Computational Geometry Learning
Exam

MPRI

Duration 2h30

You are allowed access to the official course notes, the slides, and any personal notes.

1 Minimal enclosing shell

We consider sets of points in $\mathbb{R}^2$ in generic position. A *shell* is a region in the plane between two concentric circles. An *enclosing shell* for a point set $\mathcal{P}$ is a shell that contains all the points of $\mathcal{P}$, see Figure 1. We define a *minimal enclosing shell (MES)* of $\mathcal{P}$ as an enclosing shell with minimal area.

Answer the following questions, with justification, in any convenient order.

1. Does there always exist a MES?
2. Is it unique?
3. Describe without justification a MES for the point sets in Figures 2 and 3 and give their area.
4. Give an efficient algorithm to compute a MES. Hint: find a change of variables that makes everything linear.
5. Does this generalize to higher dimensions?

2 Voronoi counting

Terminology reminder: for a set of points in the plane, its Voronoi diagram consists of 0-cells (vertices), 1-cells (segments) and 2-cells (polygons).

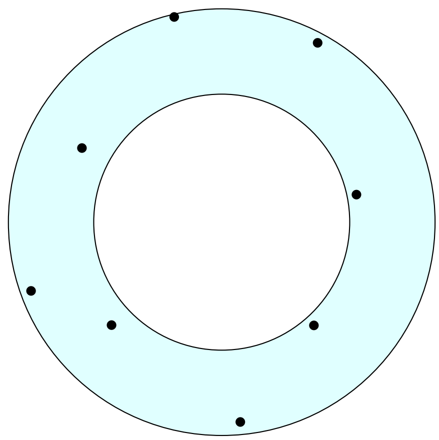

Figure 1: An enclosing shell

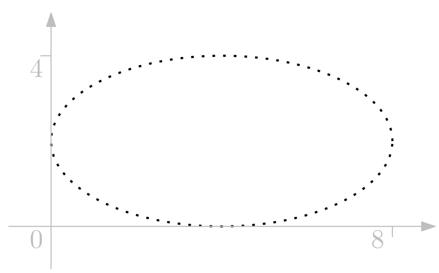

Figure 2: First point set

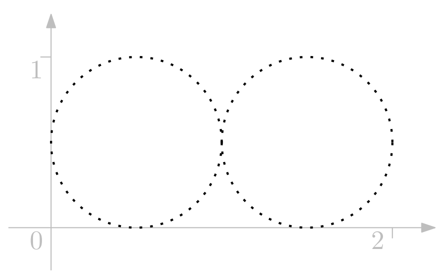

Figure 3: Second point set

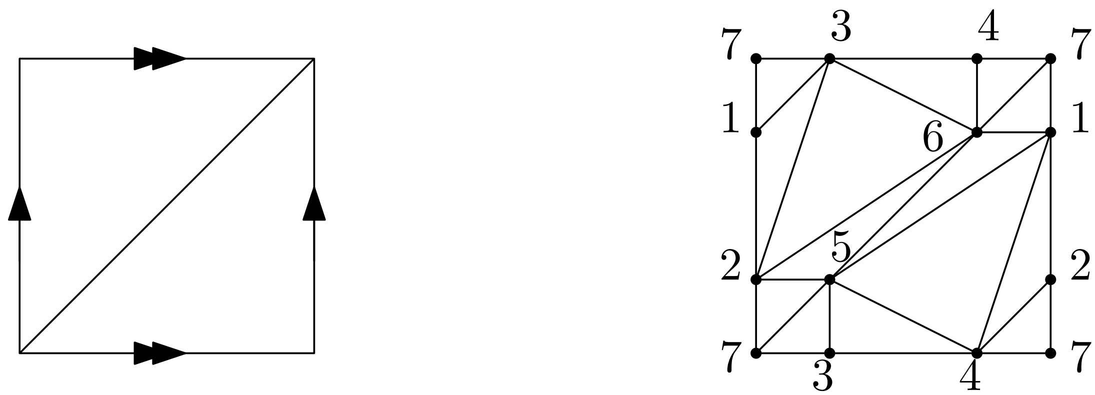

Figure 4: Left: Torus obtained by identifying the edges of two triangles. Right: simplicial complex triangulating the torus.

For a set of n points in $\mathbb{R}^d$ in generic position, what is the *maximum* number of 4-cells that a Voronoi diagram can have, as a function of n and $d \geq 1$? Give some justification.

3 Problem

A torus is the surface obtained from gluing the edges of a square respecting the gluing pattern of Figure 4 (left), i.e., by identifying edges with same arrow pattern in pairs, without reverting arrows.

