Persistence Modules vs. Quiver Representations

$\boldsymbol{k}$: field of coefficients

persistence module: $$\boldsymbol{k} \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \boldsymbol{k}^2 \xrightarrow{\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)} \boldsymbol{k} \xrightarrow{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} \boldsymbol{k}^2 \xrightarrow{\left(\begin{smallmatrix}1 & 0\\1 & 1\end{smallmatrix}\right)} \boldsymbol{k}^2$$

underlying graph: $$\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \xrightarrow{b} \underset{3}{\bullet} \xrightarrow{c} \underset{4}{\bullet} \xrightarrow{d} \underset{5}{\bullet}$$

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quiver representation: $$\boldsymbol{k} \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} \boldsymbol{k} \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}} \boldsymbol{k}^2$$

quiver: $$\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \xrightarrow{b} \underset{3}{\bullet} \xrightarrow{c} \underset{4}{\bullet} \xrightarrow{d} \underset{5}{\bullet}$$

Outline

- quivers and representations
- the category of representations
- the classification problem
- Gabriel's theorem(s)
- proof of Gabriel's theorem
- beyond Gabriel's theorem

Quivers and Representations

Definition: A *quiver* $\mathtt{Q}$ consists of two sets Q_0, Q_1 and two maps $s, t : Q_1 \to Q_0$. The elements in Q_0 are called the *vertices* of $\mathtt{Q}$, while those of Q_1 are called the *arrows*. The source map s assigns a source s_a to every arrow $a \in Q_1$, while the *target map* t assigns a target t_a.

$$\mathtt{L}_n (n \geq 1) \qquad \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$$

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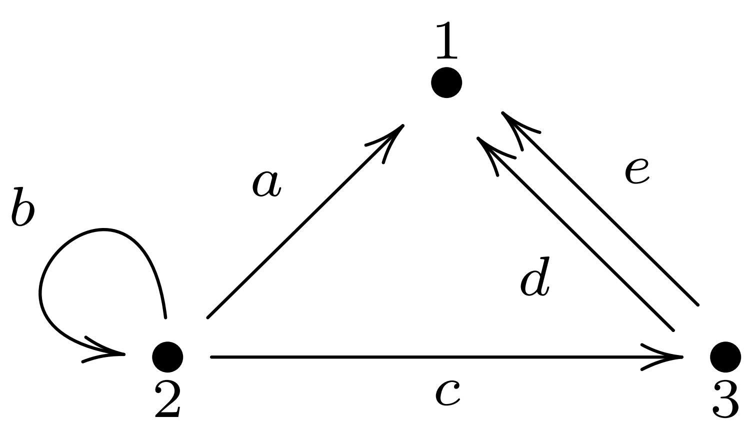
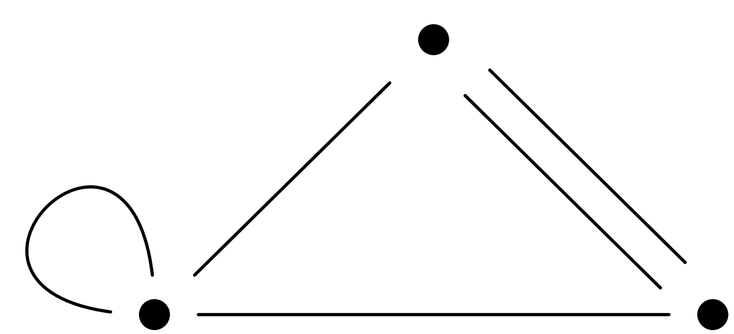


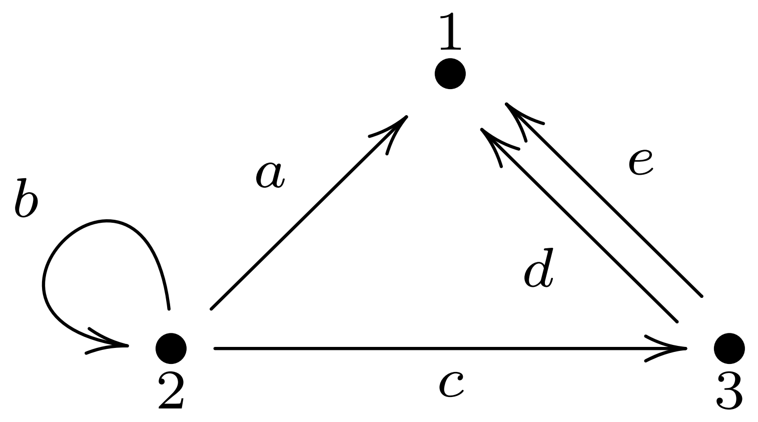
Q



$\bar{\mathtt{Q}}$

Quivers and Representations

Definition: A *representation* of Q over a field $\boldsymbol{k}$ is a pair $\mathbb{V} = (V_i, v_a)$ consisting of a set of $\boldsymbol{k}$-vector spaces $\{V_i \mid i \in Q_0\}$ together with a set of $\boldsymbol{k}$-linear maps $\{v_a : V_{s_a} \to V_{t_a} \mid a \in Q_1\}$.



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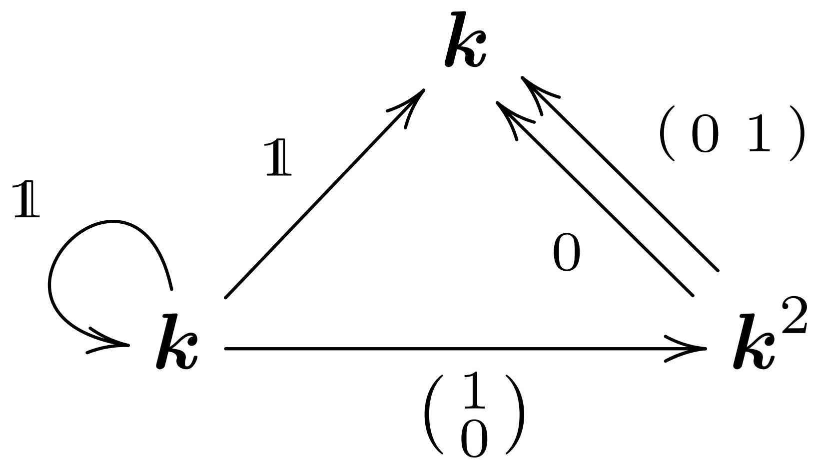
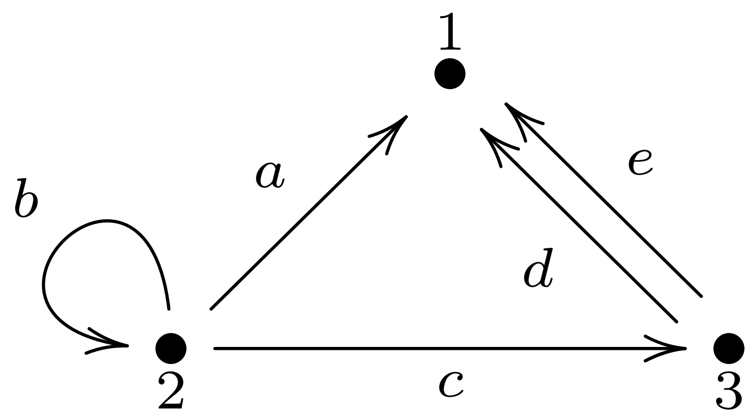


$\in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$

Note: commutativity is not required

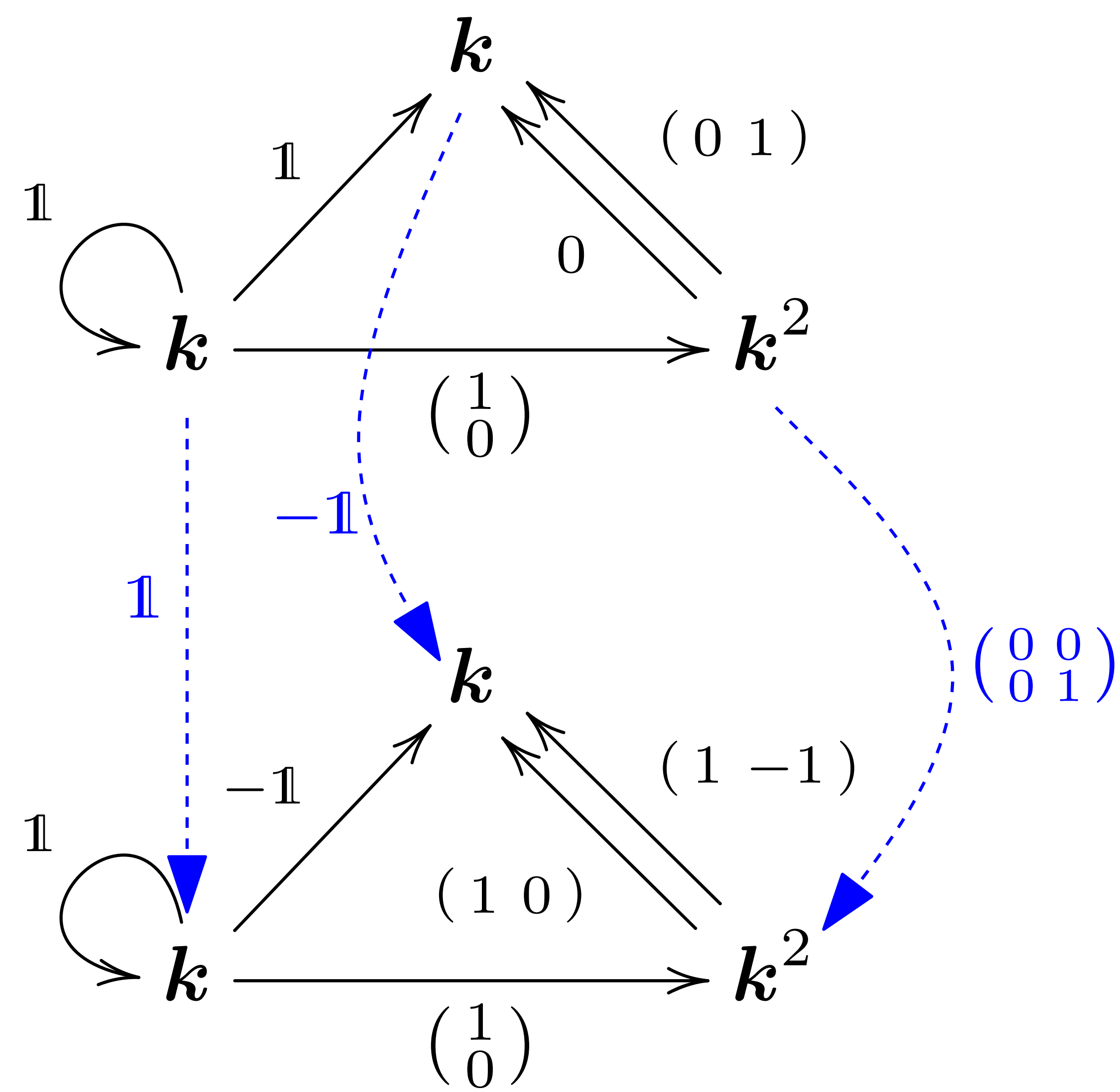
Quivers and Representations

Definition: A *morphism* ϕ between two $\boldsymbol{k}$-representations $\mathbb{V}, \mathbb{W}$ of $\mathtt{Q}$ is a set of $\boldsymbol{k}$-linear maps $\phi_i : V_i \to W_i$ such that $w_a \circ \phi_{s_a} = \phi_{t_a} \circ v_a$ for every arrow $a \in Q_1$.



