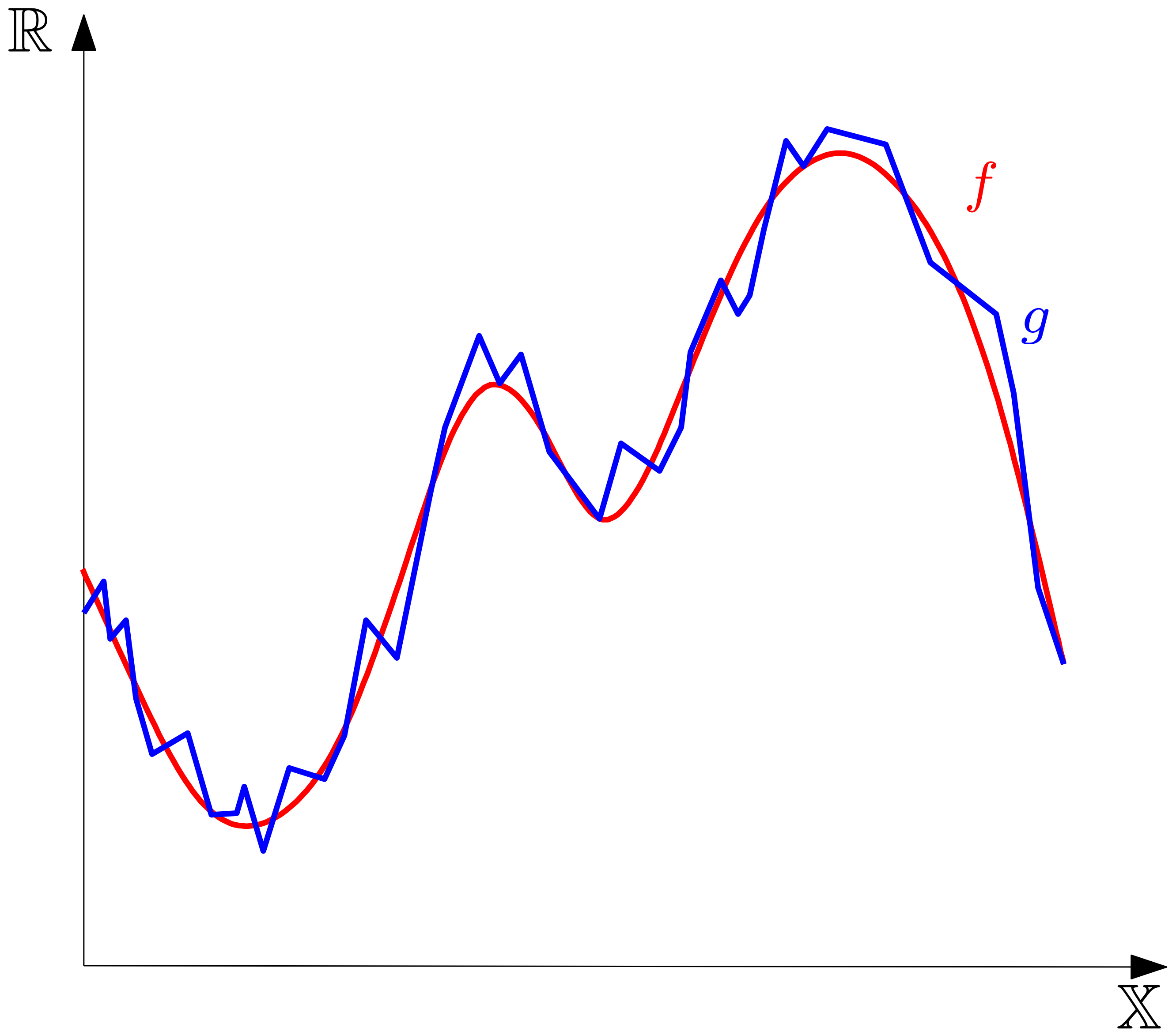
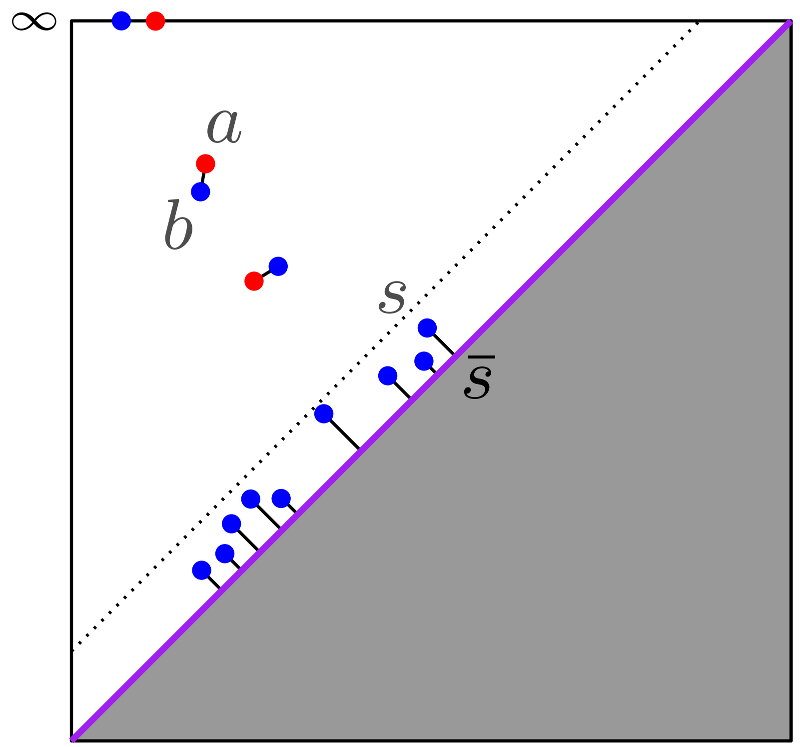
Stability Properties





