n$ so that $V(g_i')$ and $V(g_{i+1}')$ are close for any $i = 1, \cdots, 3$. If $V(g_n) \neq V(g_{n_2})$, we obtain a sequence $g_1' = g_n, g_2' = g_3, \cdots g_{k-1}' = g_{n_2}$ so that $V(g_i')$ and $V(g_{i+1}')$ are close for any $i = 1, \cdots, k-2$. In either case, the new sequence has a length less than k. Similarly, we can obtain a shorter sequence if $V(g_n)$ is close to $V(g_4)$. One can keep shortening the sequence so that its length is no longer than 5, which however has been proven to be impossible. This proves the lemma.

Now we are ready to further merge the open sets in $\mathcal{V}$ to obtain the final open cover $\tilde{\mathcal{V}}$ of X as follows. For any vertex $g_n \in N$ of G', Let $\tilde{V}(g_n)$ be the subset of $\mathcal{V}$ consisting of (1) $V(g_n)$, and (2) any critical open set $V(g) \subset \mathcal{V}$ with $g \sim_c g_n$, and (3) any regular open set $V \subset \mathcal{V}$ connecting two critical open sets which are equivalent to g_n. We abuse the notation and also denote $\tilde{V}(g_n)$ the open set of the union of the open sets in $\tilde{V}(g_n)$. What $\tilde{V}(g_n)$ represents will be clear from the context. Let $\tilde{\mathcal{V}}_N = \{V \in \mathcal{V} : V \in \tilde{V}(g_n) \text{ for some } g_n \in N\}$. The open cover $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_1 \cup \tilde{\mathcal{V}}_2 \cup \tilde{\mathcal{V}}_3$ of X consists of three types of open sets:

(1) $\tilde{\mathcal{V}}_1 = \{\tilde{V}(g_n) : g_n \in N\}$;
(2) $\tilde{\mathcal{V}}_2 = \{V(g) : g \in M \text{ and } V(g) \not\subset \tilde{\mathcal{V}}_N\}$
(3) $\tilde{\mathcal{V}}_3 = \{V \in \mathcal{V} : V \text{ is regular and } V \not\subset \tilde{\mathcal{V}}_N\}$.

Figure 4 shows different types of elements in $\tilde{\mathcal{V}}$. We summarize the properties for the open cover $\tilde{\mathcal{V}}$ in the following corollary, which follows from Lemma 7, Lemma 8, Lemma 9, and Lemma 10.

Corollary 1 *Under the hypotheses of Theorem 5, the open sets in $\tilde{\mathcal{V}}$ satisfy the following properties.*

- *$\tilde{V}(g_1)$ and $\tilde{V}(g_2)$ are disjoint for two different $g_1, g_2 \in N$.*
- *For any two open sets $\tilde{V}_1, \tilde{V}_2 \in \tilde{\mathcal{V}}_1 \cup \tilde{\mathcal{V}}_2$, any path in the 1-skeleton of the nerve $N(\tilde{\mathcal{V}})$ connecting $\tilde{V}_1, \tilde{V}_2$ consists of at least two elements from $\tilde{\mathcal{V}}_3$.*

