Voronoi diagrams in nature
The solar system (Descartes)
Growth of merystem
Euclidean Voronoi diagrams

Voronoi cell

\[ V(p_i) = \{ x : \| x - p_i \| \leq \| x - p_j \|, \ \forall j \} \]

Voronoi diagram \((P)\)

\[ = \{ \text{collection of all cells } V(p_i), p_i \in P \} \]
Voronoi diagrams and polytopes

Polytope

The intersection of a finite collection of half-spaces:

\[ V = \bigcap_{i \in I} h_i^+ \]

- Each Voronoi cell is a polytope
- The Voronoi diagram has the structure of a cell complex
- The Voronoi diagram of \( P \) is the projection of a polytope of \( \mathbb{R}^{d+1} \)
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Voronoi diagrams and polyhedra

- \text{Vor}(p_1, \ldots, p_n) \text{ is the minimization diagram of the } n \text{ functions } \delta_i(x) = (x - p_i)^2

- \text{arg min}(\delta_i) = \text{arg max}(h_i)
  \text{where } h_{p_i}(x) = 2p_i \cdot x - p_i^2

- The minimization diagram of the \( \delta_i \) is also the maximization diagram of the \text{affine functions } h_{p_i}(x)

- The faces of \text{Vor}(P) \text{ are the projections of the faces of } \mathcal{V}(P) = \bigcap_i h_{p_i}^+

  \begin{align*}
  h_{p_i}^+ &= \{x : x_{d+1} > 2p_i \cdot x - p_i^2\}
  \end{align*}

Note!

the graph of \( h_{p_i}(x) \) is the hyperplane tangent to \( Q : x_{d+1} = x^2 \) at \( (x, x^2) \)
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Lifting map

The faces of $\text{Vor}(P)$ are the projection of the faces of the polytope

$$\mathcal{V}(P) = \bigcap_i h_{p_i}^+$$

where $h_{p_i}$ is the hyperplane tangent to paraboloid $Q$ at the lifted point $(p_i, p_i^2)$

Corollaries

- The size of $\text{Vor}(P)$ is the same as the size of $\mathcal{V}(P)$
- Computing $\text{Vor}(P)$ reduces to computing $\mathcal{V}(P)$
**Lifting map**

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**Corollaries**

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Voronoi diagram and Delaunay triangulation

Finite set of points $P \in \mathbb{R}^d$

- The Delaunay complex is the nerve of the Voronoi diagram
- It is not always embedded in $\mathbb{R}^d$
Empty circumballs

An (open) $d$-ball $B$ circumscribing a simplex $\sigma \subset \mathcal{P}$ is called empty if:

1. $\text{vert}(\sigma) \subset \partial B$
2. $B \cap \mathcal{P} = \emptyset$

$\text{Del}(\mathcal{P})$ is the collection of simplices admitting an empty circumball.
Point sets in general position wrt spheres

$P = \{p_1, p_2 \ldots p_n\}$ is said to be in general position wrt spheres if

$\not\exists \ d + 2$ points of $P$ lying on a same $(d - 1)$-sphere

Theorem [Delaunay 1936]

If $P$ is in general position wrt spheres, the natural mapping

$$f : \text{vert(Del}P) \rightarrow P$$

provides a realization of Del$(P)$ called the Delaunay triangulation of $P$. 
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**Theorem [Delaunay 1936]**

If \( P \) is in general position wrt spheres, the natural mapping

\[ f : \text{vert}(\text{Del}P) \to P \]

provides a realization of \( \text{Del}(P) \) called the Delaunay triangulation of \( P \).
Proof of Delaunay’s theorem 1

Linearization

\[ S(x) = x^2 - 2c \cdot x + s, \quad s = c^2 - r^2 \]

\[ S(x) < 0 \iff \begin{cases} z < 2c \cdot x - s \\ z = x^2 \end{cases} \quad (h_S^-) \quad (P) \]

\[ \iff \hat{x} = (x, x^2) \in h_S^- \]
Proof of Delaunay’s theorem 2

$P$ general position wrt spheres $\iff \hat{P}$ in general position

$\sigma$ a simplex, $S_\sigma$ its circumscribing sphere

$\sigma \in \text{Del}(P) \iff S_\sigma$ empty

$\iff \forall i, \hat{p}_i \in h_{S_\sigma}^+$

$\iff \hat{\sigma}$ is a face of $\text{conv}^- (\hat{P})$

$\text{Del}(P) = \text{proj} (\text{conv}^- (\hat{P}))$
Proof of Delaunay’s theorem 2

Proof of Delaunay’s th.

