Persistence Modules vs. Quiver Representations

$k$: field of coefficients

Persistence module:

$$
\begin{align*}
\mathbb{k} & \rightarrow \mathbb{k}^2 \\
\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \rightarrow \\
\mathbb{k} & \rightarrow \mathbb{k}^2 \\
\begin{pmatrix} 1 \\ 1 \end{pmatrix} & \rightarrow
\end{align*}
$$

Underlying graph:

1 $\xrightarrow{a}$ 2 $\xrightarrow{b}$ 3 $\xrightarrow{c}$ 4 $\xrightarrow{d}$ 5
Persistence Modules vs. Quiver Representations

$k$: field of coefficients

quiver representation:

$$\begin{align*}
k &\rightarrow k^2 & (\begin{pmatrix} 1 \\ 0 \end{pmatrix}) &\rightarrow & k &\rightarrow & k^2 & (\begin{pmatrix} 1 \\ 1 \end{pmatrix}) &\rightarrow & k^2 & (\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}) &\rightarrow & k^2
\end{align*}$$

quiver:

$$\begin{align*}
1 &\rightarrow a & 2 &\rightarrow b & 3 &\rightarrow c & 4 &\rightarrow d & 5
\end{align*}$$
Outline

- quivers and representations
- the category of representations
- the classification problem
- Gabriel’s theorem(s)
- proof of Gabriel’s theorem
- beyond Gabriel’s theorem
Quivers and Representations

Definition: A quiver $Q$ consists of two sets $Q_0, Q_1$ and two maps $s, t : Q_1 \to Q_0$. The elements in $Q_0$ are called the vertices of $Q$, while those of $Q_1$ are called the arrows. The source map $s$ assigns a source $s_a$ to every arrow $a \in Q_1$, while the target map $t$ assigns a target $t_a$. 

$$Ln(n \geq 1) \quad \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet$$
Quivers and Representations

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Quivers and Representations

**Definition:** A representation of $Q$ over a field $k$ is a pair $V = (V_i, v_a)$ consisting of a set of $k$-vector spaces $\{V_i \mid i \in Q_0\}$ together with a set of $k$-linear maps $\{v_a : V_{s_a} \to V_{t_a} \mid a \in Q_1\}$.

Note: commutativity is not required
**Quivers and Representations**

**Definition:** A morphism $\phi$ between two $k$-representations $V, W$ of $Q$ is a set of $k$-linear maps $\phi_i : V_i \to W_i$ such that $w_a \circ \phi_s = \phi_t \circ v_a$ for every arrow $a \in Q_1$. 

![Diagrams](image)

Every quadrangle associated with a quiver edge commutes
**Definition:** A morphism $\phi$ between two $k$-representations $V, W$ of $Q$ is a set of $k$-linear maps $\phi_i : V_i \to W_i$ such that $w_a \circ \phi_{s_a} = \phi_{t_a} \circ v_a$ for every arrow $a \in Q_1$. 

![Diagram]

every quadrangle associated with a quiver edge commutes
The Category of Representations

The representations of a quiver $Q = (Q_0, Q_1)$, together with their morphisms, form a category called $\text{Rep}_k(Q)$. This category is abelian:

- it contains a zero object, namely the trivial representation

\[
\begin{array}{cccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]
The Category of Representations

The representations of a quiver $Q = (Q_0, Q_1)$, together with their morphisms, form a category called $\text{Rep}_k(Q)$. This category is abelian:

- it contains a zero object, namely the trivial representation
- it has internal and external direct sums, defined pointwise. For any $V, W$, the representation $V \oplus W$ has spaces $V_i \oplus W_i$ for $i \in Q_0$ and maps $v_a \oplus w_a = \begin{pmatrix} v_a & 0 \\ 0 & w_a \end{pmatrix}$ for $a \in Q_1$

\[
\begin{array}{c}
\text{k} & \rightarrow & \text{k}^2 & \rightarrow & \text{k} & \rightarrow & \text{k}^2 & \rightarrow & \text{k}^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{k} & \rightarrow & 0 & \rightarrow & \text{k} & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

\[
\oplus
\]

\[
\begin{array}{c}
\text{k}^2 & \rightarrow & \text{k}^2 & \rightarrow & \text{k}^2 & \rightarrow & \text{k}^3 & \rightarrow & \text{k}^2 \\
\text{=}
\end{array}
\]
The Category of Representations

The representations of a quiver $Q = (Q_0, Q_1)$, together with their morphisms, form a category called $\text{Rep}_k(Q)$. This category is **abelian**:

- it contains a zero object, namely the *trivial representation*
- it has internal and external direct sums, defined *pointwise*. For any $V, W$, the representation $V \oplus W$ has spaces $V_i \oplus W_i$ for $i \in Q_0$ and maps $v_a \oplus w_a = \begin{pmatrix} v_a & 0 \\ 0 & w_a \end{pmatrix}$ for $a \in Q_1$

- every morphism has a *kernel*, an *image* and a *cokernel*, defined *pointwise.*

  $\rightarrow$ a morphism $\phi$ is injective iff $\ker \phi = 0$, and surjective iff $\text{coker} \phi = 0$. 

\[
\begin{array}{ccccccc}
  k & \rightarrow & k^2 & \rightarrow & k & \rightarrow & k^2 \\
  \downarrow & & \downarrow & & \downarrow & & \downarrow \\
  1 & \rightarrow & (0,1) & \rightarrow & 1 & \rightarrow & (1,-1) \\
  \downarrow & & \downarrow & & \downarrow & & \downarrow \\
  k & \rightarrow & k & \rightarrow & k & \rightarrow & 0 \\
  \downarrow & & \downarrow & & \downarrow & & \downarrow \\
  \text{ker} \phi = & 0 & \rightarrow & k & \rightarrow & 0 & \rightarrow & k \\
  \downarrow & & \downarrow & & \downarrow & & \downarrow \\
  0 & \rightarrow & k & \rightarrow & 0 & \rightarrow & k & \rightarrow & k^2 \\
  \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & & & \text{coker} \phi = 0
\end{array}
\]
The Classification Problem