Proof sketch

Let $f, g : X \to \mathbb{R}$ be q-tame, and let $\varepsilon = \|f - g\|_\infty$.

$$F_\alpha := f^{-1}((-\infty, \alpha])$$

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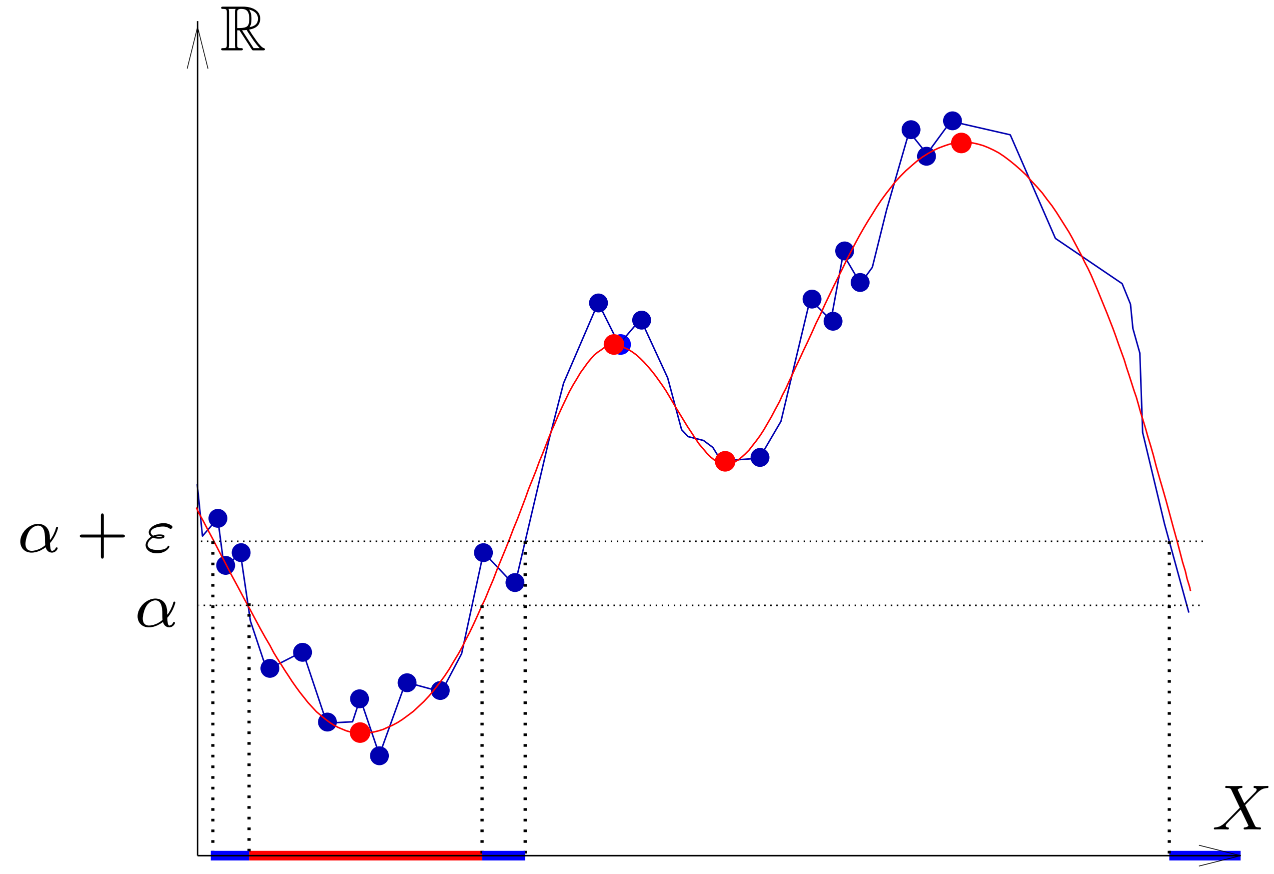
- Key observation: $\{F_\alpha\}_\alpha$ and $\{G_\alpha\}_\alpha$ are ε-**interleaved** w.r.t. inclusion:

$$\forall \alpha \in \mathbb{R},\ G_{\alpha-\varepsilon} \subseteq F_\alpha \subseteq G_{\alpha+\varepsilon}$$