$\mathtt{Q}$

every quadrangle associated with a quiver edge commutes



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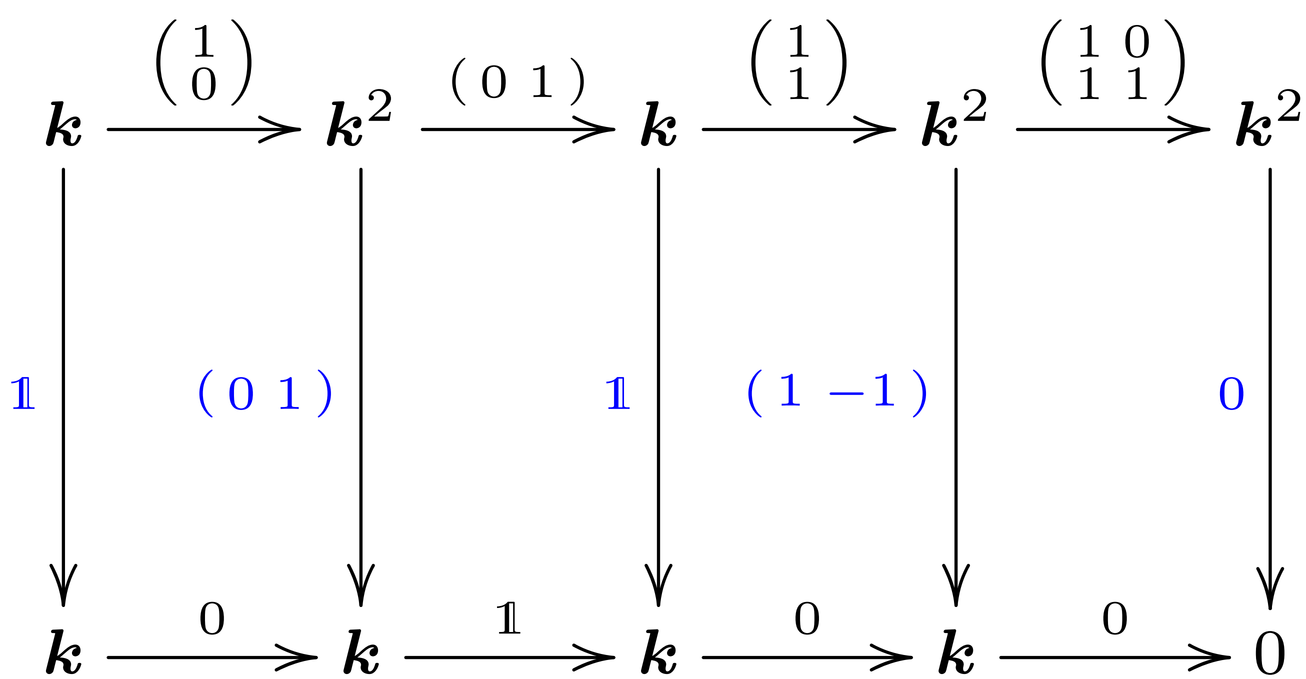

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The Category of Representations

The representations of a quiver $\mathtt{Q} = (Q_0, Q_1)$, together with their morphisms, form a category called $\mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$. This category is **abelian**:

- it contains a zero object, namely the *trivial representation*

$$\boxed{0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0}$$

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- it has internal and external direct sums, defined *pointwise*. For any $\mathbb{V}, \mathbb{W}$, the representation $\mathbb{V} \oplus \mathbb{W}$ has spaces $V_i \oplus W_i$ for $i \in Q_0$ and maps $v_a \oplus w_a = \begin{pmatrix} v_a & 0 \\ 0 & w_a \end{pmatrix}$ for $a \in Q_1$

$$\boldsymbol{k} \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} \boldsymbol{k} \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}} \boldsymbol{k}^2$$

$$\bigoplus$$

$$\boldsymbol{k} \xrightarrow{0} 0 \xrightarrow{0} \boldsymbol{k} \xrightarrow{\mathbb{1}} \boldsymbol{k} \xrightarrow{0} 0$$

$$=$$

$$\boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}1 & 0\\1 & 0\\0 & 1\end{pmatrix}} \boldsymbol{k}^3 \xrightarrow{\begin{pmatrix}1 & 0 & 0\\1 & 1 & 0\end{pmatrix}} \boldsymbol{k}^2$$

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- every morphism has a *kernel*, an *image* and a *cokernel*, defined *pointwise*.

  $\rightarrow$ a morphism ϕ is injective iff $\ker \phi = 0$, and surjective iff $\operatorname{coker} \phi = 0$.

$$\begin{array}{ccccccccc}
\boldsymbol{k} & \xrightarrow{\binom{1}{0}} & \boldsymbol{k}^2 & \xrightarrow{(0\ 1)} & \boldsymbol{k} & \xrightarrow{\binom{1}{1}} & \boldsymbol{k}^2 & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} & \boldsymbol{k}^2 \\
{\scriptstyle \mathbb{1}}\downarrow & & \downarrow{\scriptstyle (0\ 1)} & & {\scriptstyle \mathbb{1}}\downarrow & & \downarrow{\scriptstyle (1\ -1)} & & \downarrow{\scriptstyle 0} \\
\boldsymbol{k} & \xrightarrow{0} & \boldsymbol{k} & \xrightarrow{\mathbb{1}} & \boldsymbol{k} & \xrightarrow{0} & \boldsymbol{k} & \xrightarrow{0} & 0
\end{array}$$

$$\ker \phi = \quad 0 \xrightarrow{0} \boldsymbol{k} \xrightarrow{0} 0 \xrightarrow{0} \boldsymbol{k} \xrightarrow{\binom{1}{2}} \boldsymbol{k}^2 \qquad \operatorname{coker} \phi = 0$$

The Classification Problem

Goal: Classify the representations of a given quiver $\mathtt{Q} = (Q_0, Q_1)$ up to isomorphism.

Note: harder than for single vector spaces because no semisimplicity (subrepresentations may not be summands)

$$\mathbb{V} = \ \boldsymbol{k} \xrightarrow{\mathbb{1}} \boldsymbol{k}$$

$$\mathbb{W} = \ 0 \xrightarrow{0} \boldsymbol{k}$$

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$\rightarrow$ simplifying assumptions:

- $\mathtt{Q}$ is finite and connected
- study the subcategory $\mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$ of *finite-dimensional* representations

$$\underline{\dim}\,\mathbb{V} = (\dim V_1, \cdots, \dim V_n)^\top,$$
$$\dim \mathbb{V} = \|\underline{\dim}\,\mathbb{V}\|_1 = \sum_{i=1}^{n} \dim V_i.$$

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Theorem: [Krull-Remak-Schmidt-Azumaya]
$\forall \mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$, $\exists\ \mathbb{V}_1, \cdots, \mathbb{V}_r$ *indecomposable* s.t. $\mathbb{V} \cong \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_r$.
The decomposition is unique up to isomorphism and reordering.

note: $\mathbb{V}$ indecomposable iff there are no $\mathbb{U}, \mathbb{W} \neq 0$ such that $\mathbb{V} \cong \mathbb{U} \oplus \mathbb{W}$

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proof:

- existence: by induction on $\dim \mathbb{V}$
- uniqueness: show that endomorphisms ring of each $\mathbb{V}_i$ is local (it is isomorphic to $\boldsymbol{k}$ for interval representations of A_n-type quivers), then apply Azumaya's theorem

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The decomposition is unique up to isomorphism and reordering.