**Goal:** Classify the representations of a given quiver $Q = (Q_0, Q_1)$ up to isomorphism.

**Note:** harder than for single vector spaces because no semisimplicity (subrepresentations may not be summands)

$$V = \begin{array}{c} k \\ \downarrow 1 \end{array} k$$

$$W = \begin{array}{c} 0 \\ \downarrow 0 \end{array} k$$
The Classification Problem

**Goal:** Classify the representations of a given quiver $Q = (Q_0, Q_1)$ up to isomorphism.

→ simplifying assumptions:
- $Q$ is finite and connected
- study the subcategory $\text{rep}_k(Q)$ of finite-dimensional representations

$$\dim \mathcal{V} = (\dim V_1, \cdots, \dim V_n)^\top,$$

$$\dim \mathcal{V} = \|\dim \mathcal{V}\|_1 = \sum_{i=1}^{n} \dim V_i.$$
The Classification Problem

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**Theorem:** [Krull-Remak-Schmidt-Azumaya]
\[ \forall V \in \text{rep}_k(Q), \exists V_1, \ldots, V_r \text{ indecomposable s.t. } V \cong V_1 \oplus \cdots \oplus V_r. \]
The decomposition is unique up to isomorphism and reordering.

note: $V$ indecomposable iff there are no $U, W \neq 0$ such that $V \cong U \oplus W$
The Classification Problem

**Goal:** Classify the representations of a given quiver $Q = (Q_0, Q_1)$ up to isomorphism.

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\]
The decomposition is unique up to isomorphism and reordering.

proof:

- existence: by induction on $\dim V$
- uniqueness: show that endomorphisms ring of each $V_i$ is local (it is isomorphic to $k$ for interval representations of $A_n$-type quivers), then apply Azumaya’s theorem
The Classification Problem

**Goal:** Classify the representations of a given quiver \( Q = (Q_0, Q_1) \) up to isomorphism.

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- \( Q \) is finite and connected
- study the subcategory \( \text{rep}_k(Q) \) of *finite-dimensional* representations

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\]
The decomposition is unique up to isomorphism and reordering.

→ problem becomes to identify the indecomposable representations of \( Q \)

(≠ from identifying representations with no subrepresentations)

(no semisimplicity)
Gabriel’s Theorem

**Theorem:** [Gabriel I]

Assuming $Q$ is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\text{rep}_k(Q)$ iff $Q$ is Dynkin.

$A_n(n \geq 1)$

$D_n(n \geq 4)$
Gabriel’s Theorem

**Theorem:** [Gabriel I]
Assuming $Q$ is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\text{rep}_k(Q)$ iff $Q$ is Dynkin.

(does not depend on the choice of field and of arrow orientations)

$A_n (n \geq 1)$

$D_n (n \geq 4)$
Gabriel’s Theorem

**Theorem:** [Gabriel I]
Assuming $Q$ is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\text{rep}_k(Q)$ iff $Q$ is Dynkin.

Given that $Q$ is Dynkin, how to identify indecomposable representations?
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**Theorem: [Gabriel II]**
Assuming $Q$ is Dynkin with $n$ vertices, the map $\mathcal{V} \mapsto \dim \mathcal{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of $Q$ and the set of positive roots of the Tits form of $Q$. 
Gabriel’s Theorem

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Assuming $Q$ is Dynkin with $n$ vertices, the map $\nabla \mapsto \text{dim} \nabla$ induces a bijection between the set of isomorphism classes of indecomposable representations of $Q$ and the set of positive roots of the Tits form of $Q$.

(isom. classes of indecomposables are fully characterized by their dim. vectors)
Gabriel’s Theorem

**Tits form:** given $Q = (Q_0, Q_1)$ with $|Q_0| = n$ and $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$,

$$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_{s_a} x_{t_a}.$$
Gabriel’s Theorem

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$$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_{s(a)} x_{t(a)}.$$

Proposition: $q_Q$ is positive definite ($q_Q(x) > 0 \ \forall x \neq 0$) iff $Q$ is Dynkin.

example: $Q$ of type $A_n$:

\begin{align*}
q_Q(x) &= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} \\
&= \sum_{i=1}^{n-1} \frac{1}{2} (x_i - x_{i+1})^2 + \frac{1}{2} x_1^2 + \frac{1}{2} x_n^2 \\
&= 0 \text{ iff } x = (0, \cdots, 0)
\end{align*}
Gabriel’s Theorem

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$$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_s a x_t a.$$ 

**Root:** $x \in \mathbb{Z}^n \setminus \{0\}$ is a root if $q_Q(x) \leq 1$. It is positive if $x_i \geq 0 \ \forall i$. 
Gabriel’s Theorem

Tits form: given $Q = (Q_0, Q_1)$ with $|Q_0| = n$ and $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$,

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Proposition: If $Q$ is Dynkin, then the set of positive roots of $q_Q$ is finite.
Gabriel’s Theorem

Tits form: given $Q = (Q_0, Q_1)$ with $|Q_0| = n$ and $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$,

$$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_s a x_t a.$$

Root: $x \in \mathbb{Z}^n \setminus \{0\}$ is a root if $q_Q(x) \leq 1$. It is positive if $x_i \geq 0 \ \forall i$.

Proposition: If $Q$ is Dynkin, then the set of positive roots of $q_Q$ is finite.

proof outline:

- $q_Q$ can be viewed indifferently as a quadratic form on $\mathbb{Z}^n$, $\mathbb{Q}^n$ or $\mathbb{R}^n$
- $q_Q$ pos. definite on $\mathbb{Z}^n \Rightarrow q_Q$ pos. definite on $\mathbb{Q}^n$
- by taking limits, $q_Q$ is then pos. semidefinite on $\mathbb{R}^n$
- $q_Q$ invertible on $\mathbb{Q}^n$ with coeffs. in $\mathbb{Q} \Rightarrow q_Q$ invertible on $\mathbb{R}^n$
  $\Rightarrow q_Q$ pos. definite on $\mathbb{R}^n \Rightarrow \{x \in \mathbb{R}^n \mid q_Q(x) \leq 1\}$ is an ellipsoid
Gabriel’s Theorem