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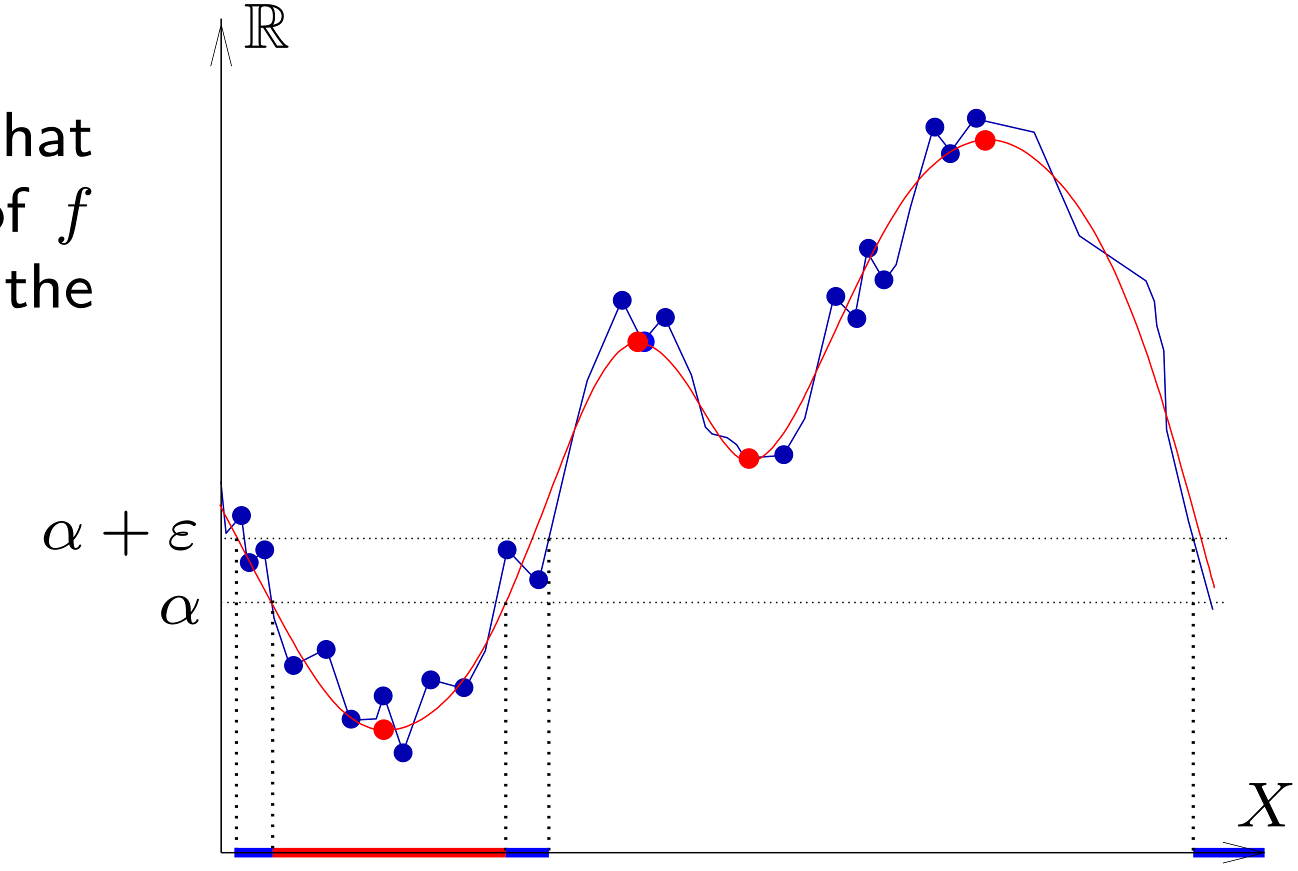
$\to$ Intuition: every homological feature that appears/dies at time α in the filtration of f appears/dies at time $\alpha+\varepsilon$ at the latest in the filtration of g, and vice versa.



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$$\cdots \subseteq F_0 \subseteq G_\varepsilon \subseteq F_{2\varepsilon} \subseteq \cdots \subseteq G_{(2n-1)\varepsilon} \subseteq F_{2n\varepsilon} \subseteq G_{(2n+1)\varepsilon} \subseteq \cdots$$

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- the filtration $\{F_{2n\varepsilon}\}_{n\in\mathbb{Z}}$ is a 2ε-*discretization* of $\{F_\alpha\}_{\alpha\in\mathbb{R}}$

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- the filtration $\{G_{(2n+1)\varepsilon}\}_{n\in\mathbb{Z}}$ is a 2ε-discretization of $\{G_\alpha\}_{\alpha\in\mathbb{R}}$

- both filtrations are 2ε-discretizations of $\{H_{n\varepsilon}\}_{n\in\mathbb{Z}}$,where $H_{n\varepsilon} = \begin{cases} F_{n\varepsilon} & \text{if } n \text{ is even} \\ G_{n\varepsilon} & \text{if } n \text{ is odd} \end{cases}$