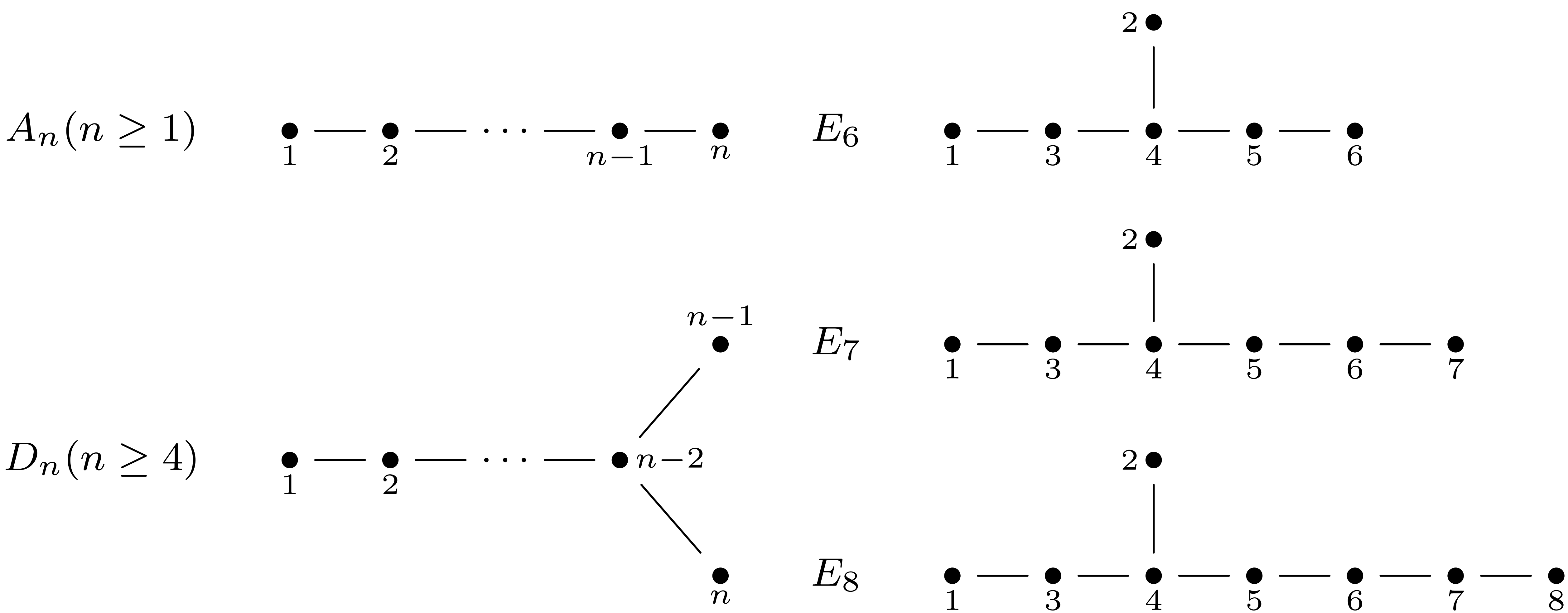
$\to$ problem becomes to identify the indecomposable representations of $\mathtt{Q}$

($\neq$ from identifying representations with no subrepresentations)

(no semisimplicity)

Gabriel's Theorem

Theorem: [Gabriel I]
Assuming Q is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$ iff Q is Dynkin.



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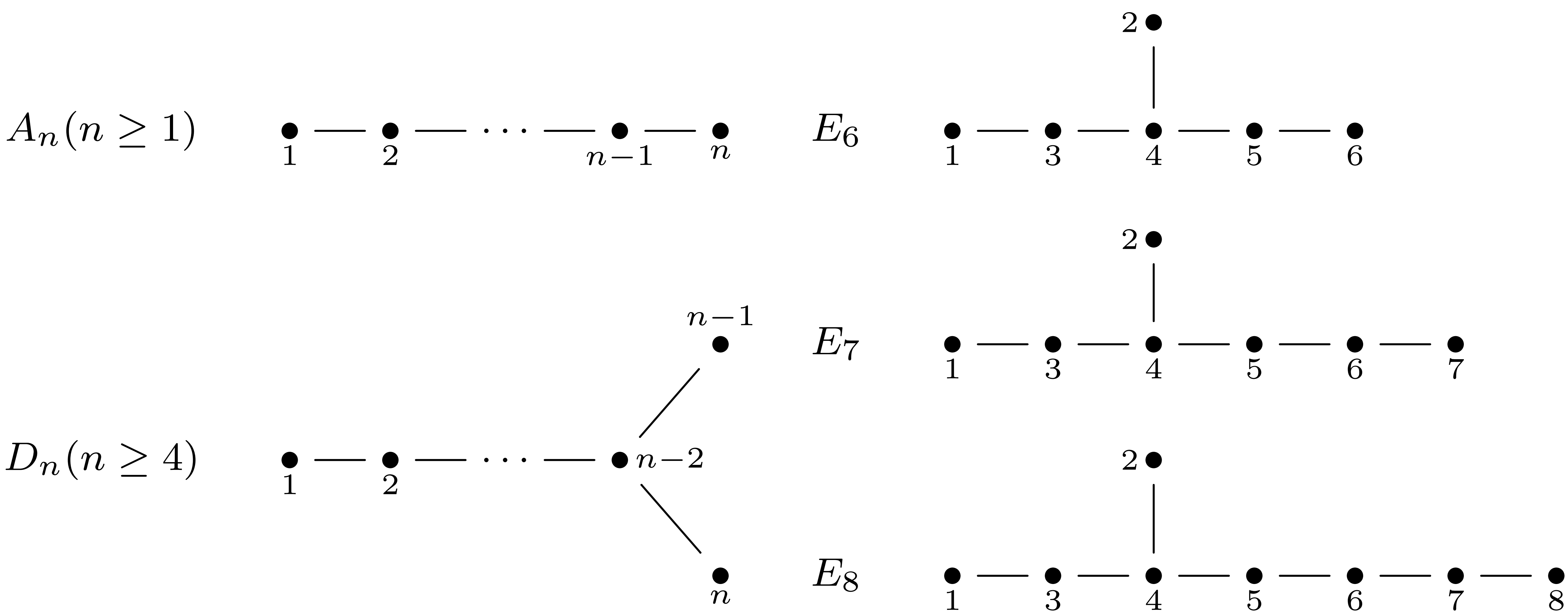
(does not depend on the choice of field and of arrow orientations)



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Assuming $\mathtt{Q}$ is Dynkin with n vertices, the map $\mathbb{V} \mapsto \underline{\dim}\,\mathbb{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of $\mathtt{Q}$ and the set of *positive roots* of the *Tits form* of $\mathtt{Q}$.

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(isom. classes of indecomposables are fully characterized by their dim. vectors)

Gabriel's Theorem

Tits form: given $\mathsf{Q} = (Q_0, Q_1)$ with $|Q_0| = n$ and $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$,

$$q_{\mathsf{Q}}(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_{s_a} x_{t_a}.$$

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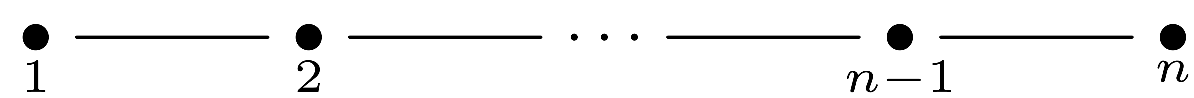
Proposition: $q_{\mathtt{Q}}$ is *positive definite* ($q_{\mathtt{Q}}(x) > 0 \ \forall x \neq 0$) iff $\mathtt{Q}$ is Dynkin.

example: $\mathtt{Q}$ of type A_n:



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proof outline:

- $q_{\mathtt{Q}}$ can be viewed indifferently as a quadratic form on $\mathbb{Z}^n$, $\mathbb{Q}^n$ or $\mathbb{R}^n$
- $q_{\mathtt{Q}}$ pos. definite on $\mathbb{Z}^n \Rightarrow q_{\mathtt{Q}}$ pos. definite on $\mathbb{Q}^n$
- by taking limits, $q_{\mathtt{Q}}$ is then pos. semidefinite on $\mathbb{R}^n$
- $q_{\mathtt{Q}}$ invertible on $\mathbb{Q}^n$ with coeffs. in $\mathbb{Q} \Rightarrow q_{\mathtt{Q}}$ invertible on $\mathbb{R}^n$

  $\Rightarrow q_{\mathtt{Q}}$ pos. definite on $\mathbb{R}^n \Rightarrow \{x \in \mathbb{R}^n \mid q_{\mathtt{Q}}(x) \leq 1\}$ is an ellipsoid

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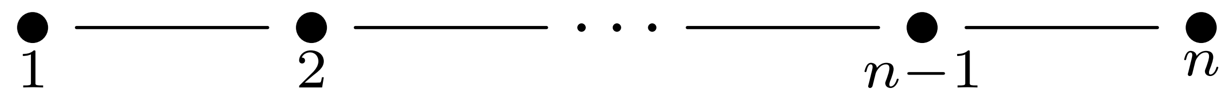

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\end{aligned}
$$

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the corresponding indecomp. representations are isomorphic to $\mathbb{I}_{\mathtt{Q}}[b, d]$:

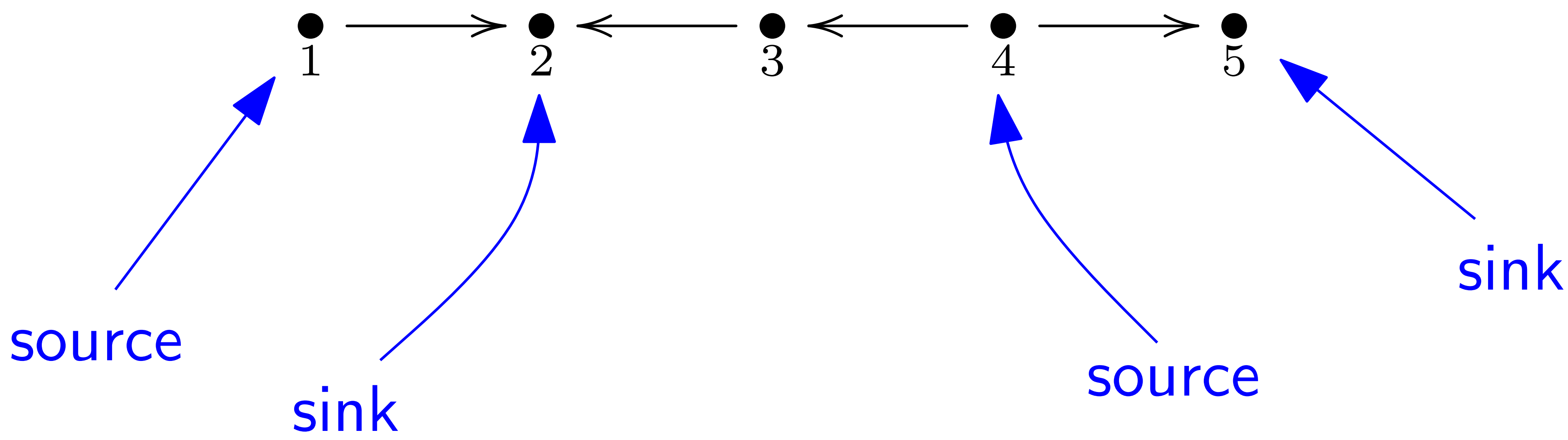
$$\underbrace{0 \overset{0}{\text{———}} \cdots \overset{0}{\text{———}} 0}_{[1,\, b-1]} \overset{0}{\text{———}} \underbrace{\boldsymbol{k} \overset{\mathbb{1}}{\text{———}} \cdots \overset{\mathbb{1}}{\text{———}} \boldsymbol{k}}_{[b,\, d]} \overset{0}{\text{———}} \underbrace{0 \overset{0}{\text{———}} \cdots \overset{0}{\text{———}} 0}_{[d+1,\, n]}$$

Reflection Functors

Advantage: explains the fact that only the dimension vectors play a role in the identification of indecomposable representations. In particular, arrow orientations are irrelevant.

idea: modify quivers by reversing arrows, and study the effect on their representations.