**Tits form:** given $Q = (Q_0, Q_1)$ with $|Q_0| = n$ and $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$,

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Assuming $Q$ is Dynkin with $n$ vertices, the map $\mathbb{V} \mapsto \dim \mathbb{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of $Q$ and the set of *positive roots* of the *Tits form* of $Q$. 
Gabriel’s Theorem

example: Q of type $A_n$:

\[
\begin{array}{c}
\bullet \quad 1 \quad 2 \quad \cdots \quad n-1 \quad n \\
\end{array}
\]

\[
q_Q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1}
\]

\[
= \sum_{i=1}^{n-1} \frac{1}{2} (x_i - x_{i+1})^2 + \frac{1}{2} x_1^2 + \frac{1}{2} x_n^2
\]

\[= 1 \text{ iff } x = (0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)\]
Gabriel’s Theorem

example: $Q$ of type $A_n$:

\[ q_Q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} \]

\[ = \sum_{i=1}^{n-1} \frac{1}{2} (x_i - x_{i+1})^2 + \frac{1}{2} x_1^2 + \frac{1}{2} x_n^2 \]

\[ = 1 \text{ iff } x = (0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0) \]

the corresponding indecomp. representations are isomorphic to $\mathbb{I}_Q[b, d]$: 

\[
\begin{array}{cccccccc}
0 & 0 & \cdots & 0 & 0 & k & 1 & \cdots & 1 & k & 0 & 0 & \cdots & 0 & 0 \\
\end{array}
\]
Reflection Functors

**Advantage:** explains the fact that only the dimension vectors play a role in the identification of indecomposable representations. In particular, arrow orientations are irrelevant.

**Idea:** modify quivers by reversing arrows, and study the effect on their representations.
Reflection Functors

Definition: sink = only incoming arrows; source = only outgoing arrows
Reflection Functors

\[ \begin{array}{c}
\bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet \\
1 & & 2 & & 3 & & 4 & & 5 \\
\end{array} \]

\( s_1 \mathcal{Q} : \)
\[ \begin{array}{c}
\bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet \\
1 & & 2 & & 3 & & 4 & & 5 \\
\end{array} \]

\( s_2 \mathcal{Q} : \)
\[ \begin{array}{c}
\bullet & \leftarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet \\
1 & & 2 & & 3 & & 4 & & 5 \\
\end{array} \]

\( s_4 \mathcal{Q} : \)
\[ \begin{array}{c}
\bullet & \rightarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet \\
1 & & 2 & & 3 & & 4 & & 5 \\
\end{array} \]

\( s_5 \mathcal{Q} : \)
\[ \begin{array}{c}
\bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet \\
1 & & 2 & & 3 & & 4 & & 5 \\
\end{array} \]

**Definition:** reflection \( s_i = \) reverse all arrows incident to sink/source \( i \)
Reflection Functors

\[ \begin{array}{ccccc}
\bullet & \rightarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet \\
1 & & 2 & & 3 & & 4 & & 5 & & \\
\end{array} \]

\[ s_1Q : \quad \begin{array}{ccccc}
\bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet \\
1 & & 2 & & 3 & & 4 & & 5 & & \\
\end{array} \]

\[ s_2Q : \quad \begin{array}{ccccc}
\bullet & \leftarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet \\
1 & & 2 & & 3 & & 4 & & 5 & & \\
\end{array} \]

\[ s_4Q : \quad \begin{array}{ccccc}
\bullet & \rightarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet \\
1 & & 2 & & 3 & & 4 & & 5 & & \\
\end{array} \]

\[ s_5Q : \quad \begin{array}{ccccc}
\bullet & \rightarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet & \leftarrow & \bullet \\
1 & & 2 & & 3 & & 4 & & 5 & & \\
\end{array} \]

**Definition:** reflection functor \( R_i^\pm = \text{functor } \text{Rep}_k(Q) \rightarrow \text{Rep}_k(s_iQ) \)
Reflection Functors

Let $\mathcal{V} = (V_i, \nu_a) \in \text{Rep}_k(Q)$, let $i$ be a sink
Reflection Functors

Let $V = (V_i, v_a) \in \text{Rep}_k(Q)$, let $i$ be a sink

Definition: $\mathcal{R}_i^+ V = (W_i, w_a)$ is defined by:

- $W_j = V_j$ for all $j \neq i$
- $w_a = v_a$ for all $a \notin Q^i_1$ (arrows incident to $i$)
Reflection Functors

Let $\mathbb{V} = (V_i, v_a) \in \text{Rep}_k(Q)$, let $i$ be a sink

**Definition:** $\mathcal{R}_i^+ \mathbb{V} = (W_i, w_a)$ is defined by:

- $W_j = V_j$ for all $j \neq i$
- $w_a = v_a$ for all $a \notin Q_i^1$
- $W_i = \ker \xi_i : \bigoplus_{a \in Q_1^i} V_{s_a} \rightarrow V_i$

\[ (x_{s_a})_{a \in Q_1^i} \mapsto \sum_{a \in Q_1^i} v_a(x_{s_a}) \]
Reflection Functors

Let $\mathbb{V} = (V_i, v_a) \in \text{Rep}_k(\mathbb{Q})$, let $i$ be a sink

**Definition:** $\mathcal{R}^+_i \mathbb{V} = (W_i, w_a)$ is defined by:

- $W_j = V_j$ for all $j \neq i$
- $w_a = v_a$ for all $a \notin Q_1^i$
- $W_i = \ker \xi_i : \bigoplus_{a \in Q_1^i} V_{sa} \rightarrow V_i$

$$
\left( x_{sa} \right)_{a \in Q_1^i} \mapsto \sum_{a \in Q_1^i} v_a(x_{sa})
$$

- for $a \in Q_1^i$, let $b$ be the opposite arrow, and let $w_b$ be the composition:

$$
W_{sb} = W_i = \ker \xi_i \hookrightarrow \bigoplus_{c \in Q_1^i} V_{sc} \rightarrow V_{sa} = W_{sa} = W_{tb}
$$

(arrows incident to $i$)

(canonical inclusion)