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$\to$ **goal**: bound distances between diagrams of filtrations and discretizations

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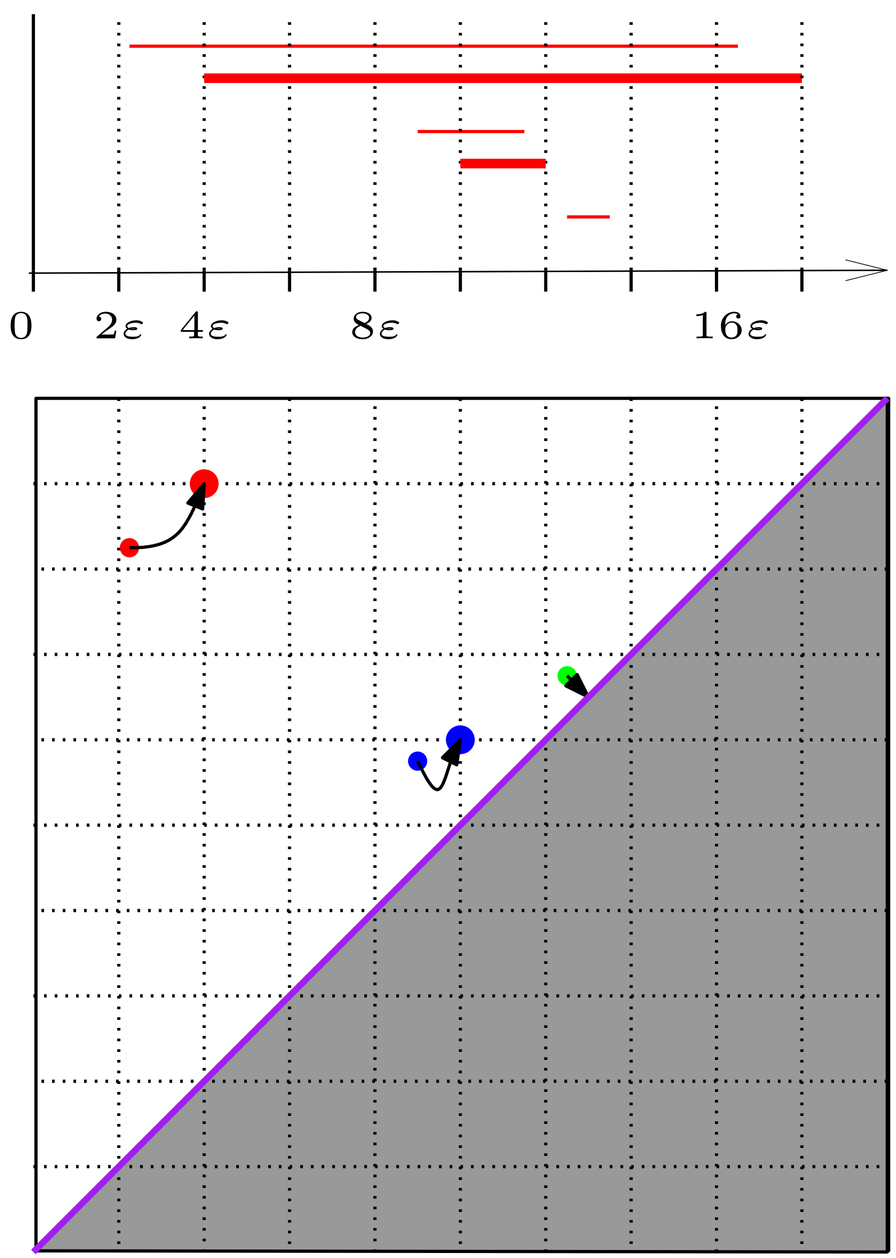
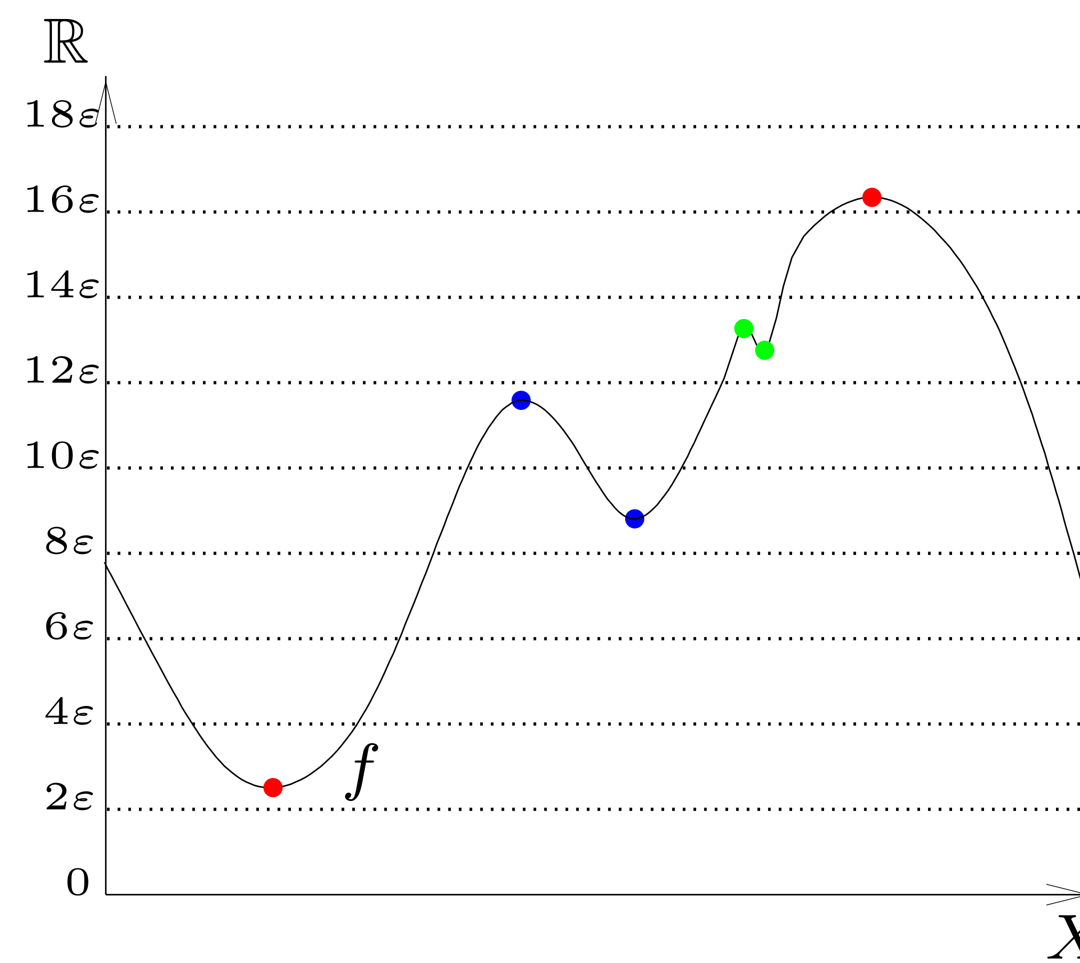
- Discretization $\Rightarrow$ pixelization effect on the barcodes / diagrams:





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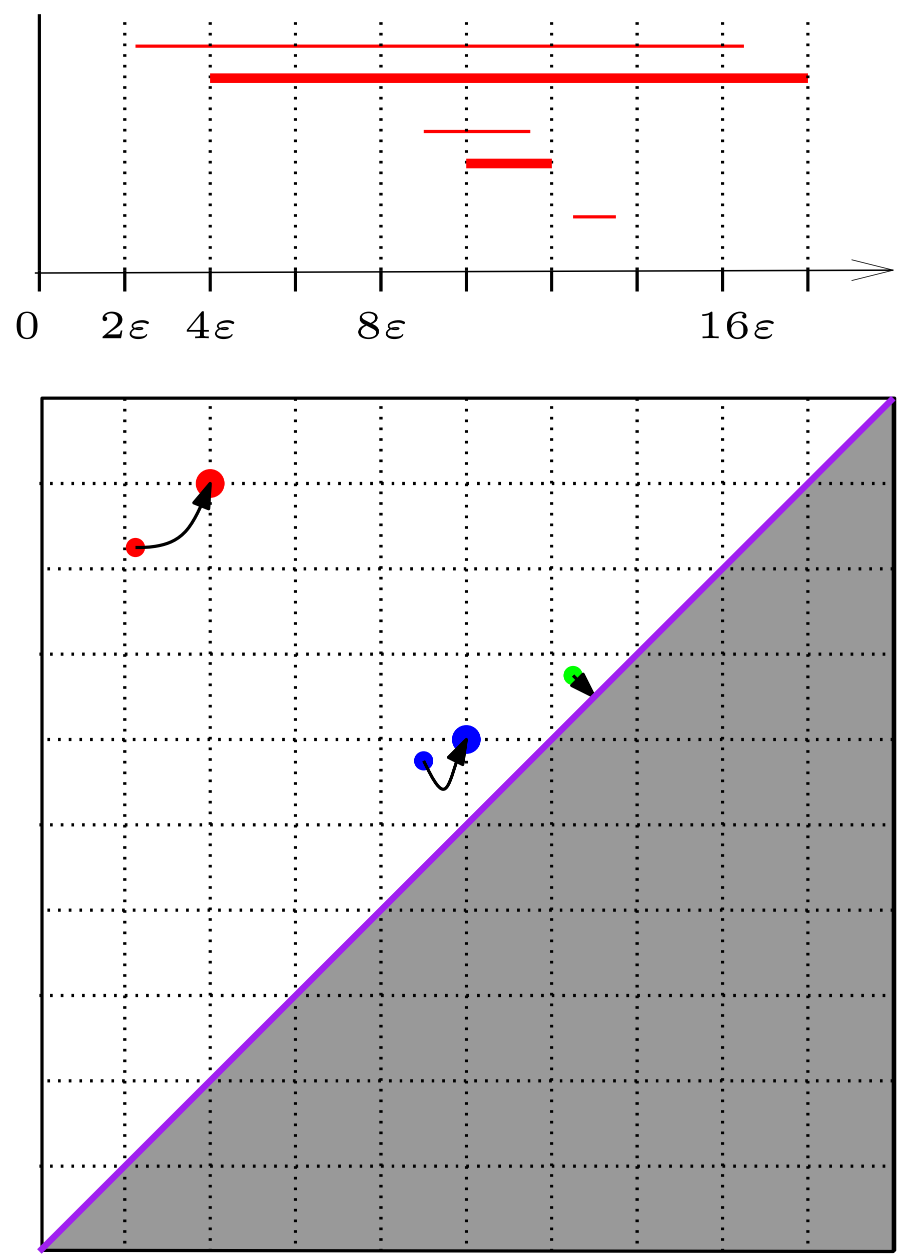