Reflection Functors



Definition: sink = only incoming arrows; source = only outgoing arrows

Reflection Functors

$$1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \longrightarrow 5$$

$$s_1\mathtt{Q}: \quad 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longrightarrow 5$$

$$s_2\mathtt{Q}: \quad 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5$$

$$s_4\mathtt{Q}: \quad 1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$$

$$s_5\mathtt{Q}: \quad 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow 5$$

Definition: reflection s_i = reverse all arrows incident to sink/source i

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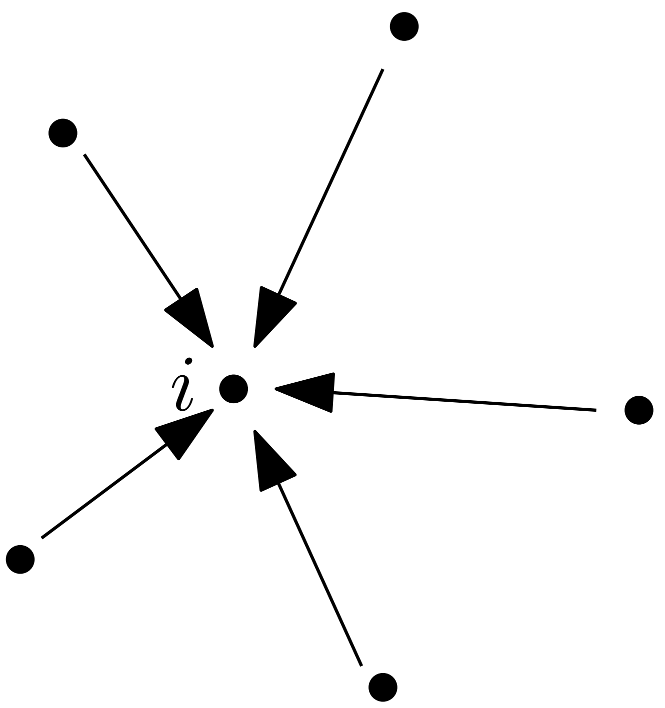
Definition: reflection functor $\mathcal{R}_i^{\pm}$ = functor $\mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q}) \to \mathrm{Rep}_{\boldsymbol{k}}(s_i\mathtt{Q})$

Reflection Functors

Let $\mathbb{V} = (V_i, v_a) \in \mathrm{Rep}_{\boldsymbol{k}}(\mathsf{Q})$, let i be a sink



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Definition: $\mathcal{R}_i^+\mathbb{V} = (W_i, w_a)$ is defined by :

- $W_j = V_j$ for all $j \neq i$
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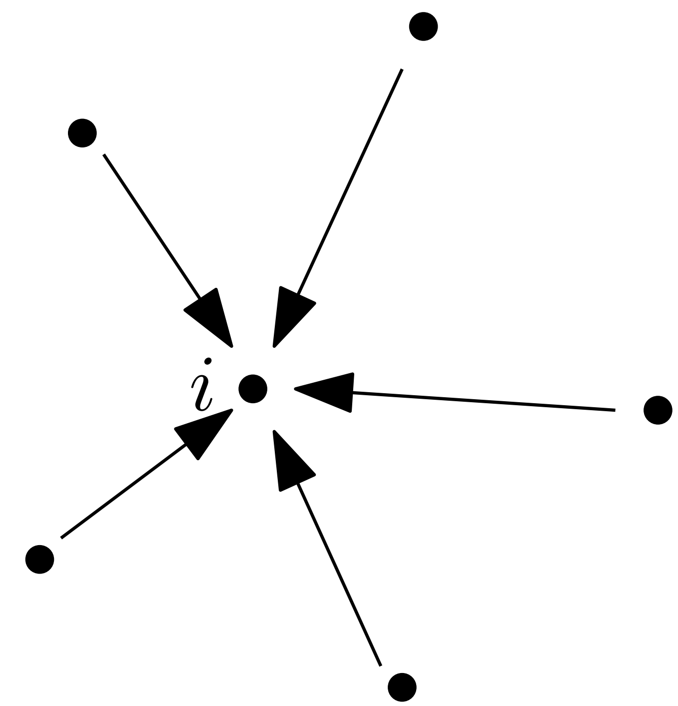

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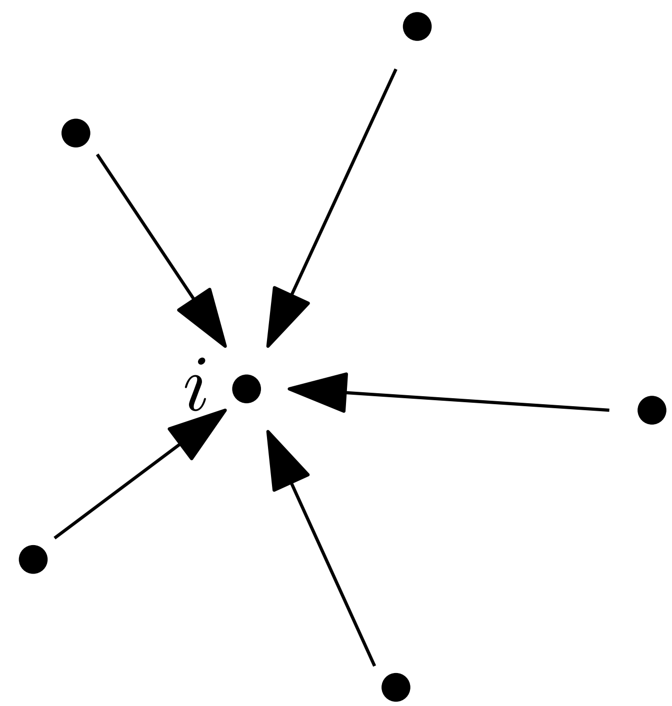
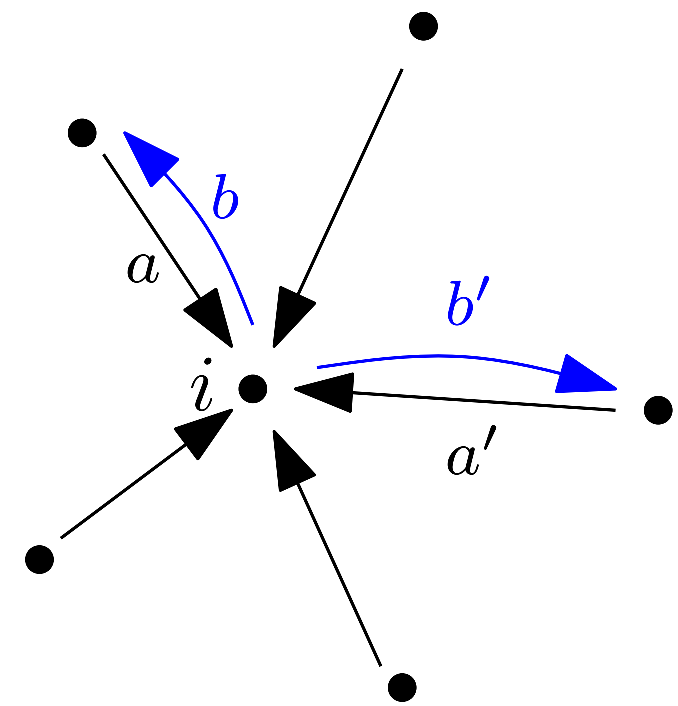
- $W_i = \ker \xi_i : \left| \begin{array}{l} \bigoplus_{a \in Q_1^i} V_{s_a} \longrightarrow V_i \\ (x_{s_a})_{a \in Q_1^i} \longmapsto \sum_{a \in Q_1^i} v_a(x_{s_a}) \end{array} \right.$



Reflection Functors

Let $\mathbb{V} = (V_i, v_a) \in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$, let i be a sink

Definition: $\mathcal{R}_i^+ \mathbb{V} = (W_i, w_a)$ is defined by :



- $W_j = V_j$ for all $j \neq i$

(arrows incident to i)

- $w_a = v_a$ for all $a \notin Q_1^i$

- $W_i = \ker \xi_i : \left| \begin{array}{ccc} \bigoplus_{a \in Q_1^i} V_{s_a} & \longrightarrow & V_i \\ (x_{s_a})_{a \in Q_1^i} & \longmapsto & \sum_{a \in Q_1^i} v_a(x_{s_a}) \end{array} \right.$

- for $a \in Q_1^i$, let b be the opposite arrow, and let w_b be the composition:

$$W_{s_b} = W_i = \ker \xi_i \hookrightarrow \bigoplus_{c \in Q_1^i} V_{s_c} \longrightarrow V_{s_a} = W_{s_a} = W_{t_b}$$

(canonical inclusion) (projection to component V_{s_a})

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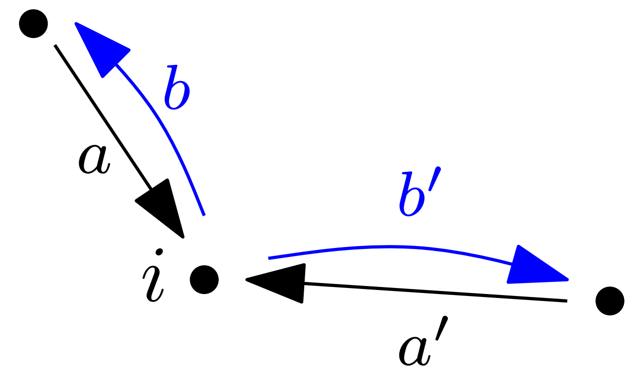