(projection to component $V_{sa}$)
Reflection Functors

Let \( \mathcal{V} = (V_i, v_a) \in \text{Rep}_k(Q) \), let \( i \) be a sink

**Definition:** \( \mathcal{R}_i^+ \mathcal{V} = (W_i, w_a) \) is defined by:

- \( W_j = V_j \) for all \( j \neq i \)
- \( w_a = v_a \) for all \( a \notin Q_1^i \)
- \( W_i = \ker \xi_i : \bigoplus_{a \in Q_1^i} V_{sa} \longrightarrow V_i \)

\[
\begin{array}{c}
\left( x_{sa} \right)_{a \in Q_1^i} \longmapsto \sum_{a \in Q_1^i} v_a(x_{sa})
\end{array}
\]

- for \( a \in Q_1^i \), let \( b \) be the opposite arrow, and let \( w_b \) be the composition:

\[
W_{sb} = W_i = \ker \xi_i \longmapsto \bigoplus_{c \in Q_1^i} V_{sc} \longrightarrow V_{sa} = W_{sa} = W_{tb}
\]

intuition: \( W_i \) carries the information passing through \( V_i \) in \( \mathcal{V} \)
Reflection Functors

Let $\mathcal{V} = (V_i, v_a) \in \text{Rep}_k(Q)$, let $i$ be a sink source.

**Definition:** $\mathcal{R}_i^+ \mathcal{V} = (W_i, w_a)$ is defined by:

- $W_j = V_j$ for all $j \neq i$
- $w_a = v_a$ for all $a \notin Q_1^i$
- $W_i = \ker \xi_i \mid \bigoplus_{a \in Q_1^i} V_{s_a} \hookrightarrow V_i$
  
  \[
  x_i \mapsto (v_a(x_i))_{a \in Q_1^i}
  \]

- for $a \in Q_1^i$, let $b$ be the opposite arrow, and let $w_b$ be the composition:

  $$W_{s_b} = W_{t_a} = V_{t_a} \hookrightarrow \bigoplus_{c \in Q_1^i} V_{t_c} \twoheadrightarrow \text{coker } \xi_i = W_i = W_{t_b}$$

  (canonical inclusion) \hspace{2cm} (quotient modulo $\text{im } \xi_i$)
Reflection Functors

Let $\mathcal{V} = (V_i, v_a) \in \text{Rep}_k(Q)$, let $i$ be a sink.

**Definition:** $\mathcal{R}_i^\pm \mathcal{V} = (W_i, w_a)$ is defined by:

- $W_j = V_j$ for all $j \neq i$
- $w_a = v_a$ for all $a \notin Q_1^i$
- $W_i = \ker \xi_i : \bigoplus_{a \in Q_1^i} V_{s_a} \leftarrow V_i$
- $x_i \mapsto (v_a(x_i))_{a \in Q_1^i}$

- for $a \in Q_1^i$, let $b$ be the opposite arrow, and let $w_b$ be the composition:

$$W_{s_b} = W_{t_a} = V_{t_a} \hookrightarrow \bigoplus_{c \in Q_1^i} V_{t_c} \rightarrow \text{coker} \xi_i = W_i = W_{t_b}$$

(intuition: this is the operation dual to the previous one (take $V_i = \ker \xi_i$))
Reflection Functors

\[ \mathbb{V} : V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5 \]

\[ \mathcal{R}_5^+ \mathbb{V} : V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xleftarrow{\ker v_d} \]

\[ \mathcal{R}_5^- \mathcal{R}_5^+ \mathbb{V} : V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{\text{mod } \ker v_d} V_4 / \ker v_d \]
Reflection Functors

\[ \mathcal{V} : \quad V_1 \xrightarrow{v_a} V_2 \leftarrow V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5 \]

\[ \mathcal{R}_5^+ \mathcal{V} : \quad V_1 \xrightarrow{v_a} V_2 \leftarrow V_3 \xleftarrow{v_c} V_4 \xleftarrow{\ker v_d} \]

\[ \mathcal{R}_5^- \mathcal{R}_5^+ \mathcal{V} : \quad V_1 \xrightarrow{v_a} V_2 \leftarrow V_3 \xleftarrow{v_c} V_4 \xrightarrow{\mod \ker v_d} V_4/\ker v_d \xrightarrow{\cong \mathcal{V}} \im v_d \]

\[ \mathcal{V} \cong \mathcal{R}_5^- \mathcal{R}_5^+ \mathcal{V} \oplus S_5^r, \text{ where } r = \dim V_5 - \rank v_d \]
Reflection Functors

\[ V : V_1 \xrightarrow{v_a} V_2 \leftarrow V_3 \leftarrow V_4 \xrightarrow{v_d} V_5 \]

\[ R^+_2 V : V_1 \xleftarrow{\ker v_a + v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5 \]

\[ V_1 \oplus V_3 \]

\[ \pi_1 \quad \pi_3 \]
Reflection Functors

\[ \mathbb{V} : V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5 \]

\[ \mathcal{R}_2^- \mathcal{R}_2^+ \mathbb{V} : V_1 \xrightarrow{-} V_1 \oplus V_3 \xleftarrow{\ker v_a + v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5 \]

\[ \ker v_a + v_b \xrightarrow{v_a + v_b} V_1 \oplus V_3 \xrightarrow{v_a + v_b} V_2 \]
\[ V : \quad V_1 \overset{v_a}{\rightarrow} V_2 \overset{v_b}{\leftarrow} V_3 \overset{v_c}{\leftarrow} V_4 \overset{v_d}{\rightarrow} V_5 \]

\[ \mathcal{R}_2^{-} \mathcal{R}_2^{+} V : \quad V_1 \rightarrow V_1 \oplus V_3 \overset{\text{ker } v_a + v_b}{\searrow} V_3 \leftarrow V_3 \overset{v_c}{\leftarrow} V_4 \overset{v_d}{\rightarrow} V_5 \]

\[ \mathcal{R}_2^{-} \mathcal{R}_2^{+} V \cong \text{im } v_a + v_b \]

\[ \ker v_a + v_b \quad V_1 \oplus V_3 \overset{v_a + v_b}{\rightarrow} V_2 \]

\[ V \cong \mathcal{R}_2^{-} \mathcal{R}_2^{+} V \oplus S_2^r, \quad \text{where } r = \dim V_2 - \text{rank } v_a + v_b \]
Theorem: [Bernstein, Gelfand, Ponomarev]
Let $Q$ be a finite connected quiver and let $\mathcal{V}$ be a representation of $Q$. If $\mathcal{V} \cong U \oplus W$, then for any source or sink $i \in Q_0$, $\mathcal{R}_i^\pm \mathcal{V} \cong \mathcal{R}_i^\pm U \oplus \mathcal{R}_i^\pm W$.