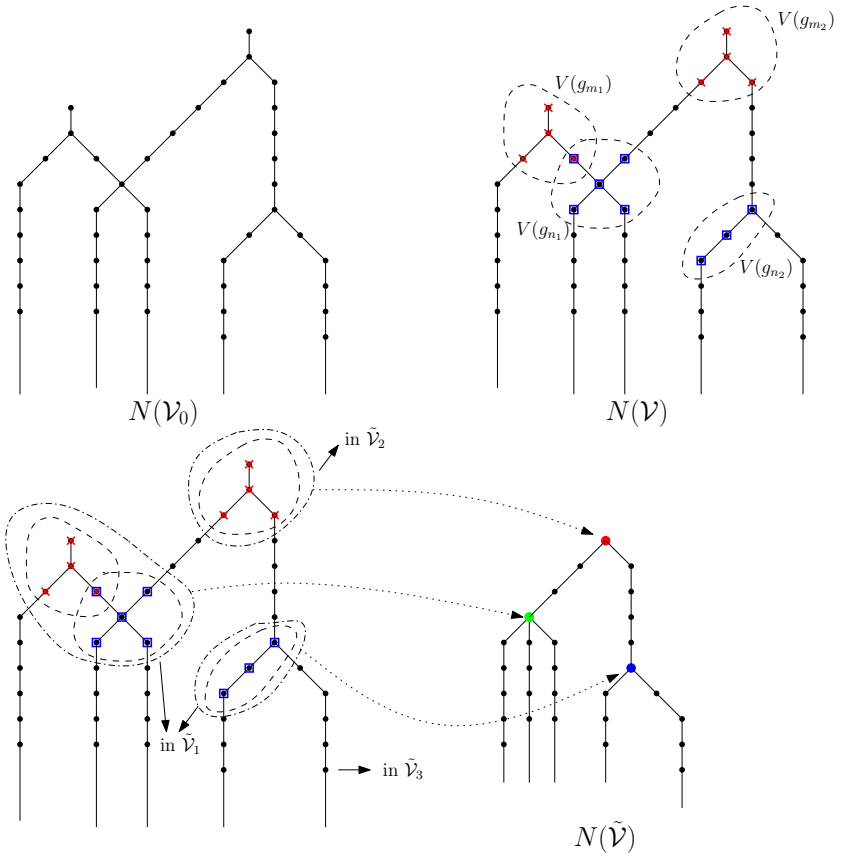


Fig. 4 Illustration of the merging strategy: From top-left to top-right (First step): We select the unions of open sets from $\mathcal{V}_0$ based on the critical points w.r.t. $b(g)$ in G' and merge them respectively; From top-right to bottom-left (Second step): We further merge the unions which are mutually close. Bottom-right: the nerve of the merging result $N(\tilde{\mathcal{V}})$.

– *Any open set $\tilde{V} \in \tilde{\mathcal{V}}_3$ is also a regular element in $\mathcal{V}$ and thus an element in $\mathcal{V}_0$. Moreover any point $g \in C(\tilde{V}) \subset G'$ is at least $\frac{\alpha}{2} - 2\epsilon$ away from any vertex of G'.*

Proposition 4 *Under the hypotheses of Theorem 5, $N(\tilde{\mathcal{V}})$ and $N(\mathcal{V}_0)$ are homotopy equivalent.*

Proof We obtain the elements in the open covering $\tilde{\mathcal{V}}$ by merging a subset of open sets in $\mathcal{V}_0$. Think of any element $\tilde{V} \in \tilde{\mathcal{V}}$ as a subset of $\mathcal{V}_0$. The nerve $N(\mathcal{V}_0)$ restricted to $\tilde{V}$ is a subgraph of $N(\mathcal{V}_0)$, whose vertex set is $\tilde{V}$ and edge set includes the edges in $N(\mathcal{V}_0)$ with both endpoints in $\tilde{V}$. We call this subgraph the nerve of $\tilde{V}$, denoted $N(\tilde{V})$. The nerve of $N(\tilde{\mathcal{V}})$ as a topological space is the quotient space $N(\mathcal{V}_0)/\bigcup_{\tilde{V}\in\tilde{\mathcal{V}}} N(\tilde{V})$. From Proposition 0.17 in [8], it is sufficient to show that $N(\tilde{V})$ is a tree for any $\tilde{V} \in \tilde{\mathcal{V}}$.

For $\tilde{V} \in \tilde{\mathcal{V}}_2 \cup \tilde{\mathcal{V}}_3$, $N(\tilde{V})$ is obviously a tree. Consider $\tilde{V} \in \tilde{\mathcal{V}}_1$. There exists a $g_n \in N$ so that $V(g_n) \subset \tilde{V}(g_n) \in \tilde{V}$. Let V_s^t be the center of $V(g_n)$. For any $g_m \in M$ and $V(g_m) \subset \tilde{V}(g_n)$, if let $v_{s'}^{t'}$ be the center of $V(g_m)$, from Lemma 9, $|s - s'| < 5$. Therefore, if V_l^i is the element in $\tilde{V}$ with the smallest sub-index and V_h^j is the element in $\tilde{V}$ with the largest sub-index, then we have $|h - l| \leq 5 + 5 + 2 + 2 = 14$. From Lemma 6, there is no loop in the subgraph $N(\tilde{V})$. This proves the proposition.