- $W_j = V_j$ for all $j \neq i$
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intuition: W_i carries the information passing through V_i in $\mathbb{V}$

- for $a \in Q_1^i$, let b be the opposite arrow, and let w_b be the composition:

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Reflection Functors

Let $\mathbb{V} = (V_i, v_a) \in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$, let i be a ~~sink~~ source

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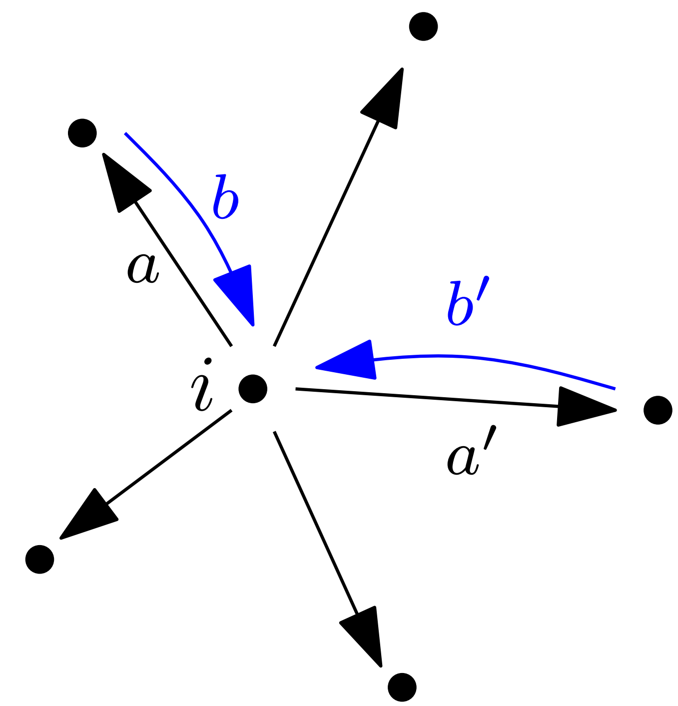

- $W_j = V_j$ for all $j \neq i$
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- $W_i =$ ~~$\ker \zeta_i$~~ $\operatorname{coker} \zeta_i$: $\bigoplus_{a \in Q_1^i} V_{s_a} \longleftarrow V_i$

  $x_i \longmapsto (v_a(x_i))_{a \in Q_1^i}$

- for $a \in Q_1^i$, let b be the opposite arrow, and let w_b be the composition:

$$W_{s_b} = W_{t_a} = V_{t_a} \hookrightarrow \bigoplus_{c \in Q_1^i} V_{t_c} \longrightarrow \operatorname{coker} \zeta_i = W_i = W_{t_b}$$

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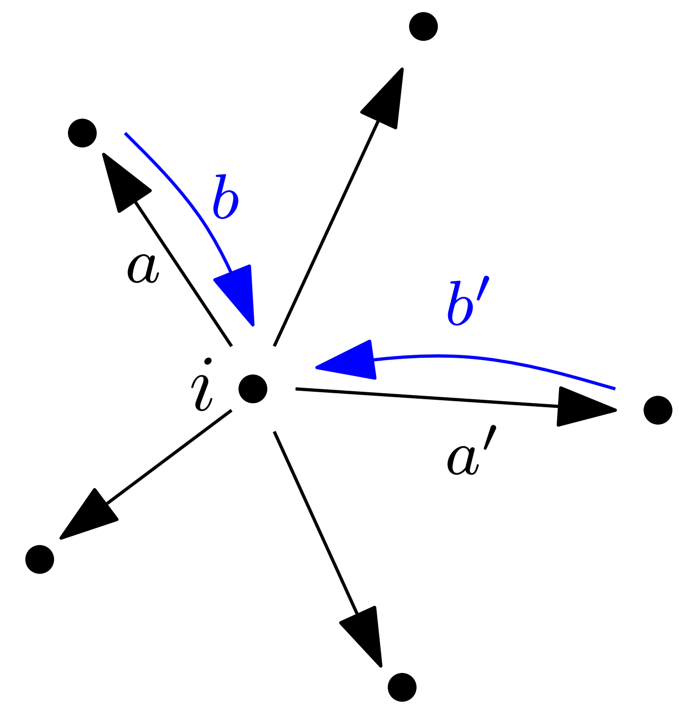

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intuition: this is the operation dual to the previous one (take $V_i = \ker \xi_i$)

- for $a \in Q_1^i$, let b be the opposite arrow, and let w_b be the composition:

$$W_{s_b} = W_{t_a} = V_{t_a} \hookrightarrow \bigoplus_{c \in Q_1^i} V_{t_c} \longrightarrow \operatorname{coker} \zeta_i = W_i = W_{t_b}$$

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Reflection Functors

$$\mathbb{V}: \quad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5$$

$$\mathcal{R}_5^+\mathbb{V}: \quad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \hookleftarrow \ker v_d$$

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$\cong \operatorname{im} v_d$

$\mathbb{V} \cong \mathcal{R}_5^-\mathcal{R}_5^+\mathbb{V} \oplus \mathbb{S}_5^r$, where $r = \dim V_5 - \operatorname{rank} v_d$

Reflection Functors

$$\mathbb{V}: \quad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5$$

$$\begin{array}{ccccccccc}
\mathcal{R}_2^+\mathbb{V}: \quad V_1 & \longleftarrow & \ker v_a + v_b & \longrightarrow & V_3 & \xleftarrow{v_c} & V_4 & \xrightarrow{v_d} & V_5 \\
& {\scriptstyle \pi_1} \nwarrow & \big\downarrow & \nearrow {\scriptstyle \pi_3} & & & & & \\
& & V_1 \oplus V_3 & & & & & &
\end{array}$$

Reflection Functors

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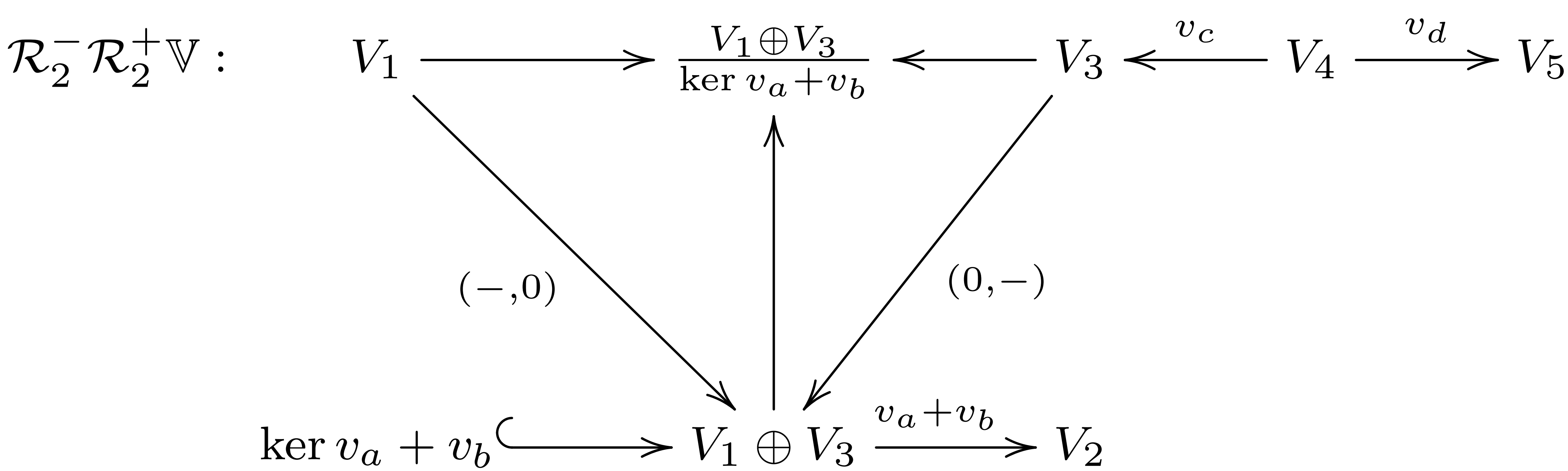

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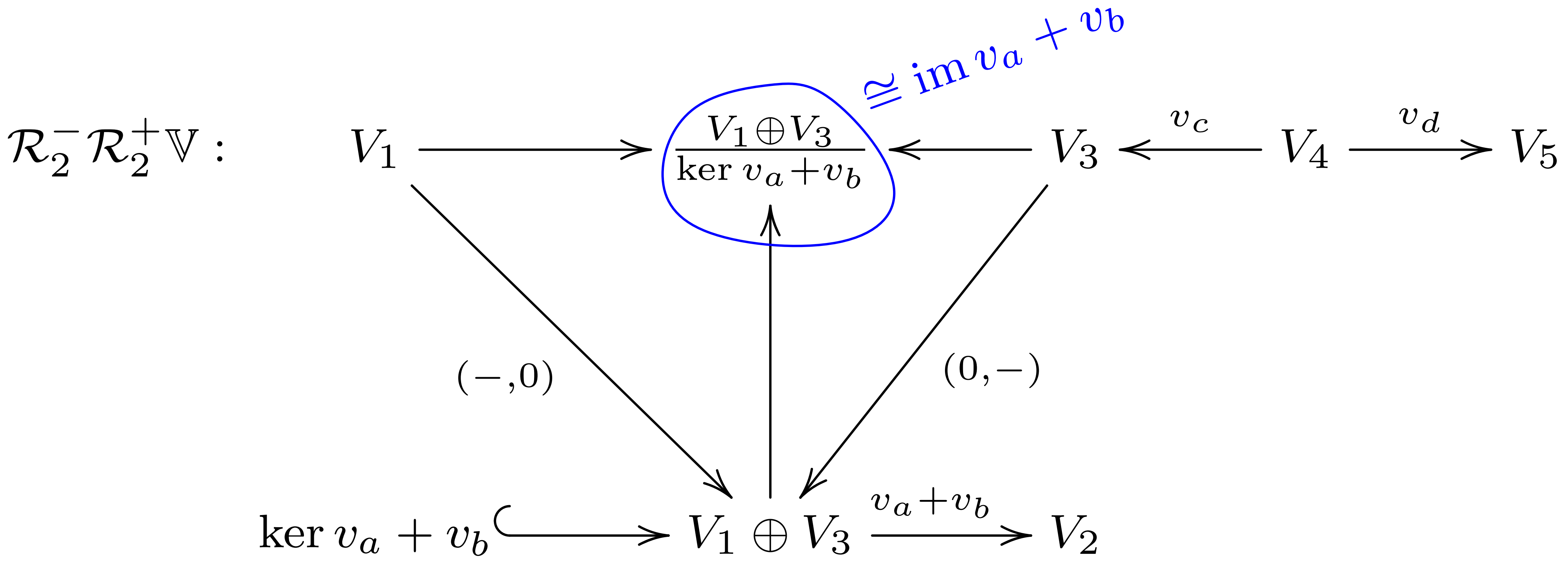