If now $\mathcal{V}$ is indecomposable:
1. If $i \in Q_0$ is a sink, then two cases are possible:
   - $\mathcal{V} \cong S_i$: in this case, $\mathcal{R}_i^+ \mathcal{V} = 0$.
   - $\mathcal{V} \ncong S_i$: in this case, $\mathcal{R}_i^+ \mathcal{V}$ is nonzero and indecomposable, $\mathcal{R}_i^- \mathcal{R}_i^+ \mathcal{V} \cong \mathcal{V}$, and the dimension vectors $x$ of $\mathcal{V}$ and $y$ of $\mathcal{R}_i^+ \mathcal{V}$ are related to each other by the following formula:

   $$y_j = \begin{cases} x_j & \text{if } j \neq i; \\ -x_i + \sum_{\substack{a \in Q_1 \\t_a = i}} x_{s_a} & \text{if } j = i. \end{cases}$$
Reflection Functors

**Theorem:** [Bernstein, Gelfand, Ponomarev]

Let $Q$ be a finite connected quiver and let $\mathcal{V}$ be a representation of $Q$. If $\mathcal{V} \cong \mathcal{U} \oplus \mathcal{W}$, then for any source or sink $i \in Q_0$, $\mathcal{R}_i^\pm \mathcal{V} \cong \mathcal{R}_i^\pm \mathcal{U} \oplus \mathcal{R}_i^\pm \mathcal{W}$.

If now $\mathcal{V}$ is indecomposable:

2. If $i \in Q_0$ is a source, then two cases are possible:

- $\mathcal{V} \cong S_i$: in this case, $\mathcal{R}_i^- \mathcal{V} = 0$.
- $\mathcal{V} \not\cong S_i$: in this case, $\mathcal{R}_i^- \mathcal{V}$ is nonzero and indecomposable, $\mathcal{R}_i^+ \mathcal{R}_i^- \mathcal{V} \cong \mathcal{V}$, and the dimension vectors $x$ of $\mathcal{V}$ and $y$ of $\mathcal{R}_i^- \mathcal{V}$ are related to each other by the following formula:

$$ y_j = \begin{cases} 
  x_j & \text{if } j \neq i; \\
  -x_i + \sum_{a \in Q_1, s_a = i} x_{t_a} & \text{if } j = i.
\end{cases} $$
Reflection Functors

**Theorem:** [Bernstein, Gelfand, Ponomarev]
Let $Q$ be a finite connected quiver and let $V$ be a representation of $Q$. If $V \cong U \oplus W$, then for any source or sink $i \in Q_0$, $\mathcal{R}_i \pm V \cong \mathcal{R}_i \pm U \oplus \mathcal{R}_i \pm W$. [...]

**Corollary:** Reflection Functors preserve the Tits form values except at simple representations:

For $i$ source/sink and $V$ indecomposable,

- either $V \cong S_i$, in which case $q_{si \cdot Q}(\dim \mathcal{R}_i \pm V) = 0$,
- or $q_{si \cdot Q}(\dim \mathcal{R}_i \pm V) = q_Q(V)$.

For $V$ arbitrary,

$V \cong V_1 \oplus \cdots \oplus V_r \oplus S_i \implies q_{si \cdot Q}(\dim \mathcal{R}_i \pm V) = q_Q(\dim V_1 \oplus \cdots \oplus V_r)$
Reflection Functors

Example: Q of type $A_n$, $i$ sink, $\mathcal{V} \cong \bigoplus_{j=1}^{r} \mathbb{I}_Q[b_j, d_j] \in \text{rep}_k(Q)$:

\[
\begin{array}{c}
V_i \\
V_1 \cdots V_{i-1} \quad V_{i+1} \cdots V_n \\
W_i
\end{array}
\]
Reflection Functors

Example: Q of type $A_n$, $i$ sink, $\nabla \cong \bigoplus_{j=1}^{r} \mathbb{I}_Q[b_j, d_j] \in \text{rep}_k(Q)$:

$$
\begin{align*}
\mathcal{R}_i^+ \nabla &\cong \bigoplus_{j=1}^{r} \mathcal{R}_i^+ \mathbb{I}_Q[b_j, d_j], \text{ where } \\
\mathcal{R}_i^+ \mathbb{I}_Q[b_j, d_j] &= \begin{cases} 
0 & \text{if } i = b_j = d_j; \\
\mathbb{I}_{s_i Q}[i+1, d_j] & \text{if } i = b_j < d_j; \\
\mathbb{I}_{s_i Q}[i, d_j] & \text{if } i+1 = b_j \leq d_j; \\
\mathbb{I}_{s_i Q}[b_j, i-1] & \text{if } b_j < d_j = i; \\
\mathbb{I}_{s_i Q}[b_j, i] & \text{if } b_j \leq d_j = i - 1; \\
\mathbb{I}_{s_i Q}[b_j, d_j] & \text{otherwise.}
\end{cases}
\end{align*}
$$
Reflection Functors

Example: Q of type $A_n$, i sink, $\mathbb{V} \cong \bigoplus_{j=1}^{r} \mathbb{I}_Q[b_j, d_j] \in \text{rep}_k(Q)$:

Diamond (forced exact by $\mathcal{R}_i^+$)

$$\mathcal{R}_i^+ \mathbb{V} \cong \bigoplus_{j=1}^{r} \mathcal{R}_i^+ \mathbb{I}_Q[b_j, d_j],$$

where

$$\mathcal{R}_i^+ \mathbb{I}_Q[b_j, d_j] = \begin{cases} 0 & \text{if } i = b_j = d_j; \\ \mathbb{I}_s_i Q[i + 1, d_j] & \text{if } i = b_j < d_j; \\ \mathbb{I}_s_i Q[i, d_j] & \text{if } i + 1 = b_j \leq d_j; \\ \mathbb{I}_s_i Q[b_j, i - 1] & \text{if } b_j < d_j = i; \\ \mathbb{I}_s_i Q[b_j, i] & \text{if } b_j \leq d_j = i - 1; \\ \mathbb{I}_s_i Q[b_j, d_j] & \text{otherwise.} \end{cases}$$