5.2 Construction of open cover for G'

In this section, we construct an open cover G' based on the open cover $\tilde{\mathcal{V}}$ of X. For an open set $V \in \mathcal{V}_0$, we construct a connected open set $U_V \subset G'$ so that $C(V) \subset U_V$ as follows. Let $l = \min\{d(V)\}$ and $u = \max\{d(V)\}$. We have $u - l \leq 2\alpha$. Let $\bar{U} = b^{-1}([l - 2\epsilon, u + 2\epsilon])$, and then $C(V) \subset \bar{U}$. Since $u - l + 4\epsilon < 2\alpha + 4\epsilon < \frac{L}{4}$, one can verify that there is no loop in $\bar{U}$ and thus $\bar{U}$ consists of a set of trees. We claim $C(V)$ is contained in one of the trees. Indeed, for any two $g_1, g_2 \in C(V)$, we have $l - \epsilon < b(g_1), b(g_2) < u + \epsilon$. Now let $x_i \in V$ so that $g_i \in C(x_i)$ for $i = 1, 2$. Let β be a path in V connecting x_1 and x_2. Following Lemma 5, β can trace out a path γ in $\bar{U}$ connecting g_1 and g_2, which implies that $C(V)$ is contained in a tree in $\bar{U}$. Let U_V denote that tree. Let $\mathcal{U}_0 = \{U_V : V \in \mathcal{V}_0\}$. It is obvious that $\mathcal{U}_0$ is an open cover of G'. We now merge the elements in $\mathcal{U}_0$ to construct

a new open cover according to the way in which the elements in $\mathcal{V}_0$ are merged to obtain $\tilde{\mathcal{V}}$. Specifically, from our construction of $\tilde{\mathcal{V}}$, any open set $\tilde{V} \in \tilde{\mathcal{V}}$ is the union of a subset of open sets in $\mathcal{V}_0$. We also denote this subset using $\tilde{V}$. Let $U_{\tilde{V}} = \{U_V : V \in \tilde{V} \subset \mathcal{V}_0\}$. We also denote $U_{\tilde{V}}$ is the open set of the union of the open sets in $U_{\tilde{V}}$.

Consider an open set $\tilde{V} \in \tilde{\mathcal{V}}_3$. As it is also a regular open set in $\mathcal{V}$ and thus an open set in $\mathcal{V}_0$, $d(\tilde{V}) = (p\alpha, (p+2)\alpha)$ for some integer $p > 0$. From Corollary 1, any point in $C(\tilde{V})$ is at least $\frac{\alpha}{2} - 2\epsilon$ away from any vertex in $M \cup N$ and any point in $U_{\tilde{V}}$ is at least $\frac{\alpha}{2} - 4\epsilon$ away from any vertex in $M \cup N$. Thus $U_{\tilde{V}}$ is a segment in G' without any branches. We shrink $U_{\tilde{V}}$ to obtain a new open set $\tilde{U}_{\tilde{V}} = U_{\tilde{V}} \cap b^{-1}(p\alpha + 2\epsilon, (p+2)\alpha - 2\epsilon)$, which is also a segment in G'. For any open set $\tilde{V} \in \tilde{\mathcal{V}}_1 \cup \tilde{\mathcal{V}}_2$, let $\tilde{U}_{\tilde{V}} = U_{\tilde{V}}$. Thus we obtain

$$\tilde{\mathcal{U}} = \{\tilde{U}_{\tilde{V}} : \tilde{V} \in \tilde{\mathcal{V}}\}.$$

One can verify that $\tilde{\mathcal{U}}$ is an open cover of G'. Moreover we have the following two lemmas which relate the nerve $N(\tilde{\mathcal{V}})$ to G'.

Proposition 5 *Under the hypotheses of Theorem 5, the nerve $N(\tilde{\mathcal{V}})$ and the nerve $N(\tilde{\mathcal{U}})$ are isomorphic as graphs.*

Proof It suffices to prove the following three claims.