Pixelization map: $\forall \alpha \leq \beta$,

$$\pi_{2\varepsilon}(\alpha, \beta) = \begin{cases} (\lceil \frac{\alpha}{2\varepsilon} \rceil 2\varepsilon, \lceil \frac{\beta}{2\varepsilon} \rceil 2\varepsilon) & \text{if } \lceil \frac{\beta}{2\varepsilon} \rceil > \lceil \frac{\alpha}{2\varepsilon} \rceil \\ (\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}) & \text{if } \lceil \frac{\beta}{2\varepsilon} \rceil = \lceil \frac{\alpha}{2\varepsilon} \rceil \end{cases}$$

Theorem: If $f : X \to \mathbb{R}$ is q-tame, then $\pi_{2\varepsilon}$ induces a bijection $\mathrm{Dg}\, f \to \mathrm{Dg}\, f^{2\varepsilon}$.

$$\Rightarrow d_B^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, f^{2\varepsilon}) \leq 2\varepsilon$$

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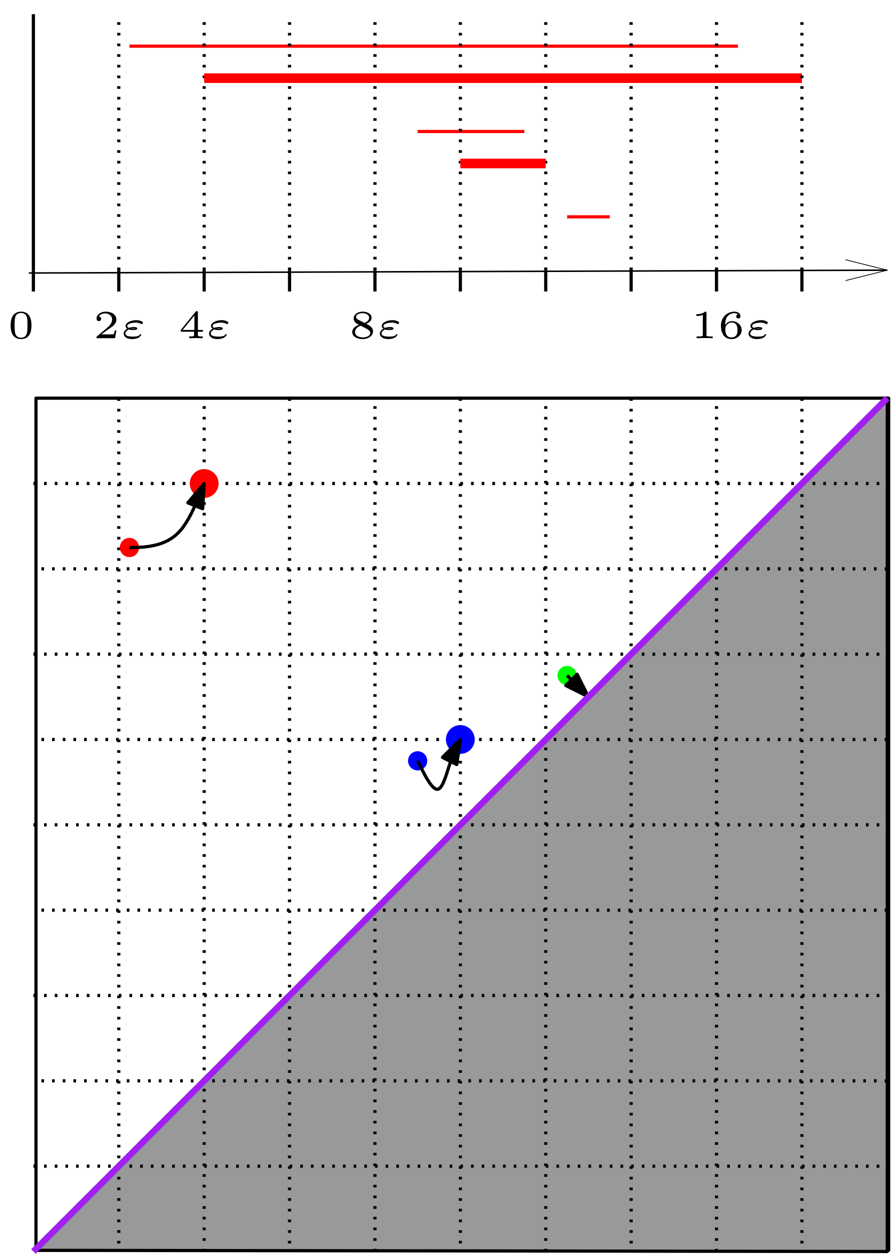