$\mathbb{V} \cong \mathcal{R}_2^- \mathcal{R}_2^+ \mathbb{V} \oplus \mathbb{S}_2^r$, where $r = \dim V_2 - \operatorname{rank} v_a + v_b$

Reflection Functors

Theorem: [Bernstein, Gelfand, Ponomarev]
Let `Q` be a finite connected quiver and let $\mathbb{V}$ be a representation of `Q`. If $\mathbb{V} \cong \mathbb{U} \oplus \mathbb{W}$, then for any source or sink $i \in Q_0$, $\mathcal{R}_i^{\pm}\mathbb{V} \cong \mathcal{R}_i^{\pm}\mathbb{U} \oplus \mathcal{R}_i^{\pm}\mathbb{W}$.

If now $\mathbb{V}$ is indecomposable:
1. If $i \in Q_0$ is a sink, then two cases are possible:

- $\mathbb{V} \cong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{+}\mathbb{V} = 0$.
- $\mathbb{V} \ncong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{+}\mathbb{V}$ is nonzero and indecomposable, $\mathcal{R}_i^{-}\mathcal{R}_i^{+}\mathbb{V} \cong \mathbb{V}$, and the dimension vectors x of $\mathbb{V}$ and y of $\mathcal{R}_i^{+}\mathbb{V}$ are related to each other by the following formula:

$$y_j = \begin{cases} x_j & \text{if } j \neq i; \\ -x_i + \sum_{\substack{a \in Q_1 \\ t_a = i}} x_{s_a} & \text{if } j = i. \end{cases}$$

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If now $\mathbb{V}$ is indecomposable:
2. If $i \in Q_0$ is a source, then two cases are possible:

- $\mathbb{V} \cong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{-}\mathbb{V} = 0$.
- $\mathbb{V} \ncong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{-}\mathbb{V}$ is nonzero and indecomposable, $\mathcal{R}_i^{+}\mathcal{R}_i^{-}\mathbb{V} \cong \mathbb{V}$, and the dimension vectors x of $\mathbb{V}$ and y of $\mathcal{R}_i^{-}\mathbb{V}$ are related to each other by the following formula:

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[...]

Corollary: Reflection Functors preserve the Tits form values except at simple representations:

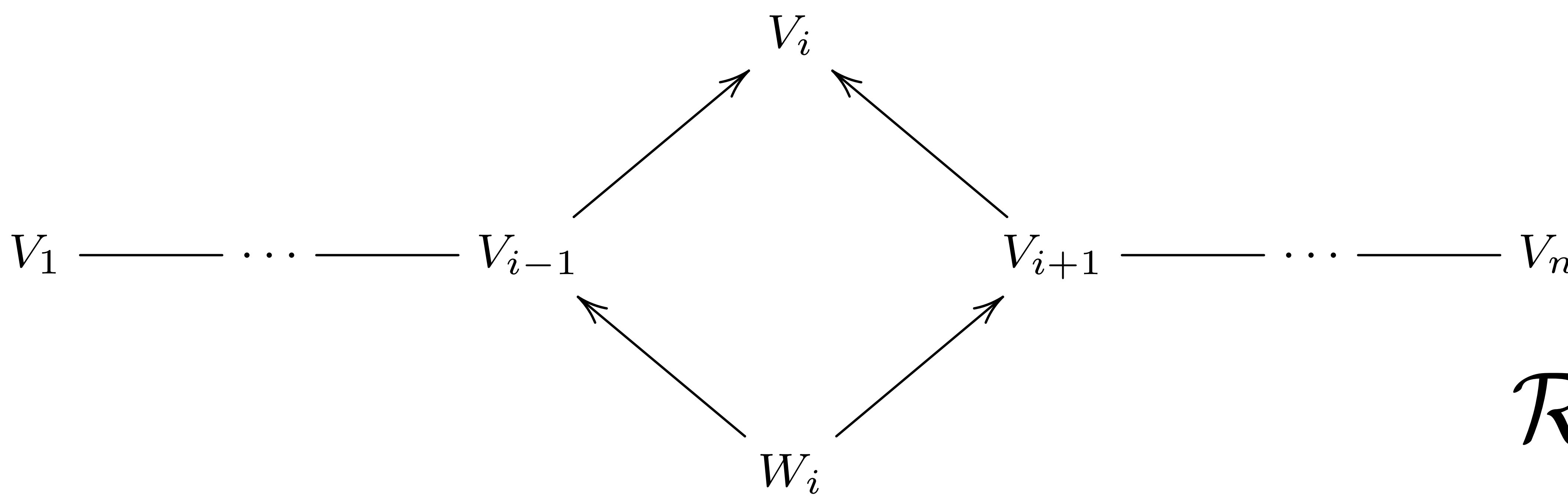
For i source/sink and $\mathbb{V}$ indecomposable,

- either $\mathbb{V} \cong \mathbb{S}_i$, in which case $q_{s_i\mathtt{Q}}(\underline{\dim}\,\mathcal{R}_i^{\pm}\mathbb{V}) = 0$,
- or $q_{s_i\mathtt{Q}}(\underline{\dim}\,\mathcal{R}_i^{\pm}\mathbb{V}) = q_{\mathtt{Q}}(\mathbb{V})$.

For $\mathbb{V}$ arbitrary,
$\mathbb{V} \cong \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_r \oplus \mathbb{S}_i^s \implies q_{s_i\mathtt{Q}}(\underline{\dim}\,\mathcal{R}_i^{\pm}\mathbb{V}) = q_{\mathtt{Q}}(\underline{\dim}\,\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_r)$

Reflection Functors

Example: $\mathtt{Q}$ of type A_n, i sink, $\mathbb{V} \cong \bigoplus_{j=1}^{r} \mathbb{I}_{\mathtt{Q}}[b_j, d_j] \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$:



$\mathbb{V}$

$\mathcal{R}_i^+\mathbb{V}$

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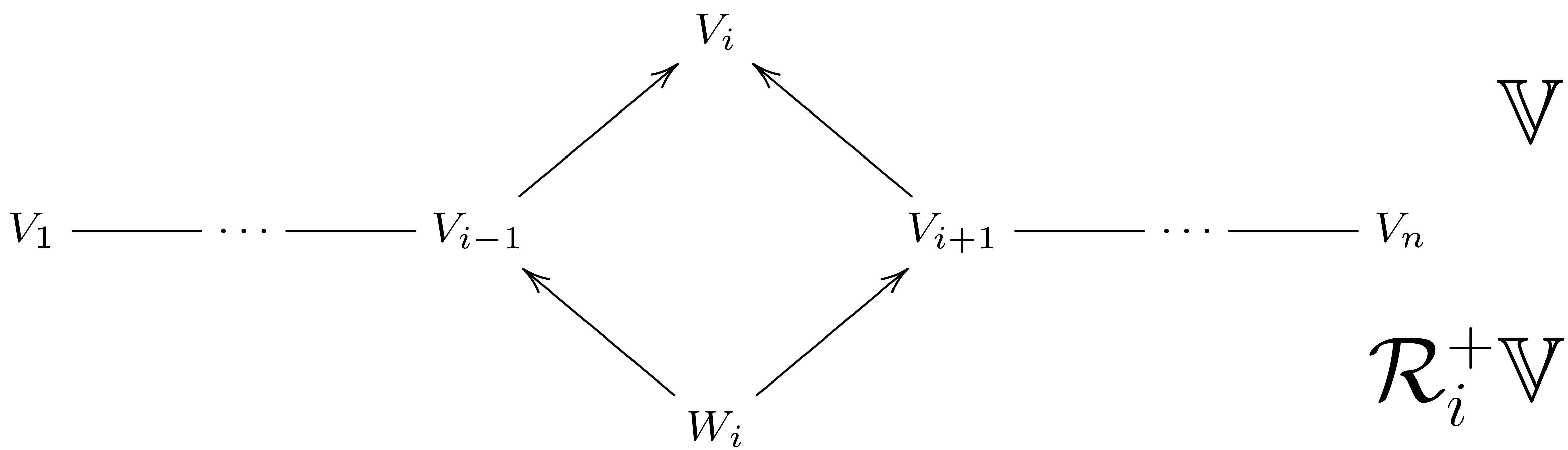

$\mathcal{R}_i^+\mathbb{V} \cong \bigoplus_{j=1}^{r} \mathcal{R}_i^+ \mathbb{I}_{\mathtt{Q}}[b_j, d_j]$, where

$$\mathcal{R}_i^+ \mathbb{I}_{\mathtt{Q}}[b_j, d_j] = \begin{cases} 0 & \text{if } i = b_j = d_j; \\ \mathbb{I}_{s_i\mathtt{Q}}[i+1, d_j] & \text{if } i = b_j < d_j; \\ \mathbb{I}_{s_i\mathtt{Q}}[i, d_j] & \text{if } i+1 = b_j \leq d_j; \\ \mathbb{I}_{s_i\mathtt{Q}}[b_j, i-1] & \text{if } b_j < d_j = i; \\ \mathbb{I}_{s_i\mathtt{Q}}[b_j, i] & \text{if } b_j \leq d_j = i-1; \\ \mathbb{I}_{s_i\mathtt{Q}}[b_j, d_j] & \text{otherwise.} \end{cases}$$

