Topological Data Analysis — Exam

April 07, 2016

1 Bottleneck distance

**Question 1.** Show that the bottleneck distance satisfies the triangle inequality.

**Question 2.** Show that the bottleneck distance is only a pseudodistance, that is: exhibit a pair of distinct persistence diagrams whose distance is zero.

2 Euler characteristic

Given a topological space $X$ and a field $k$, the *Euler characteristic* is the quantity:

$$
\chi(X; k) = \sum_{i=0}^{+\infty} (-1)^i \dim H_i(X; k).
$$

**Question 3.** Show that $\chi$ is a topological invariant, that is: for any spaces $X, Y$ that are homotopy equivalent, $\chi(X; k) = \chi(Y; k)$.

**Hint:** look at what happens to each individual homology group.

Now we want to prove the Euler-Poincaré theorem:

**Theorem 1.** For any simplicial complex $X$ and any field $k$:

$$
\chi(X; k) = \sum_{i=0}^{+\infty} (-1)^i n_i(X),
$$

where $n_i(X)$ denotes the number of simplices of $X$ of dimension $i$.

For this we will use topological persistence. Consider an arbitrary filtration of $X$:

$$
\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_m = X.
$$

Assume without loss of generality that a single simplex $\sigma_j$ is inserted at each step $j$:

$$
\forall j = 1, \cdots, m, \ X_j \setminus X_{j-1} = \{\sigma_j\}.
$$

Note that $m$ is then equal to the number of simplices of $X$, that is:

$$
m = \sum_{i=0}^{+\infty} n_i(X).
$$

Let us apply the persistence algorithm to this simplicial filtration. Recall that we have the following property:
Lemma 1. At each step $j$, the insertion of simplex $\sigma_j$ either creates an independent $d_j$-dimensional cycle (i.e. increases the dimension of $H_{d_j}(X_{j-1};k)$ by 1) or kills a $(d_j - 1)$-dimensional cycle (i.e. decreases the dimension of $H_{d_j-1}(X_{j-1};k)$ by 1), where $d_j$ is the dimension of $\sigma_j$.

Question 4. Using Lemma 1, prove Theorem 1. 
*Hint:* proceed by induction on $j$.

Question 5. Deduce that the Euler characteristic of a triangulable space is independent of the choice of field $k$.

3 Reeb graph and Mapper

Consider the function $f$ depicted on the left-hand side of Figure 1. Note that its domain $X$ (the gray area) is a subset of the plane, not a torus.

![Figure 1: Left: the height function $f$ on a planar domain. Right: the interval cover.](image)

Question 6. Compute the Reeb graph of $f$. You may simply draw it with labels.

Question 7. Compute the extended persistence diagram of the map $f$.
*Hints:* homology of dimension 2 and above is trivial; the values $a, \cdots, l$ are paired up to form the diagram points.

Question 8. Prune the diagram to obtain the extended persistence diagram of the quotient map $\tilde{f}$. What do you observe? Explain.

Question 9. Consider now the interval cover of $\text{Im } f$ depicted on the right-hand side of Figure 1. Compute the corresponding Mapper.

Question 10. Relate the Mapper to the Reeb graph: in your opinion, to which points of the extended persistence diagram of $\tilde{f}$ do the topological features of the Mapper correspond?
4 The category of persistence modules

A morphism between two persistence modules $V$ and $W$ over $T$ is a family $\Phi$ of linear maps $(\phi_t : V_t \rightarrow W_t)_{t \in T}$ such that the following diagram commutes for all $t \leq t' \in T$:

$$
\begin{array}{c}
V_t & \xrightarrow{v_t'} & V_{t'} \\
\downarrow{\phi_t} & & \downarrow{\phi_{t'}} \\
W_t & \xrightarrow{w_{t'}} & W_{t'}
\end{array}
$$

$\Phi$ is called an isomorphism if each map $\phi_t$ is itself an isomorphism $V_t \rightarrow W_t$.

**Question 11.** Show that the persistence modules over a fixed index set $T$, together with the morphisms between them, form a category, that is:

- for each persistence module $V$ there exist isomorphisms $V \rightarrow V$,
- for any morphisms $U \xrightarrow{\Phi} V$ and $V \xrightarrow{\Psi} W$ there exists a morphism $U \xrightarrow{\Psi \circ \Phi} W$.

A submodule $W$ of a persistence module $V$ is composed of subspaces $W_t \subseteq V_t$ for all $t \in T$, and of maps $w_{t'}^t = v_{t'}^t |_{W_t}$ for all $t \leq t' \in T$. In particular, $v_{t'}^t(W_t) \subseteq W_{t'}$ for all $t \leq t' \in T$. A simple example of submodule is the null module $W = 0$ (defined by $W_t = 0$ for all $t \in T$ and $w_{t'}^t = 0$ for all $t \leq t' \in T$), which is a submodule of any module $V$ over $T$.

**Question 12.** Show that the null module is both an initial and a terminal object in the category, that is: for any module $V$ there is a unique morphism $0 \rightarrow V$ and a unique morphism $V \rightarrow 0$.

Direct sums in the category are defined pointwise, that is: for any persistence modules $V$ and $W$ over $T$, the direct sum $V \oplus W$ is composed of the spaces $V_t \oplus W_t$ for all $t \in T$, and of the maps $v_{t'}^t \oplus w_{t'}^t$ for all $t \leq t' \in T$. We say that $V$ and $W$ are summands of the direct sum $V \oplus W$. In particular, we have $V = 0 \oplus V = V \oplus 0$ for any persistence module $V$, so $V$ is always a summand of itself. A persistence module $V$ is called indecomposable if its only summands are itself or the null module. The decomposition theorem that we saw in class asserts that the only indecomposable persistence modules are the so-called interval modules over $T$, and that (under some conditions on $T$ or on the dimensions of the spaces) any module decomposes as a direct sum of interval modules.

**Question 13.** Prove the decomposition theorem in the case where $|T| = 1$.

In the general case however, the result is much more complicated to prove as the persistence modules over $T$ do not share the same properties of vector spaces, in particular they are not semisimple.

**Question 14.** In the case where $|T| \geq 2$, exhibit a counterexample showing that a submodule of a persistence module $V$ over $T$ may not always be a summand of $V$. 

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