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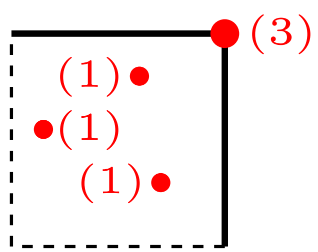
Theorem: If $f : X \to \mathbb{R}$ is q-tame, then $\pi_{2\varepsilon}$ induces a bijection $\mathrm{Dg}\, f \to \mathrm{Dg}\, f^{2\varepsilon}$.

$\to$ proof: show that the multiplicities of $\mathrm{Dg}\, f$ and $\mathrm{Dg}\, f^{2\varepsilon}$ are the same inside each grid cell that does not intersect the diagonal.

The case of diagonal cells is trivial.



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- Back to interleaved filtrations:

$$\cdots \subseteq F_0 \subseteq G_\varepsilon \subseteq F_{2\varepsilon} \subseteq \cdots \subseteq G_{(2n-1)\varepsilon} \subseteq F_{2n\varepsilon} \subseteq G_{(2n+1)\varepsilon} \subseteq \cdots$$

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Previous theorem + triangle inequality $\Rightarrow \mathrm{d}_B^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 8\varepsilon$

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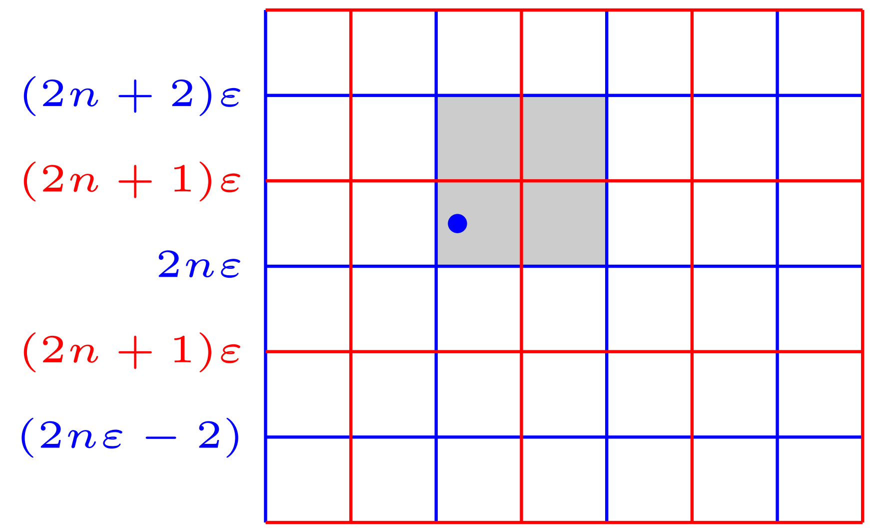
Improvement:



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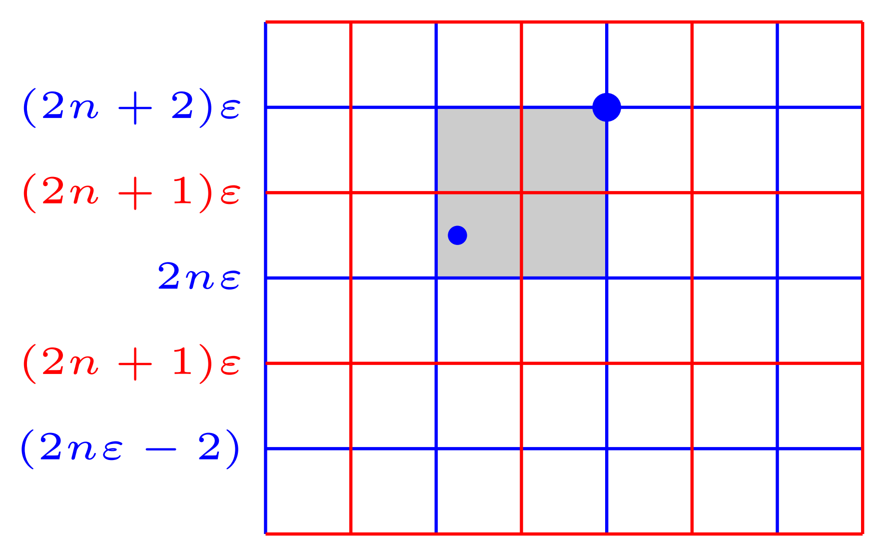