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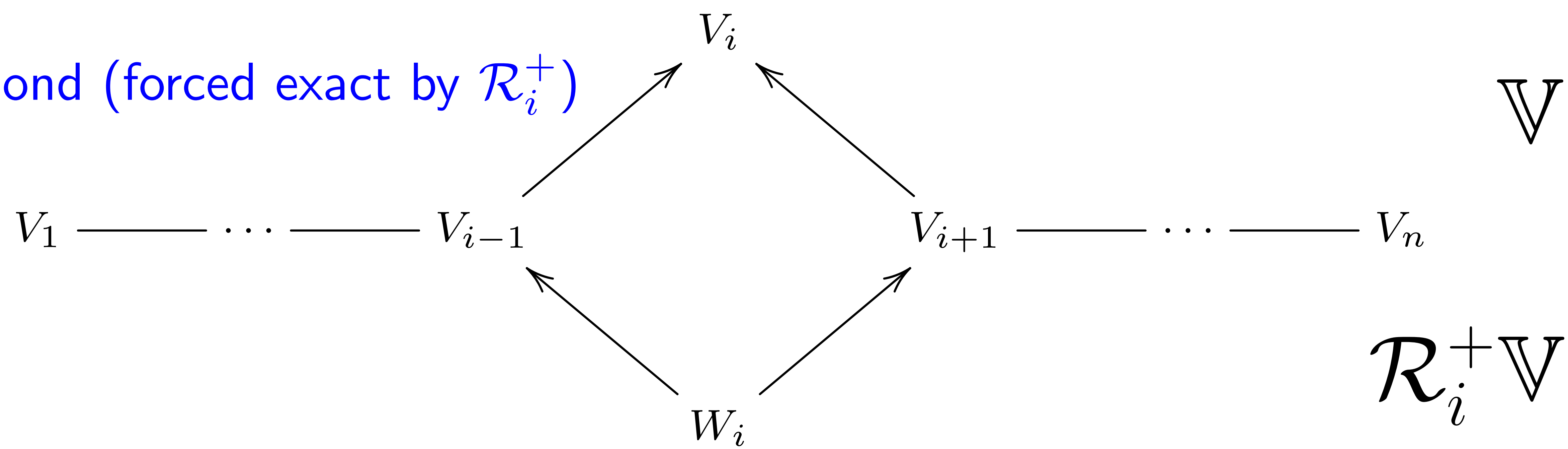
Diamond (forced exact by $\mathcal{R}_i^+$)



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Diamond Principle [Carlsson, de Silva]

Proof of Gabriel's Theorem (A_n case)

Theorem: [Gabriel I, A_n type]
Assuming $\mathtt{Q}$ is of type A_n, every isomorphism class of indecomposable representations in $\mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$ contains $\mathbb{I}_{\mathtt{Q}}[b,d]$ for some $1 \leq b \leq d \leq n$.

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What we can do:

- turn indecomposable representations of $\mathtt{Q}$ into indecomposable representations of reflections of $\mathtt{Q}$ (or zero)
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What we can do:

- turn indecomposable representations of $\mathtt{Q}$ into indecomposable representations of reflections of $\mathtt{Q}$ (or zero)

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$\rightarrow$ idea: turn $\mathtt{Q}$ into itself via sequences of reflections, and observe the evolution of the indecomposables and their Tits form values

Proof of Gabriel's Theorem (A_n case)

Special case: linear quiver $\mathtt{L}_n$: $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$

Let $\mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{L}_n)$ indecomposable, $\underline{\dim}\,\mathbb{V} = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

$\rightarrow$ apply reflections $s_1 s_2 \cdots s_{n-1} s_n \mathtt{L}_n$ and observe evolution of $\underline{\dim}\,\mathbb{V}$

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$$\underline{\dim}\,\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_2, \cdots, x_{n-1}, x_{n-1} - x_n)^\top$$

$$\underline{\dim}\,\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_2, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

$$\cdots$$

$$\underline{\dim}\,\mathcal{R}_2^+ \cdots \mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

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≤ 0

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$$\Longrightarrow \mathcal{C}^+\mathbb{V} = \mathcal{R}_1^+\mathcal{R}_2^+\cdots\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } x_n = 0$$

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$$\cdots$$

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$\Rightarrow \exists i_1, i_2, \cdots, i_{s-1}, i_s$ s.t. $\mathcal{R}^+_{i_s}\mathcal{R}^+_{i_{s-1}}\cdots\mathcal{R}^+_{i_2}\mathcal{R}^+_{i_1}\mathbb{V} = 0$

$$\mathcal{R}^+_{i_{s-1}}\cdots\mathcal{R}^+_{i_2}\mathcal{R}^+_{i_1}\mathbb{V} \neq 0$$

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$\Longrightarrow \mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2}\mathcal{R}^+_{i_1}\mathbb{V}$ is indecomposable and isomorphic to $\mathbb{S}_r$ for some $1 \leq r \leq n$

(Reflection Functor Thm)

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$\Longrightarrow \mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2}\mathcal{R}^+_{i_1}\mathbb{V}$ is indecomposable and isomorphic to $\mathbb{S}_r$ for some $1 \leq r \leq n$

$\Longrightarrow q_{\mathtt{L}_n}(\underline{\dim}\,\mathbb{V}) = q_{s_{i_{s-1}} \cdots s_{i_1}\mathtt{L}_n}(\underline{\dim}\,\mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2}\mathcal{R}^+_{i_1}\mathbb{V}) = q_{s_{i_{s-1}} \cdots s_{i_1}\mathtt{L}_n}(\underline{\dim}\,\mathbb{S}_r) = 1$

(Corollary)

Proof of Gabriel's Theorem (A_n case)

Special case: linear quiver $\mathtt{L}_n$: $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$

Let $\mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{L}_n)$ indecomposable, $\underline{\dim}\,\mathbb{V} = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

$\exists i_1, i_2, \cdots, i_{s-1}, i_s$ s.t. $\mathcal{R}^+_{i_s}\mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2}\mathcal{R}^+_{i_1}\mathbb{V} = 0$

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$\Longrightarrow \underline{\dim}\,\mathbb{V} = \underline{\dim}\,\mathbb{I}_{\mathtt{L}_n}[b,d]$ for some $1 \leq b \leq d \leq n \Longrightarrow \mathbb{V} \cong \mathbb{I}_{\mathtt{L}_n}[b,d]$ $\square$

(Example)

Proof of Gabriel's Theorem (A_n case)

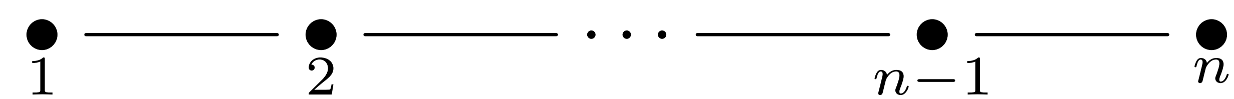
A_n-type quiver Q: $\underset{1}{\bullet} \;\text{———}\; \underset{2}{\bullet} \;\text{———}\; \cdots \;\text{———}\; \underset{n-1}{\bullet} \;\text{———}\; \underset{n}{\bullet}$

$\rightarrow$ goal: find a sequence of indices $i_1, i_2, \cdots, i_{s-1}, i_s$ s.t.

$$\mathcal{R}^+_{i_s}\mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2}\mathcal{R}^+_{i_1}\mathbb{V} = 0 \text{ for all } \mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$$

Proof of Gabriel's Theorem (A_n case)

A_n-type quiver $\mathtt{Q}$:



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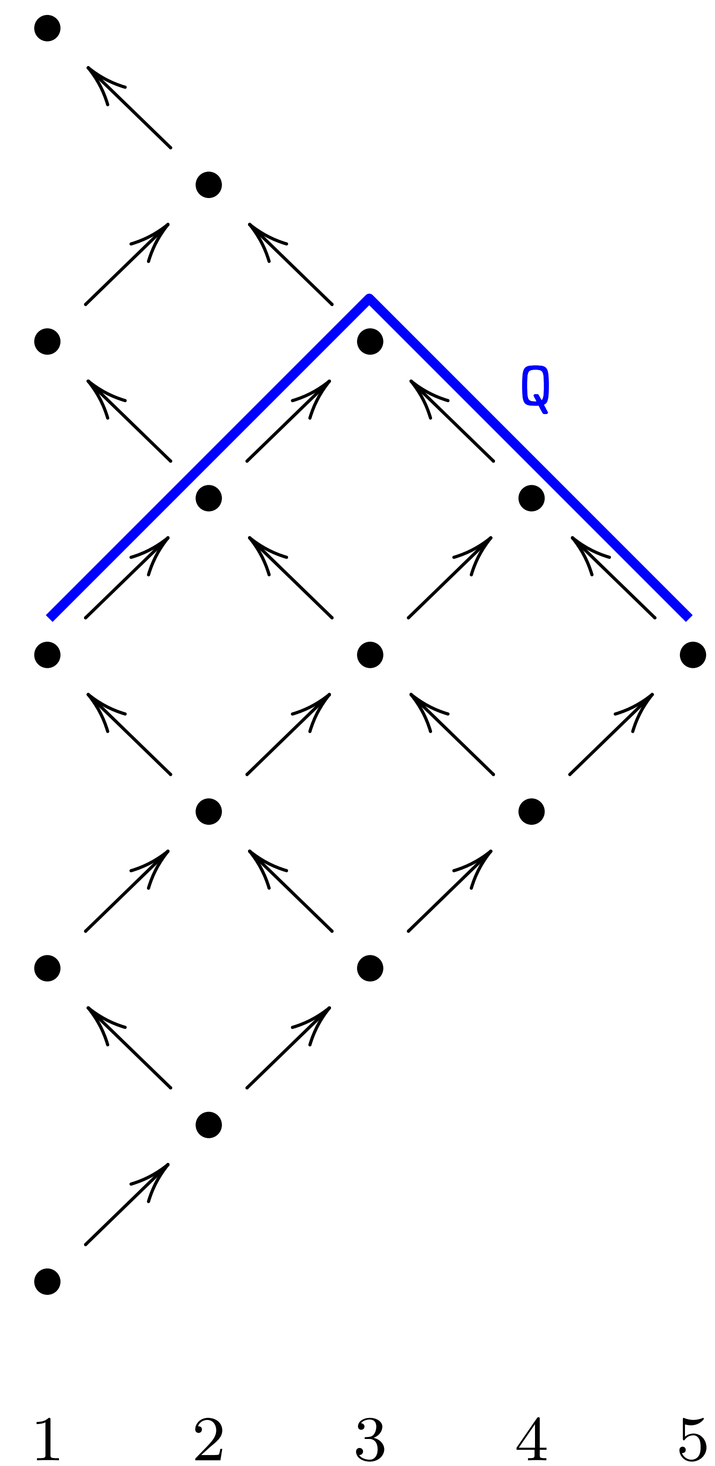
$\rightarrow$ idea: turn $\mathtt{Q}$ into $\mathtt{L}_n$, then use the same sequence a before