- Claim (i): For any two $\tilde{V}_i, \tilde{V}_j \in \tilde{\mathcal{V}}_1 \cup \tilde{\mathcal{V}}_2$, $\tilde{U}_{\tilde{V}_i} \cap \tilde{U}_{\tilde{V}_j} = \emptyset$.
  Any path in $N(\tilde{\mathcal{V}})$ connecting $\tilde{V}_i$ and $\tilde{V}_j$ must pass through at least two open sets in $\tilde{\mathcal{V}}_3$, which are regular open sets in $\mathcal{V}$. From Lemma 8, any regular set has two neighbors in the nerve $N(\mathcal{V})$ one lower and one higher, WLOG, assume $\tilde{V}_i$ is higher than $\tilde{V}_j$. We have $\inf\{d(x) : x \in \tilde{V}_i\} \geq \alpha + sup\{d(x) | x \in \tilde{V}_j\}$, which implies $\inf\{b(g) | g \in \tilde{U}_{\tilde{V}_i}\} \geq \alpha + sup\{b(g) | g \in \tilde{U}_{\tilde{V}_j}\} - 2\epsilon > sup\{b(g) | g \in \tilde{U}_{\tilde{V}_j}\}$. Thus $\tilde{U}_{\tilde{V}_i} \cap \tilde{U}_{\tilde{V}_j} = \emptyset$.
- Claim (ii): For any two $\tilde{V}_i, \tilde{V}_j \in \tilde{\mathcal{V}}_3$, $\tilde{V}_i \cap \tilde{V}_j = \emptyset$ if and only if $\tilde{U}_{\tilde{V}_i} \cap \tilde{U}_{\tilde{V}_j} = \emptyset$.
  If $\tilde{V}_i \cap \tilde{V}_j \neq \emptyset$, assume $\tilde{V}_i$ is the only neighboring vertex in the nerve $N(\mathcal{V})$ higher than $\tilde{V}_j$. Let $d(\tilde{V}_j) = (p\alpha, (p+2)\alpha)$ and $d(\tilde{V}_i) = ((p+1)\alpha, (p+3)\alpha)$. Choose a point x from $\tilde{V}_i \cap \tilde{V}_j$ so that $d(x) = (p + \frac{3}{2})\alpha$. We have $C(x) \in \tilde{U}_{\tilde{V}_i} \cap \tilde{U}_{\tilde{V}_j}$, which shows $\tilde{U}_{\tilde{V}_i} \cap \tilde{U}_{\tilde{V}_j} \neq \emptyset$.
  If $\tilde{V}_i \cap \tilde{V}_j = \emptyset$. Let $d(\tilde{V}_i) = (p\alpha, (p+2)\alpha)$ and $d(\tilde{V}_i) = (q\alpha, (q+2)\alpha)$. If $|p - q| \geq 2$, it is obvious that $\tilde{U}_{\tilde{V}_i} \cap \tilde{U}_{\tilde{V}_j} = \emptyset$. Now assume that $q - p \leq 1$, which forces the shortest path connecting $\tilde{V}_i$ and $\tilde{V}_j$ in $N(\tilde{\mathcal{V}})$ must pass through some open set $\tilde{V} \in \tilde{\mathcal{V}}_1 \cup \tilde{\mathcal{V}}_2$. By Lemma 7, for any $g_i \in C(\tilde{V}_i)$ $d_{G'}(g_i, g) \geq \frac{\alpha}{2} - 2\epsilon$ and for any $g_j \in C(\tilde{V}_j)$ $d_{G'}(g_j, g) \geq \frac{\alpha}{2} - 2\epsilon$ for any vertex $g \in M \cup N$ such that $V(g) \in \tilde{V}$. Thus $d_{G'}(g_i, g_j) \geq \alpha - 4\epsilon$, which implies $\tilde{U}_{\tilde{V}_i} \cap \tilde{U}_{\tilde{V}_j} = \emptyset$.
- Claim (iii): For any $\tilde{V}_i \in \tilde{\mathcal{V}}_1 \cup \tilde{\mathcal{V}}_2$ and any $\tilde{V}_j \in \tilde{\mathcal{V}}_3$, $\tilde{V}_i \cap \tilde{V}_j = \emptyset$ if and only if $\tilde{U}_{\tilde{V}_i} \cap \tilde{U}_{\tilde{V}_j} = \emptyset$.
  First assume that $\tilde{V}_i$ and $\tilde{V}_j$ have a non-empty intersection. As $\tilde{V}_j \in \tilde{\mathcal{V}}_3$, it is a regular open set in $\mathcal{V}$ which has one higher neighboring vertex and one lower neighboring vertex in $N(\mathcal{V}_0)$. Since $\tilde{V}_j$ is regular, we have $d(\tilde{V}_j) = (p\alpha, (p+2)\alpha)$ for some integer $p > 0$. We know $\tilde{V}_i$ consists of a subset of open sets in $\mathcal{V}_0$ and let $V \in \tilde{V}_i$ be the open set in $\mathcal{V}_0$ so that $V \cap \tilde{V}_j \neq \emptyset$. WLOG, assume V is the higher neighboring vertex of $\tilde{V}_j$ and we have $d(V \cap \tilde{V}_j) \supset ((p+1)\alpha, (p+2)\alpha)$. We choose a point in $x \in \tilde{V}_j \cap V$ so that $d(x) = (p+2)\alpha - 4\epsilon$. Since $b(C(x)) \subset ((p+2)\alpha - 5\epsilon, (p+2)\alpha - 3\epsilon)$, $C(x) \in \tilde{U}_{\tilde{V}_j} \cap \tilde{U}_{\tilde{V}_i}$ and thus $\tilde{U}_{\tilde{V}_i} \cap \tilde{U}_{\tilde{V}_j} \neq \emptyset$.
  Second assume $\tilde{V}_i \cap \tilde{V}_j = \emptyset$. If any path in the nerve $N(\tilde{\mathcal{V}})$ connecting $\tilde{V}_i$ and $\tilde{V}_j$ passes through some open set in $\tilde{\mathcal{V}}_1 \cup \tilde{\mathcal{V}}_2$, then we are done based on Claim (i). Now assume there is a path β in the nerve $N(\tilde{\mathcal{V}})$ connecting $\tilde{V}_i$ and $\tilde{V}_j$ only passing through open sets in $\tilde{\mathcal{V}}_3$. Since any open set in $\tilde{\mathcal{V}}_3$ is a regular set in $\mathcal{V}$, the worst scenery is that β contains no intermediate open sets. In this worst scenery, due to the shrinking operation on $\tilde{U}_{\tilde{V}_j}$, one can verify that $\tilde{U}_{\tilde{V}_i} \cap \tilde{U}_{\tilde{V}_j} = \emptyset$.