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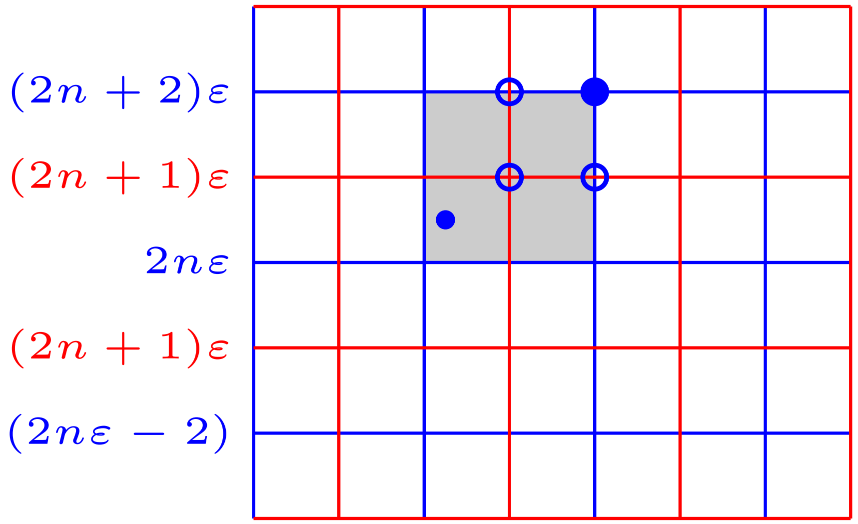

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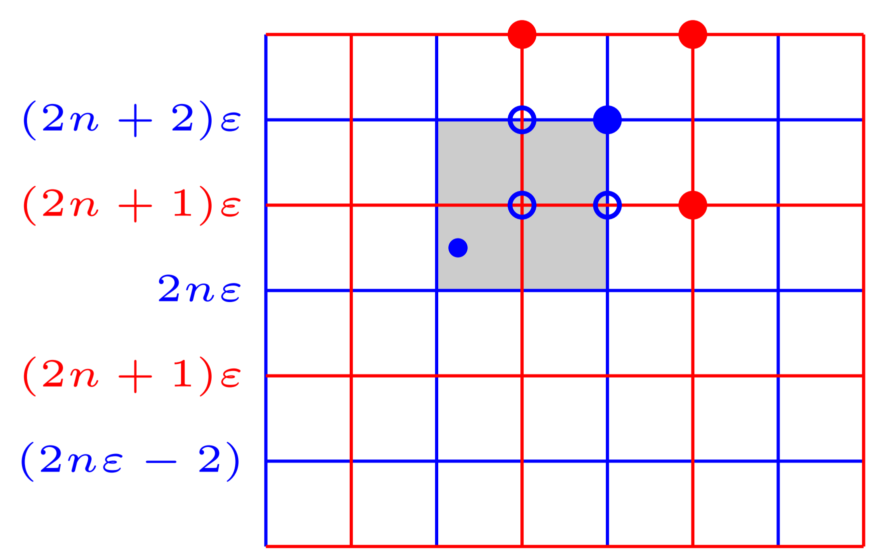

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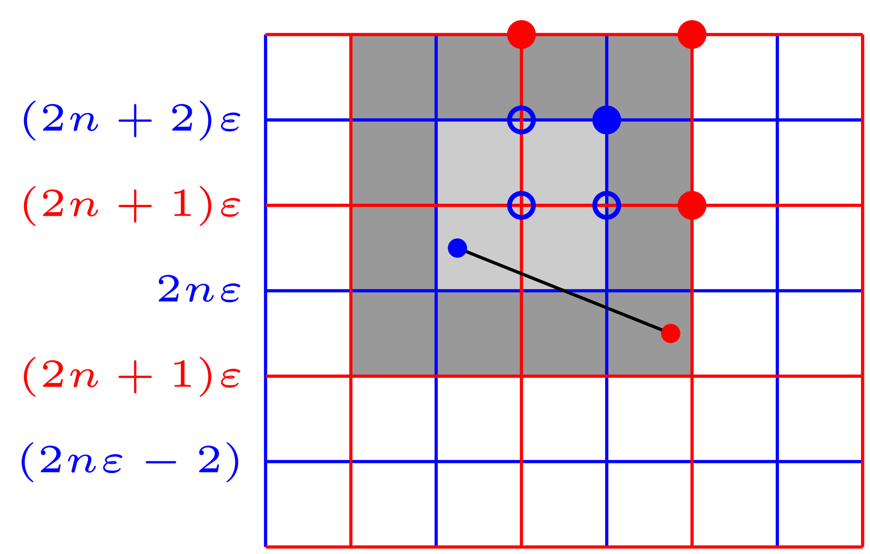
Improvement: $\boxed{\mathrm{d}_B^\infty(\mathrm{Dg}\, f, \mathrm{Dg}\, g) \leq 3\varepsilon}$



Proof sketch

- Comments:
  - sketch of proof based on [Chazal, Cohen-Steiner, Glisse, Guibas, O. 2009].
  - uses only the fact that F, G are interleaved over the scale $\varepsilon\mathbb{Z}$.
  - bound 3ε is tight under this assumption.