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$\iff \hat{\sigma}$ is a face of $\text{conv}^-(\hat{P})$

$\text{Del}(P) = \text{proj}(\text{conv}^-(\hat{P}))$
\[ \mathcal{V}(P) = \cap_i h^+_{p_i} \]
\[ \mathcal{D}(P) = \text{conv}^-(\hat{P}) \]
If $P$ is in general position wrt spheres:

$$V(P) = h_{p_1}^+ \cap \ldots \cap h_{p_n}^+ \quad \text{duality} \quad D(P) = \text{conv}^{-}(\{\hat{p}_1, \ldots, \hat{p}_n\})$$

**Voronoi Diagram of** $P$ \quad \text{nerve} \quad **Delaunay Complex of** $P$
The combinatorial complexity of the Delaunay triangulation diagram of $n$ points of $\mathbb{R}^d$ is the same as the combinatorial complexity of a convex hull of $n$ points of $\mathbb{R}^{d+1}$.

$$\Theta\left(n \left\lceil \frac{d}{2} \right\rceil \right)$$

Quadratic in $\mathbb{R}^3$
Constructing $\text{Del}(P)$, $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$

Algorithm

1. Lift the points of $P$ onto the paraboloid $x_{d+1} = x^2$ of $\mathbb{R}^{d+1}$:
   \[ p_i \rightarrow \hat{p}_i = (p_i, p_i^2) \]
2. Compute $\text{conv}(\{\hat{p}_i\})$
3. Project the lower hull $\text{conv}^-(\{\hat{p}_i\})$ onto $\mathbb{R}^d$

Complexity: $\Theta(n \log n + n^{\left\lfloor \frac{d+1}{2} \right\rfloor})$
Direct algorithm: insertion of a new point $p_i$

1. Location: find all the $d$-simplices that conflict with $p_i$
i.e. whose circumscribing ball contains $p_i$

2. Update: construct the new $d$-simplices
Updating the adjacency graph

We look at the $d$-simplices to be removed and at their neighbors.

Each $d$-simplex is considered $\leq \frac{d(d+1)}{2}$ times.

Update cost $= O(\# \text{ created and deleted simplices} )$

$= O(\# \text{ created simplices} )$
Some optimality properties of Delaunay triangulations

Among all possible triangulations of $\mathcal{P}$, $\text{Del}(\mathcal{P})$

1. (2d) maximizes the smallest angle [Lawson]

2. (2d) Linear interpolation of $\{(p_i, f(p_i))\}$ that minimizes [Rippa]

$$R(T) = \sum_i \int_{T_i} \left( \left( \frac{\partial \phi_i}{\partial x} \right)^2 + \left( \frac{\partial \phi_i}{\partial y} \right)^2 \right) \, dx \, dy$$

(*Dirichlet energy*)

$\phi_i =$ linear interpolation of the $f(p_j)$ over triangle $T_i \in T$

3. minimizes the radius of the maximal smallest ball enclosing a simplex ) [Rajan]
Optimizing the angular vector \((d = 2)\)

Angular vector of a triangulation \(T(\mathcal{P})\)

\[
\text{ang} (T(\mathcal{P})) = (\alpha_1, \ldots, \alpha_{3t}), \quad \alpha_1 \leq \ldots \leq \alpha_{3t}
\]

Optimality  Any triangulation of a given point set \(\mathcal{P}\) whose angular vector is maximal (for the lexicographic order) is a Delaunay triangulation of \(\mathcal{P}\)

Good for matrix conditioning in FE methods
Local characterization of Delaunay complexes

Pair of regular simplices

\[ \sigma_2(q_1) \geq 0 \quad \text{and} \quad \sigma_1(q_2) \geq 0 \]

\[ \Leftrightarrow \hat{c}_1 \in h_{\sigma_2}^+ \quad \text{and} \quad \hat{c}_2 \in h_{\sigma_1}^+ \]