Diamond Principle [Carlsson, de Silva]
Proof of Gabriel’s Theorem ($A_n$ case)

**Theorem:** [Gabriel I, $A_n$ type]
Assuming $Q$ is of type $A_n$, every isomorphism class of indecomposable representations in $\text{rep}_k(Q)$ contains $\mathbb{I}_Q[b, d]$ for some $1 \leq b \leq d \leq n$. 
Proof of Gabriel’s Theorem ($A_n$ case)

**Theorem:** [Gabriel I, $A_n$ type]
Assuming $Q$ is of type $A_n$, every isomorphism class of indecomposable representations in $\text{rep}_k(Q)$ contains $I_Q[b, d]$ for some $1 \leq b \leq d \leq n$.

What we can do:

- turn indecomposable representations of $Q$ into indecomposable representations of reflections of $Q$ (or zero)
- while doing so, preserve the value of the Tits form (or zero)
Proof of Gabriel’s Theorem ($A_n$ case)

**Theorem:** [Gabriel I, $A_n$ type]
Assuming $Q$ is of type $A_n$, every isomorphism class of indecomposable representations in $\text{rep}_k(Q)$ contains $\mathbb{I}_Q[b, d]$ for some $1 \leq b \leq d \leq n$.

What we can do:

- turn indecomposable representations of $Q$ into indecomposable representations of reflections of $Q$ (or zero)
- while doing so, preserve the value of the Tits form (or zero)

→ idea: turn $Q$ into itself via sequences of reflections, and observe the evolution of the indecomposables and their Tits form values
Proof of Gabriel’s Theorem \((A_n \text{ case})\)

Special case: linear quiver \(L_n:\)

\[\bullet \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n\]

Let \(\mathbb{V} \in \text{rep}_k(L_n)\) indecomposable, \(\dim \mathbb{V} = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top\)

\[\rightarrow \text{apply reflections } s_1s_2\cdots s_{n-1}s_nL_n \text{ and observe evolution of } \dim \mathbb{V}\]
Proof of Gabriel’s Theorem \((A_n \text{ case})\)

Special case: linear quiver \(L_n\):

\[
\bullet \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n
\]

Let \(V \in \text{rep}_k(L_n)\) indecomposable, \(\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top\)

\[
\dim \mathcal{R}_n^+ V = 0 \text{ or } (x_1, x_2, \cdots, x_{n-1}, x_{n-1} - x_n)^\top
\]

\[
\dim \mathcal{R}_{n-1}^+ \mathcal{R}_n^+ V = 0 \text{ or } (x_1, x_2, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top
\]

\[
\cdots
\]

\[
\dim \mathcal{R}_2^+ \cdots \mathcal{R}_{n-1}^+ \mathcal{R}_n^+ V = 0 \text{ or } (x_1, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top
\]

\[
\dim \mathcal{R}_1^+ \mathcal{R}_2^+ \cdots \mathcal{R}_{n-1}^+ \mathcal{R}_n^+ V = 0 \text{ or } (-x_n, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top
\]
Proof of Gabriel’s Theorem ($A_n$ case)

Special case: linear quiver $L_n$: \[ \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet \]

Let $V \in \text{rep}_k(L_n)$ indecomposable, $\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

\[ \dim R_n^+ V = 0 \text{ or } (x_1, x_2, \cdots, x_{n-1}, x_{n-1} - x_n)^\top \]

\[ \dim R_{n-1}^+ R_n^+ V = 0 \text{ or } (x_1, x_2, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top \]

\[ \cdots \]

\[ \dim R_2^+ \cdots R_{n-1}^+ R_n^+ V = 0 \text{ or } (x_1, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top \]

\[ \dim R_1^+ R_2^+ \cdots R_{n-1}^+ R_n^+ V = 0 \text{ or } (x_n, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top \]

\[ \Rightarrow C^+ V = R_1^+ R_2^+ \cdots R_{n-1}^+ R_n^+ V = 0 \text{ or } x_n = 0 \]
Proof of Gabriel’s Theorem \((A_n \text{ case})\)

Special case: linear quiver \(L_n:\)

\[
\begin{array}{ccccccc}
\bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet \\
1 & & 2 & & \cdots & & n-1 \\
& & & & \rightarrow & & n
\end{array}
\]

Let \(V \in \text{rep}_k(L_n)\) indecomposable, \(\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)\top\)

\[
\dim C^+ V = 0 \text{ or } (0, x_1, x_2, \cdots, x_{n-2}, x_{n-1})\top
\]

\[
\dim C^+ C^+ V = 0 \text{ or } (0, 0, x_1, \cdots, x_{n-3}, x_{n-2})\top
\]

\[
\cdots
\]

\[
\dim C^+ \cdots C^+ V = 0 \text{ or } (0, 0, 0, \cdots, 0, x_1)\top
\]

\[
\dim \underbrace{C^+ \cdots C^+}_n V = 0
\]
Proof of Gabriel’s Theorem \((A_n \text{ case})\)

Special case: linear quiver \(L_n\):

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\downarrow \\
\cdots \\
\downarrow \\
\bullet \\
\downarrow \\
\bullet \\
\end{array}
\]

Let \(V \in \text{rep}_k(L_n)\) indecomposable, \(\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top\)

\[\dim C^+ V = 0 \text{ or } (0, x_1, x_2, \cdots, x_{n-2}, x_{n-1})^\top\]

\[\dim C^+ C^+ V = 0 \text{ or } (0, 0, x_1, \cdots, x_{n-3}, x_{n-2})^\top\]

\[\vdots\]

\[\dim C^+ \cdots C^+ V = 0 \text{ or } (0, 0, 0, \cdots, 0, x_1)^\top\]

\[\text{\(n-1\) times} \Rightarrow \exists i_1, i_2, \cdots, i_{s-1}, i_s \text{ s.t. } R_{i_s}^+ R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ V = 0\]

\[\dim C^+ \cdots C^+ V = 0 \text{ \(n\) times} \Rightarrow R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ V \neq 0\]
Proof of Gabriel’s Theorem \((A_n\text{ case})\)