Proposition 6 *Under the hypotheses of Theorem 5, $N(\tilde{\mathcal{U}})$ is homotopy equivalent to G'.*

Proof As we have proved, $\tilde{\mathcal{U}}$ is an open covering of G'. Since any edge on the original G' has a length longer than L, one can verify that any element of $\tilde{\mathcal{U}}$ contains no loop and thus is a tree, and in particular is contractible. Furthermore, the union of any two elements of $\tilde{\mathcal{U}}$ does not contains a loop. This means that if two elements of $\tilde{\mathcal{U}}$ intersect with each other, their intersection is connected and thus contractible. Following from Nerve lemma, we have $N(\tilde{\mathcal{U}})$ is homotopy equivalent to G'.

Proof of Theorem 5. From Proposition 4, Proposition 5, Proposition 6, it remains to show that the nerve $N(\mathcal{V}_0)$ is homotopy equivalent to the α-Reeb graph G. Indeed, we represent each node V_k^l in $N(\mathcal{V}_0)$ using a copy of the interval I_k. If $V_{k_1}^{l_1}$ and $V_{k_2}^{l_2}$ with $k_1 < k_2$ are the endpoints of an edge in $N(\mathcal{V}_0)$, then we glue the upper half of I_{k_1} to the lower half of I_{k_2}. We identify any two points which are glued together directly or indirectly. By definition, the

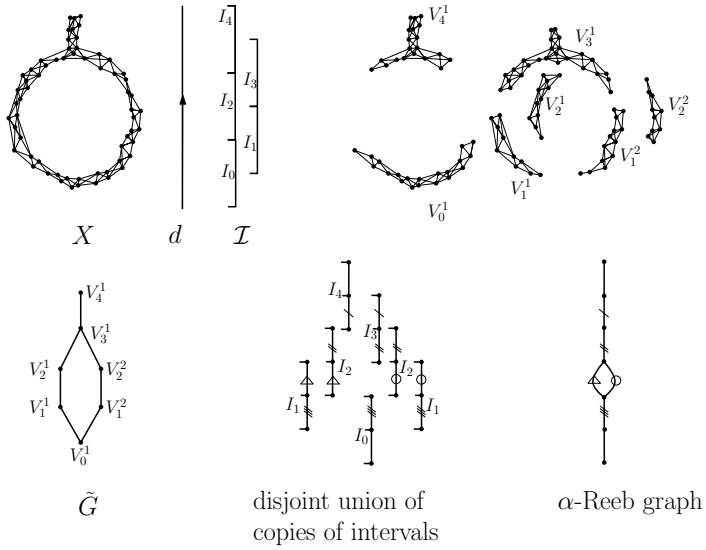


Fig. 5 Illustration of the different steps of the algorithm for computing α-Reeb graph. In the disjoint union of copies of intervals, the subintervals marked with same labels are identified in the α-Reeb graph.

α-Reeb graph is the quotient space of the disjoint union of these intervals - see Figure 5. From Lemma 6, there are more than one node between the top node and the bottom node of any loop in $N(\mathcal{V}_0)$. Thus, we have a one-to-one correspondence between the loops in $N(\mathcal{V}_0)$ and the loops in the α-Reeb graph. This proves the theorem.