**Theorem** A triangulation \( T(P) \) such that all pairs of simplexes are regular is a Delaunay triangulation \( \text{Del}(P) \)

**Proof** The PL function whose graph \( G \) is obtained by lifting the triangles is locally convex and has a convex support

\[ \Rightarrow \quad G = \text{conv}^-(Q) \quad \Rightarrow \quad T(Q) = \text{Del}(Q) \]
Lawson’s proof using flips

While \( \exists \) a non regular pair \((t_3, t_4)\)
/* \( t_3 \cup t_4 \) is convex */
replace \((t_3, t_4)\) by \((t_1, t_2)\)

**Regularize \(\Leftrightarrow\) improve \(\text{ang}(T(P))\)**
\[
\text{ang}(t_1, t_2) \geq \text{ang}(t_3, t_4)
\]
\[
a_1 = a_3 + a_4, \quad d_2 = d_3 + d_4,
\]
\[
c_1 \geq d_3, \quad b_1 \geq d_4, \quad b_2 \geq a_4, \quad c_2 \geq a_3
\]

- The algorithm terminates since the number of triangulations of \(P\) is finite and \(\text{ang}(T(P))\) cannot decrease
- The obtained triangulation is a Delaunay triangulation of \(P\) since all its edges are regular
Meshing surfaces and 3D domains

- visualization and graphics applications
- CAD and reverse engineering
- geometric modelling in medicine, geology, biology etc.
- autonomous exploration and mapping (SLAM)
- scientific computing: meshes for FEM
Grid methods

Lorensen & Cline [87] : marching cube
Lopez & Brodlie [03] : topological consistency
Plantiga & Vegter [04] : certified topology using interval arithmetic

Morse theory

Stander & Hart [97]
B., Cohen-Steiner & Vegter [04] : certified topology

Delaunay refinement

Hermeline [84]
Ruppert [95]
Shewchuk [02]

Chew [93]
B. & Oudot [03,04]
Cheng et al. [04]
Main issues

Sampling
- How do we choose points in the domain?
- What information do we need to know/measure about the domain?

Meshing
- How do we connect the points?
- Under what sampling conditions can we compute a good approximation of the domain?
Scale and dimension
Sampling conditions

[Federer 1958], [Amenta & Bern 1998]

Medial axis of $\mathcal{M}$: $\text{axis}(\mathcal{M})$

set of points with at least two closest points on $\mathcal{M}$

Local feature size and reach

$\forall x \in \mathcal{M}, \ lfs(x) = d(x, \text{axis}(\mathcal{M}))$

$rch(\mathcal{M}) = \inf_{x \in \mathcal{M}} lfs(x)$

$(\epsilon, \bar{\eta})$-net of $\mathcal{M}$

1. $\mathcal{P} \subset \mathcal{M}, \forall x \in \mathcal{M}: \ d(x, \mathcal{P}) \leq \epsilon \ lfs(x)$
2. $\forall p, q \in \mathcal{P}, \ \|p - q\| \geq \bar{\eta} \epsilon \ \min(lfs(p), lfs(q))$
Definition

The restricted Delaunay triangulation $\text{Del}_X(P)$ to $X \subset \mathbb{R}^d$ is the nerve of $\text{Vor}(P) \cap X$.

If $P$ is an $\varepsilon$-sample, any ball centered on $X$ that circumscribes a facet $f$ of $\text{Del}_X(P)$ has a radius $\leq \varepsilon \text{Ifs}(c_f)$. 

[Chew 93]
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Restricted Delaunay triangulation
Restricted Delaunay triang. of \((\varepsilon, \bar{\eta})\)-nets

[Amenta et al. 1998-], [B. & Oudot 2005]

If

- \(S \subset \mathbb{R}^3\) is a compact surface of positive reach without boundary
- \(\mathcal{P}\) is an \((\varepsilon, \bar{\eta})\)-net with \(\bar{\eta} \geq \text{cst}\) and \(\varepsilon\) small enough

then

- \(\text{Del}_{|S}(S)\) provides good estimates of normals
- There exists a homeomorphism \(\phi: \text{Del}_{|S}(\mathcal{P}) \rightarrow S\)
- \(\sup_x (\|\phi(x) - x\|) = O(\varepsilon^2)\)
Schwarz lantern