- Show that Figure 4 (left) is not a simplicial complex.
- Figure 4 (right) gives a simplicial complex of the torus. Give the number of simplices of each dimension of Figure 4 (right), and compute explicitly the Euler characteristic of the torus.
- Denote by v, e, and f the number of vertices, edges, and triangles of a simplicial complex, and χ its Euler characteristic. Prove that any simplicial complex triangulating a compact surface with no boundary satisfies:
  - $2e = 3f$,
  - $e = 3(v - \chi)$,
  - $e \leq \frac{1}{2}v(v-1)$.
- Prove that the simplicial complex triangulating the torus in Figure 4 has the minimal number of triangles.

- We say that a simplex $\sigma_1 \leq_{\text{lex}} \sigma_2$ iff the string $x_1 x_2 \ldots x_d \leq_{\text{lex}} y_1 y_2 \ldots y_{d'}$ in lexicographic order, where the x_i are the labels of vertices of σ_1 (i.e., $\sigma_1 = \{x_1, x_2, \ldots, x_d\}$), the y_j are the labels of vertices of σ_2 (i.e., $\sigma_2 = \{y_1, \ldots, y_{d'}\}$), and the labels of vertices satisfy $x_i < x_{i+1}$ and $y_j < y_{j+1}$ for all appropriate i and j. We define similarly $<_{\text{lex}}$ and $=_{\text{lex}}$.

  We say that pairs of simplices $(\tau_1, \sigma_1) \leq_{\text{lex}} (\tau_2, \sigma_2)$ iff, either $\tau_1 <_{\text{lex}} \tau_2$, or $\tau_1 =_{\text{lex}} \tau_2$ and $\sigma_1 \leq_{\text{lex}} \sigma_2$.

  Consider the following heuristic to compute a Morse matching (which is a small adaptation of the one seen in class):

  - A set of *available simplices*; $\quad A \hookleftarrow \mathbf{K}, \quad X, T, S \hookleftarrow \emptyset$
  - **While** $A \neq \emptyset$:
    - **if** there is a <u>free pair</u> (τ, σ), $\tau \subset \sigma$, in A, pick the minimal such pair (τ_0, σ_0) in lex order, and match τ_0 with σ_0:
      - $A \hookleftarrow A \setminus \{\tau_0, \sigma_0\}$,
      - $T \hookleftarrow T \cup \{\tau_0\}$,
      - $S \hookleftarrow S \cup \{\sigma_0\} \qquad \Rightarrow \omega(\tau_0) = \sigma_0$
    - **else** pick the <u>*maximal*</u> (w.r.t. to inclusion) simplex $\zeta_0 \in A$ that is minimal in lex order, and make it critical:
      - set $X \hookleftarrow X \cup \{\zeta_0\}$,
      - $A \hookleftarrow A \setminus \{\zeta_0\}$.

  Explain why this computes a Morse matching.

- Run the heuristic above by hand to compute a Morse matching for the simplicial complex in Figure 4 [left]. Detail explicitly the first five iterations of the **while** loop, with the following syntax: for $1 \leq i \leq 5$,

  "i: make xyz critical/pair $xy \leftrightarrow xyz$".

  Give explicitly the sets X and all the pairs $\tau \leftrightarrow \omega(\tau)$ at the end of computation.

- Give the boundary matrices ∂_d^X in $\mathbb{Z}/2\mathbb{Z}$ for the Morse complex obtained by the heuristic, for all $d \geq 0$.

- Give an explicit basis for the kernel and images of each boundary map ∂_d^X, and prove that for the torus, $\beta_0 = 1$, $\beta_1 = 2$, and $\beta_2 = 1$ (with $\mathbb{Z}/2\mathbb{Z}$ coefficients).

- Deduce from the construction above the $\mathbb{Z}/2\mathbb{Z}$ homology of the torus, minus the inside of one triangle (any triangle). Justify.

- Give a triangulation of the circle with minimal number of edges, and compute its homology.

- A *short exact sequence of chain complexes* is a sequence:

$$0 \xrightarrow{\quad 0 \quad} C \xrightarrow{\quad \phi \quad} D \xrightarrow{\quad \psi \quad} E \xrightarrow{\quad 0 \quad} 0$$

of five chain complexes and four chain maps, where for any two consecutive chain maps:

$$X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad g \quad} Z,$$

we have $\mathrm{im} f = \ker g$.

Let $\mathbf{K}$ be a simplicial complex, and $\mathbf{K}_1, \mathbf{K}_2 \subseteq \mathbf{K}$ two sub complexes of $\mathbf{K}$. Prove that $\mathbf{K}_1 \cup \mathbf{K}_2$ and $\mathbf{K}_1 \cap \mathbf{K}_2$ are simplicial complexes, and design a short exact sequence for the chain complexes:

$$0 \xrightarrow{\quad 0 \quad} \mathbf{C}(\mathbf{K}_1 \cap \mathbf{K}_2) \xrightarrow{\quad \phi \quad} \mathbf{C}(\mathbf{K}_1) \oplus \mathbf{C}(\mathbf{K}_2) \xrightarrow{\quad \psi \quad} \mathbf{C}(\mathbf{K}_1 \cup \mathbf{K}_2) \xrightarrow{\quad 0 \quad} 0,$$

where the boundary maps of $\mathbf{C}(\mathbf{K}_1) \oplus \mathbf{C}(\mathbf{K}_2)$ is $\partial_1 \oplus \partial_2$, for ∂_1 the boundary map of $\mathbf{K}_1$, and ∂_2 the boundary map for $\mathbf{K}_2$.