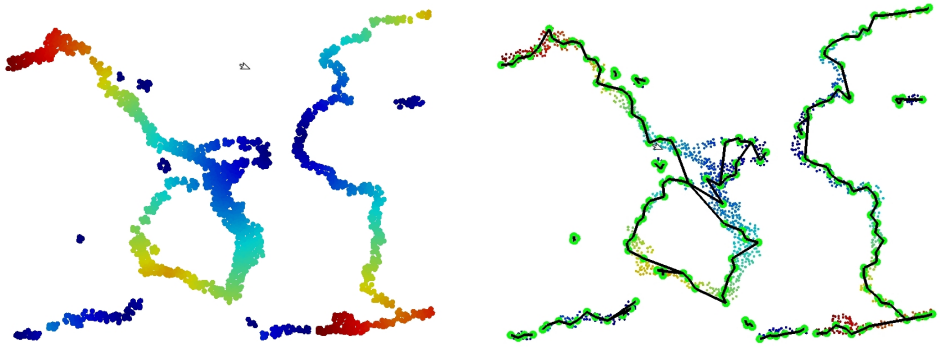
6 Algorithm

In this section, we describe an algorithm for computing the α-Reeb graph for some $\alpha > 0$. We assume the input of the algorithm includes a neighboring graph $X = (V, E)$, a function $l : E \to \mathbb{R}^+$ specifying the edge length and a parameter α. In the applications where the input is given as a set of points together with pairwise distances, i.e., a finite metric space, one can generate the neighboring graph X as a Rips graph of the input points with the parameter chosen as a fraction of α. We assume X is connected as one can apply the algorithm to each connected component otherwise.

Our algorithm, whose different steps are illustrated in Figure 5, can be described as follows. In the first step, we fix a node of X as the root r and then obtain the distance function $d : V \to \mathbb{R}^+$ by computing $d(v)$ as the graph distance from the node v to r. In the second step, we apply the Mapper algorithm [18] to the nodes V with filter d to construct a graph $\tilde{G}$. Specifically, let $\mathcal{I} = \{(i\alpha, (i+1)\alpha), ((i+0.5)\alpha, (i+1.5)\alpha) | 0 \leq i \leq m\}$ so that $\bigcup_{I_k \in \mathcal{I}} I_k$ covers the range of the function d. We say an interval $I_{k_1} \in \mathcal{I}$ is lower than another interval $I_{k_2} \in \mathcal{I}$ if the midpoint of I_{k_1} is smaller than that of I_{k_2}. Now let $V_k = d^{-1}(I_k)$ and V_k^l be the lth component of V_k Then of $\{V_k^l\}_{k,l}$ is a cover of H and the graph $\tilde{G}$ constructed by the Mapper algorithm is the 1-skeleton of the nerve of that cover. Namely, each node in $\tilde{G}$ represents an element in $\{V_k^l\}_{k,l}$. Two nodes $V_{k_1}^{l_1}$ and $V_{k_2}^{l_2}$ are connected with an edge if $V_{k_1}^{l_1} \cap V_{k_2}^{l_2} \neq \emptyset$. In fact, when we check if $V_{k_1}^{l_1} \cap V_{k_2}^{l_2} \neq \emptyset$, we only need to check if their vertices are overlapped or not as we assume the lengths of the edges in H are fractions of α.

In the final step, we represent each node V_k^l in $\tilde{G}$ using a copy of the interval I_k. As mentioned in the Section 3, α-Reeb graph is a quotient space of the disjoint union of those copies of intervals. Specifically, for an edge in $\tilde{G}$, let $V_{k_1}^{l_1}$ and $V_{k_2}^{l_2}$ be its endpoints. Then I_{k_1} and I_{k_2} must be partially overlapped. We identify the overlap part of these two intervals. After identifying the overlapped intervals for all edges in $\tilde{G}$, the resulting quotient space is the α-Reeb graph. Algorithmically, the identification is performed as follows. We split each copy of internal I_k into two by adding a point in the middle. Now think of it as a graph with two edges and label one of them upper and the other lower. Notice that two overlapped intervals I_{k_1} and I_{k_2} can not be exactly the same. One must be lower than the other. To identify their overlapped part, we identify the upper edge of the lower interval with the lower edge of the upper interval.

The time complexity of the above algorithm is dominated by the computation of the distance function in the first step, which is $O(|E| + |V| \log |V|)$. The computation of the connected components in the second step is $O(|V| \log |V|)$ based on union-find data structure. In the final step, there are at most $O(|V|)$ number of the copies of the intervals. Based on union-find data structure, the identification can also be performed in $O(|V| \log |V|)$ time.



(a) (b)

Fig. 6 Earthquake data - (a) The distance functions d on each connected components. The value increases from cold to warm colors. (b) The reconstructed α-Reeb graph.