Special case: linear quiver \(L_n\):

\[
\begin{array}{c}
\bullet \\
1 \\
\rightarrow \\
2 \\
\rightarrow \\
\cdots \\
\rightarrow \\
n-1 \\
\rightarrow \\
n \\
\end{array}
\]

Let \(\mathbb{V} \in \text{rep}_k(L_n)\) indecomposable, \(\dim \mathbb{V} = (x_1, x_2, \cdots, x_{n-1}, x_n)\top\)

\[\exists i_1, i_2, \cdots, i_{s-1}, i_s \text{ s.t. } R_{i_s}^+ R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ \mathbb{V} = 0\]

\[R_{i_s}^+ \cdots R_{i_2}^+ R_{i_1}^+ \mathbb{V} \neq 0\]

\[\implies R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ \mathbb{V} \text{ is indecomposable and isomorphic to } S_r \text{ for some } 1 \leq r \leq n\]

(Reflection Functor Thm)
Proof of Gabriel’s Theorem ($A_n$ case)

Special case: linear quiver $L_n$: \[ \bullet \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n \]

Let $V \in \text{rep}_k(L_n)$ indecomposable, $\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

\[ \exists i_1, i_2, \cdots, i_{s-1}, i_s \text{ s.t. } R_{i_s}^+ R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ V = 0 \]

\[ R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ V \neq 0 \]

\[ \Rightarrow R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ V \text{ is indecomposable and isomorphic to } S_r \text{ for some } 1 \leq r \leq n \]

\[ \Rightarrow q_{L_n}(\dim V) = q_{S_{i_{s-1}} \cdots i_1 L_n}(\dim R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ V) = q_{S_{i_{s-1}} \cdots i_1 L_n}(\dim S_r) = 1 \]

(Corollary)
Proof of Gabriel’s Theorem ($A_n$ case)

Special case: linear quiver $L_n$:

$$\textbullet \rightarrow \textbullet \rightarrow \cdots \rightarrow \textbullet \rightarrow \textbullet \rightarrow \textbullet$$

Let $V \in \text{rep}_k(L_n)$ indecomposable, $\dim V = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

$$\exists i_1, i_2, \cdots, i_{s-1}, i_s \text{ s.t. } R_{i_s}^+ R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ V = 0$$

$$R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ V \neq 0$$

$$\implies R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ V \text{ is indecomposable and isomorphic to } S_r \text{ for some } 1 \leq r \leq n$$

$$\implies q_{L_n} (\dim V) = q_{s_{i_{s-1}} \cdots s_{i_1}} L_n (\dim R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ V) = q_{s_{i_{s-1}} \cdots s_{i_1}} L_n (\dim S_r) = 1$$

$$\implies \dim V = \dim \mathbb{I}_{L_n} [b, d] \text{ for some } 1 \leq b \leq d \leq n \implies V \cong \mathbb{I}_{L_n} [b, d]$$

(Example)
Proof of Gabriel’s Theorem ($A_n$ case)

$A_n$-type quiver $Q$: $\bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet$

$\rightarrow$ goal: find a sequence of indices $i_1, i_2, \cdots, i_{s-1}, i_s$ s.t.

$$\mathcal{R}^{+}_{i_s} \mathcal{R}^{+}_{i_{s-1}} \cdots \mathcal{R}^{+}_{i_2} \mathcal{R}^{+}_{i_1} \mathbb{V} = 0 \text{ for all } \mathbb{V} \in \text{rep}_k(Q)$$
Proof of Gabriel’s Theorem \((A_n\text{ case})\)

\(A_n\)-type quiver \(Q\):

\[
\begin{array}{c}
\bullet \\
1 \\
\bullet \\
2 \\
\bullet \\
n-1 \\
\bullet \\
n \\
\end{array}
\]

→ goal: find a sequence of indices \(i_1, i_2, \cdots, i_{s-1}, i_s\) s.t.

\[
R_{i_s}^+ R_{i_{s-1}}^+ \cdots R_{i_2}^+ R_{i_1}^+ V = 0 \text{ for all } V \in \text{rep}_k(Q)
\]

→ idea: turn \(Q\) into \(L_n\), then use the same sequence as before
Proof of Gabriel’s Theorem ($A_n$ case)

$A_n$-type quiver $Q$:

- embed $Q$ in a giant pyramid
Proof of Gabriel’s Theorem ($A_n$ case)

$A_n$-type quiver $Q$:

- embed $Q$ in a giant pyramid
- travel down the pyramid to its bottom $L_n$

→ travelling one level down reverses the leftmost backward arrow

e.g. $s_1 s_2 s_3$ reverses $3 \leftrightarrow 4$
Proof of Gabriel’s Theorem \((A_n\text{ case})\)

\(A_n\)-type quiver \(Q\):

- embed \(Q\) in a giant pyramid

- travel down the pyramid to its bottom \(L_n\)
  
  \(→\) travelling one level down reverses the leftmost backward arrow
  
  e.g. \(s_1 s_2 s_3\) reverses \(\cdot 3 \leftarrow \cdot 4\)

- each diamond

  \[
  \begin{array}{c}
  i-1 \rightarrow \cdot i \rightarrow \cdot i+1 \\
  \end{array}
  \]

  is travelled down using \(\mathcal{R}^+_i\)
Proof of Gabriel’s Theorem \((A_n\) case)\)

**Theorem:** [Gabriel II]
Assuming \(Q\) is Dynkin with \(n\) vertices, the map \( \mathbb{V} \mapsto \dim \mathbb{V} \) induces a bijection between the set of isomorphism classes of indecomposable representations of \(Q\) and the set of *positive roots* of the *Tits form* of \(Q\).
Proof of Gabriel’s Theorem ($A_n$ case)

**Theorem:** [Gabriel II]
Assuming $Q$ is Dynkin with $n$ vertices, the map $\mathbb{V} \mapsto \dim \mathbb{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of $Q$ and the set of *positive roots* of the *Tits form* of $Q$.