Prove that the maps ϕ and ψ you have designed are chain maps, and prove that the sequence is a short exact sequence.

- A *long exact sequence of vector spaces* is a (possibly infinite) sequence of vector spaces and linear maps:

$$\cdots \xrightarrow{\quad f_i \quad} V_i \xrightarrow{\quad f_{i-1} \quad} V_{i-1} \xrightarrow{\quad f_{i-2} \quad} \cdots$$

satisfying $\mathrm{im} f_i = \ker f_{i-1}$.

We accept the following lemma:

Lemma. If:

$$0 \xrightarrow{\quad 0 \quad} C \xrightarrow{\quad \phi \quad} D \xrightarrow{\quad \psi \quad} E \xrightarrow{\quad 0 \quad} 0$$

is a short exact sequence of chain complexes, then there exists the following long exact sequence of their homology groups:

$$\cdots \longrightarrow \mathbf{H}_d(C) \longrightarrow \mathbf{H}_d(D) \longrightarrow \mathbf{H}_d(E) \longrightarrow \mathbf{H}_{d-1}(C) \longrightarrow \cdots,$$

for all integers $d \in \mathbb{Z}$.

Compute the $\mathbb{Z}/2\mathbb{Z}$-Betti numbers of the double torus in Figure 5 using the lemma above, and all results obtained so far. Prove formally your statement. Recall that for a finite simplicial complex, $\mathbf{H}_d = 0$ for all $d < 0$.

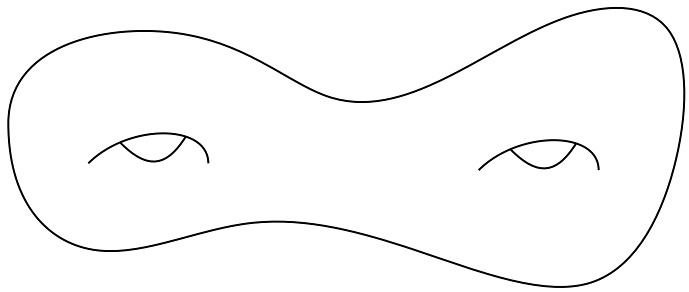

Figure 5: A double torus is obtained by taking two tori, removing a disk in each of them, and gluing the two circle boundaries together.

4 Exercise

We call a *persistence diagram* a multi-set of points $\{(x_i, y_i) : i \in I\} \cup \{(x, x) : x \in \mathbb{R}_{\geq 0}\}$, for a set of indices I possibly infinite and every pair (x_i, y_i) satisfying $0 \leq x_i \leq y_i < +\infty$ for all $i \in I$.

- Show that the bottleneck distance between two persistence diagrams satisfies the triangle inequality.
- Prove that two *distinct* persistence diagrams (as multi-sets) can be at bottleneck distance 0 from one another.
- Prove that two persistence diagrams:

  $$D_1 = \{(x_i^{(1)}, y_i^{(1)}) : i \in I\} \cup \{(x, x) : x \in \mathbb{R}_{\geq 0}\} \quad \text{and} \quad D_2 = \{(x_j^{(2)}, y_j^{(2)}) : j \in J\} \cup \{(x, x) : x \in \mathbb{R}_{\geq 0}\}$$

  that are distinct as multi-sets, and for which additionally $x_i^{(1)} < y_i^{(1)}$ and $x_j^{(2)} < y_j^{(2)}$ for all $i \in I$ and $j \in J$, can be at bottleneck distance 0 from one another.

5 Exercise

Give the persistence diagram (with filtration values based on distances) of the Rips filtration given by the point cloud of Figure 6 in the Euclidean plane. Detail your calculation.

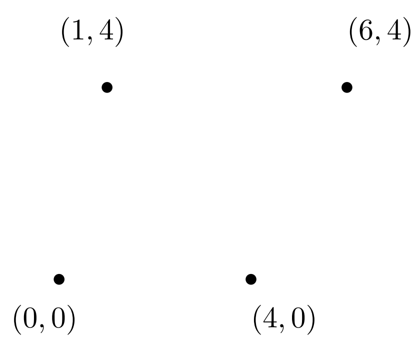

Figure 6: Point cloud in the Euclidean plane.