	#OP	#OE	#N	#E	GRT	ODT	ADT	Mean	Median
GPS traces	82541	313415	21644	21554	46.8	15.27	0.96	6.5%	5.3%
Earthquake	1600	26996	147	137	0.32	1.12	0.01	14.1%	12.5%

Table 1 #OP (#OE, #N, #E) stands for the number of original points (original edges, nodes, edges in α-Reeb graph). The graph reconstruction time (GRT) is the total time of computing distance function and reconstructing the graph. The original (ODT), respectively approximate (ADT), distance computation time shows the total time of computing these distances using the original, respectively reconstructed, graph. All times are in seconds. The last two columns show the mean and median metric distortions.

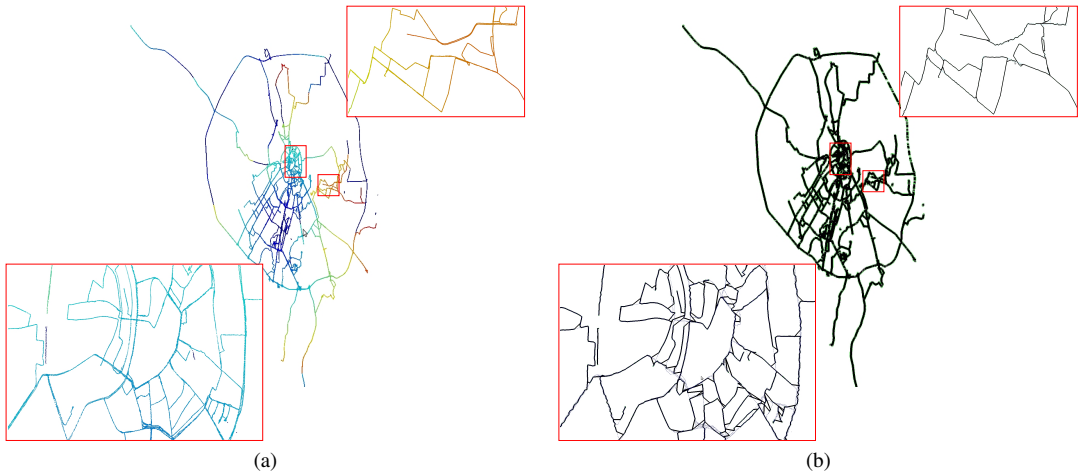
7 Experiments

In this section, we illustrate the performances of our algorithm on three different data sets. The first data set was obtained from USGS Earthquake Search [36]. It consists of earthquake epicenter locations collected, between 01/01/1970 and 01/01/2010 in the rectangular area between latitudes -75 degrees and 75 degrees and longitude -170 degrees and 10 degrees with magnitude greater than 5.0. This raw earthquake data set contains the coordinates of the epicenters of 12790 earthquakes that are mainly located around geological faults. We follow the procedure described in [28] to remove outliers and randomly sampled 1600 landmarks. Finally, we computed a neighboring graph from these landmarks with parameter 4. The length of an edge in this graph is the Euclidean distance between its endpoints. For each connected component, we fix a root point and compute the graph distance function d to the root point as shown in Figure 6(a). We also set $\alpha = 4$ and apply our algorithm to the above data to obtain the α-Reeb graph. In general, the α-Reeb graph is an abstract metric graph. In this example, for the purpose of visualization, we use the coordinates of the landmarks to embed the graph into the plane as follows. Recall that for a copy of interval I_k representing the node V_k^l in $\tilde{G}$, we split it into two by adding a point in the middle. We embed the endpoints of the interval to the landmarks of the minimum and the maximum of the function d in V_k^l, and the point in the middle to the landmark of the median of the function d in V_k^l. Figure 6(b) shows the embedding of the α-Reeb graph. Note this embedding may introduce metric distortion, i.e., the Euclidean length of the edge may not reflect the length of the corresponding edge in the α-Reeb graph.

The second data set is that of 500 GPS traces tagged "Moscow" from OpenStreetMap [37]. Since cars move on roads, we expect the locations of cars to provide information about the metric graph structure of the Moscow road network. We first selected a metric ϵ-net on the raw GPS locations with $\epsilon = 0.0001$ using furthest point sampling. Then, we computed a neighboring graph from the samples with parameter 0.0004. Again for each connected component, we fix a root point and compute the graph distance function d to the root point as shown in Figure 7(a). Set α also equals 0.0004 and compute the α-Reeb graph. Again, we use the same method as above to embed the α-Reeb graph into the plane, as shown in Figure 7(b).