What we know:

- the positive roots of $q_Q$ are the dimension vectors of interval modules $\mathbb{I}_Q[b, d]$
- each isomorphism class $C$ of indecomposables contains $\geq 1$ interval module
Proof of Gabriel’s Theorem ($A_n$ case)

**Theorem:** [Gabriel II]
Assuming $Q$ is Dynkin with $n$ vertices, the map $\mathcal{V} \mapsto \dim \mathcal{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of $Q$ and the set of *positive roots* of the *Tits form* of $Q$.

What we know:

- the positive roots of $q_Q$ are the dimension vectors of interval modules $\mathbb{I}_Q[b, d]$
- each isomorphism class $C$ of indecomposables contains $\geq 1$ interval module

Additional observations:

- $\neq$ interval modules are $\not\sim$, therefore each class $C$ contains 1 interval module
- each interval module is indecomposable (endomorphism ring isom. to $k$)
Proof of Gabriel’s Theorem (general case)

**Theorem:** [Gabriel I]

Assuming $Q$ is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\text{rep}_k(Q)$ iff $Q$ is Dynkin.
Proof of Gabriel’s Theorem (general case)

**Theorem:** [Gabriel I]
Assuming $Q$ is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\text{rep}_k(Q)$ iff $Q$ is Dynkin.

$\Leftarrow$ same process as before (proves Gabriel II as well):

- Define Coxeter functors for arbitrary (finite, connected, loop-free) quivers
- Iterate Coxeter functor to eventually send every indecomposable to zero
- Derive bijection between isom. classes of indecomposables to positive roots of $Q$ via simple representations
Proof of Gabriel’s Theorem (general case)

Theorem: [Gabriel I]
Assuming $Q$ is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\text{rep}_k(Q)$ iff $Q$ is Dynkin.

$\implies$ every connected quiver that is not Dynkin contains one of these:

$\tilde{A}_0$

$\tilde{A}_n(n \geq 1)$

$\tilde{D}_n(n \geq 4)$

$\tilde{E}_6$

$\tilde{E}_7$

$\tilde{E}_8$
Proof of Gabriel’s Theorem (general case)

**Theorem:** [Gabriel I]
Assuming $Q$ is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\text{rep}_k(Q)$ iff $Q$ is Dynkin.

$\Rightarrow$ every connected quiver that is not Dynkin contains one of these:

- $\tilde{A}_0$
- contains indecomp. representations of arbitrary dimensions

\[ V = (k^r, f) \text{ isomorphic to } W = (k^s, g) \]

$\iff$

- $r = s$ and $\exists \phi, \psi \in \text{Aut}(k^r)$ s.t. $f = \psi^{-1} \circ g \circ \phi$
Proof of Gabriel’s Theorem (general case)

**Theorem:** [Gabriel I]
Assuming $Q$ is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\text{rep}_k(Q)$ iff $Q$ is Dynkin.

$\Rightarrow$ 

every connected quiver that is not Dynkin contains one of these:

$\tilde{A}_0$

contains indecomp. representations of arbitrary dimensions

$\rightarrow$ injection from conjugacy classes of Jordan block matrices to isomorphism classes of indecomposables

(injection becomes bijection when $k$ is algebraically closed)
Beyond Gabriel’s Theorem

Gabriel’s theorem is about:

- Dynkin quivers
- finite-dimensional representations
Beyond Gabriel’s Theorem

Gabriel’s theorem is about:

- Dynkin quivers
- finite-dimensional representations

Finite connected quivers:

**Theorem:** [Kac]
The set of dimension vectors of finite-dimensional indecomposable representations of a finite connected quiver $Q$ is precisely the set of positive roots of its Tits form. In particular, this set is independent of the arrow orientations in $Q$ and of the base field.

(catch: the map $\nabla \mapsto \text{dim } \nabla$ may not be injective)
Beyond Gabriel’s Theorem

Gabriel’s theorem is about:

- Dynkin quivers
- finite-dimensional representations

Finite disconnected quivers:

\[ Q = Q_1 \sqcup Q_2 \implies \text{Rep}_k(Q) \cong \text{Rep}_k(Q_1) \times \text{Rep}_k(Q_2) \]
Beyond Gabriel’s Theorem

Gabriel’s theorem is about:

- Dynkin quivers
- finite-dimensional representations

Finite quivers, infinite-dimensional representations:

→ path algebras, modules, Auslander-Reiten theory

**Theorem**: [Auslander+Gabriel]
For a Dynkin quiver $Q$, every indecomposable representation in $\text{Rep}_k(Q)$ has finite dimension, and every representation in $\text{Rep}_k(Q)$ is a direct sum of indecomposable representations. In particular, $Q$ has finitely many isomorphism classes of indecomposable representations, and all of them are finite-dimensional.
Beyond Gabriel’s Theorem

Gabriel’s theorem is about:

- Dynkin quivers
- finite-dimensional representations

Infinite quivers:

→ research in progress

**Theorem**: [Webb] Decomposition of *pfd* rep. of the \( \mathbb{Z} \) quiver

\[ \cdots \rightarrow \bullet_{n-1} \rightarrow \bullet_n \rightarrow \bullet_{n+1} \rightarrow \cdots \]

**Theorem**: [Crawley-Boevey] Decomposition of *pfd* rep. of the \( \mathbb{R} \) quiver

\[ \cdots \rightarrow \bullet_x \rightarrow \bullet_y \rightarrow \bullet_z \rightarrow \cdots \]
Beyond Gabriel’s Theorem

Gabriel’s theorem is about:

- Dynkin quivers
- finite-dimensional representations

Infinite quivers:

→ research in progress

**Theorem:** [Webb] Decomposition of pfd rep. of the \( \mathbb{Z} \) quiver

\[
\cdots \rightarrow n-1 \rightarrow n \rightarrow n+1 \rightarrow \cdots
\]

**Theorem:** [Botnan 2015]
Arrow orientations don’t matter

**Theorem:** [Crawley-Boevey] Decomposition of pfd rep. of the \( \mathbb{R} \) quiver

\[
\cdots \rightarrow x \rightarrow y \rightarrow z \rightarrow \cdots
\]

**Theorem:** [Cochoy, O. 2016]
Arrow orientations don’t matter*

* under exactness conditions