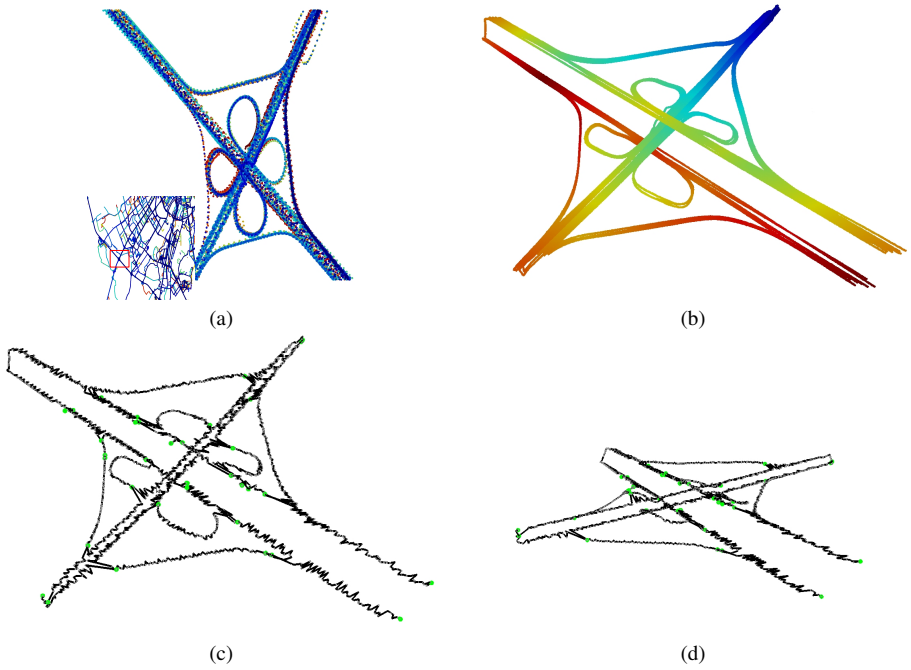
To evaluate the quality of our α-Reeb graph for each data set, we computed both original pairwise distances, and pairwise distances approximated from the constructed α-Reeb graph. For GPS traces, we randomly select 100 points as the data set is too big to compute all pairwise distances. We also evaluated the use of α-Reeb graph to speed up distance computations by showing reductions in computation time. Only pairs of points in the same connected component are included because we obtain zero error for the pairs of vertices that are not. Statistics for the size of the reconstructed graph, error of approximate distances, and reduction in computation time are given in Table 1.

The third data set we consider is also obtained from GPS traces. Roads are often split so that cars in different directions run in different lanes. In particular, this is the true for highways. In addition, when two roads cross in



(a) (b)

Fig. 7 GPS data - (a) The distance functions d on each connected components. The value increases from cold to warm colors. (b) The reconstructed α-Reeb graph.



(a) (b)

(c) (d)

Fig. 8 (a) GPS traces passing through a highway crossing in Moscow. (b) The distance function. (c) and (d)The reconstructed α-Reeb graph viewed from two perspectives.

GPS coordinates, they may bypass through a tunnel or an evaluated bridge and thus the road network itself may not cross. Such directional information is contained in the GPS traces. We encode this directional information by stacking several consecutive GPS coordinates to form a point in a higher dimensional space. In this way, we obtain a new set of points in this higher dimension space. Then we build a neighboring graph for this new set of points based on L_2 norm and apply our algorithm to recover the road network. In particular, although the paths intersect at the cross in GPS coordinates, the road network does not and this should be detected by our algorithm. To test the above strategy, we extract those GPS traces from the above "Moscow" dataset which pass through a highway crossing as shown in Figure 8(a). Since GPS records the position based on time, we resample the traces so that the distances between any two consecutive samples is the same among all traces. Then we apply the above algorithm to the resampled traces. Figure 8(c) and (d) show the reconstructed graph which recovers the road network of this highway crossing.

8 Discussion

We have proposed a method to approximate path metric spaces using metric graphs with bounded Gromov-Hausdorff distortion, and illustrated the performances of our method on a few data sets. Here we point out a few possible directions for future work. First, notice that the α-Reeb graph is a quotient space where the quotient map is 1-Lipschitz and thus the metric only gets contracted. In addition, the distance from a point to the chosen root is exactly preserved. Therefore, one always reduces the metric distortion by taking the maximum of the graph metrics of different root points. It is interesting to study the strategy of sampling root points to obtain the smallest metric distortion with the fixed number of root points. Second, our method is sensitive to the noise. One can preprocess the data and remove the noise and then apply our algorithm. Nevertheless, it is interesting to see if the algorithm can be improved to handle noise.

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