Proof of Gabriel's Theorem (A_n case)

A_n-type quiver Q:

- embed Q in a giant pyramid



Proof of Gabriel's Theorem (A_n case)

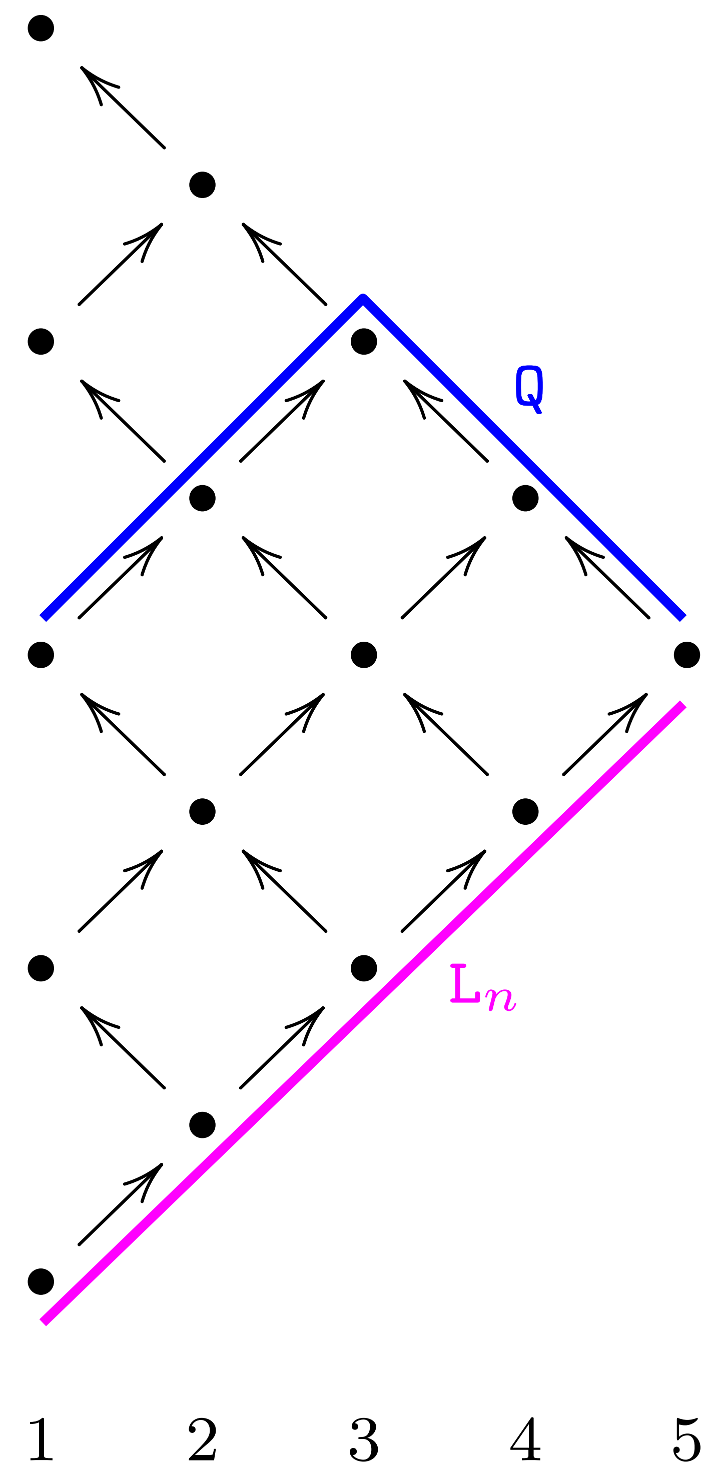
A_n-type quiver `Q`:

- embed `Q` in a giant pyramid

- travel down the pyramid to its bottom $\mathtt{L}_n$

  $\rightarrow$ travelling one level down reverses the leftmost backward arrow

  e.g. $s_1 s_2 s_3$ reverses $\underset{3}{\bullet} \longleftarrow \underset{4}{\bullet}$



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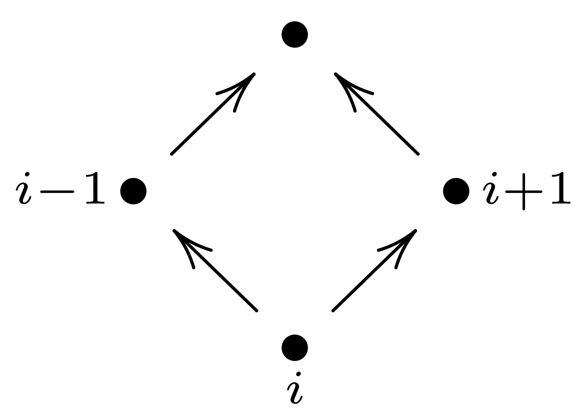
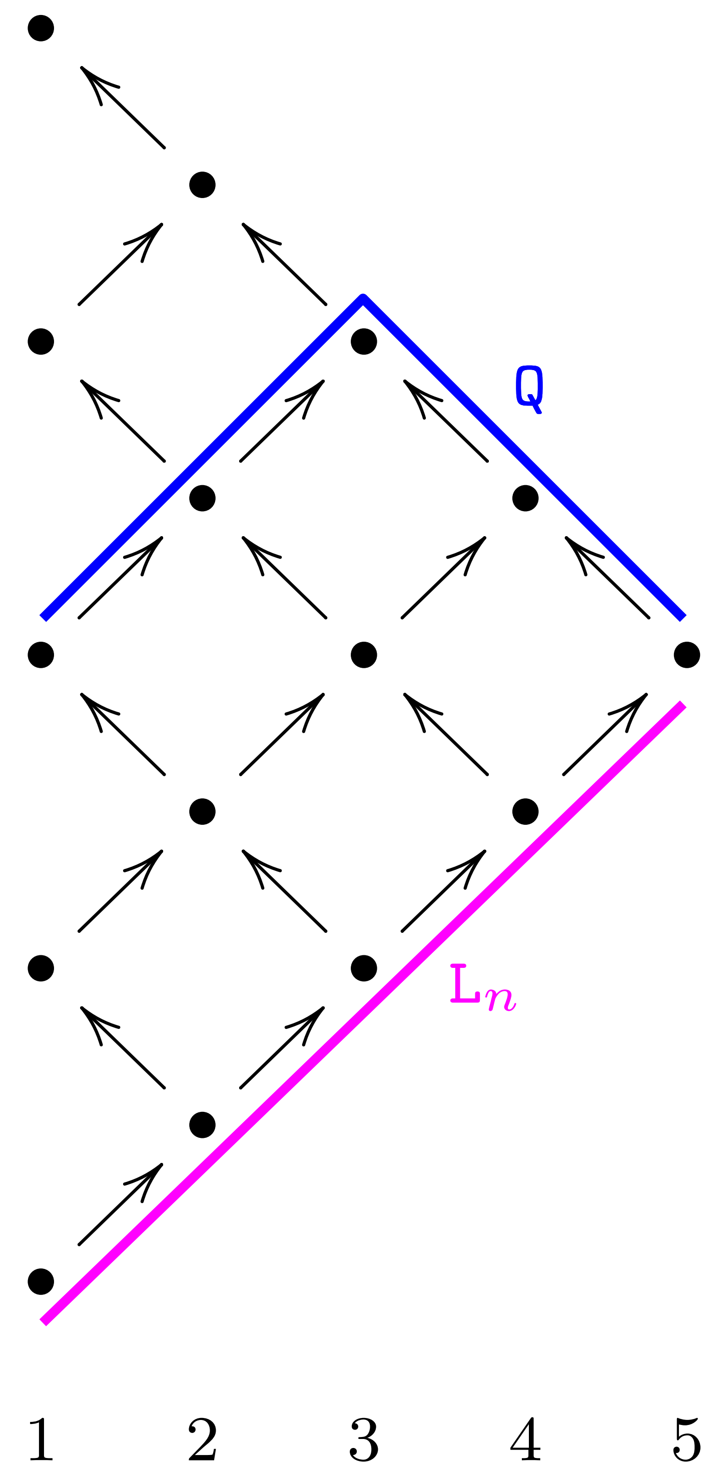
- each diamond



is travelled down using $\mathcal{R}_i^+$



Proof of Gabriel's Theorem (A_n case)

Theorem: [Gabriel II]
Assuming Q is Dynkin with n vertices, the map $\mathbb{V} \mapsto \underline{\dim}\,\mathbb{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of Q and the set of *positive roots* of the *Tits form* of Q.

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What we know:

- the positive roots of $q_{\mathtt{Q}}$ are the dimension vectors of interval modules $\mathbb{I}_{\mathtt{Q}}[b,d]$
- each isomorphism class C of indecomposables contains ≥ 1 interval module

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Additional observations:

- $\neq$ interval modules are $\ncong$, therefore each class C contains 1 interval module
- each interval module is indecomposable (endomorphism ring isom. to $\boldsymbol{k}$)

□

Proof of Gabriel's Theorem (general case)

Theorem: [Gabriel I]
Assuming $\mathtt{Q}$ is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$ iff $\mathtt{Q}$ is Dynkin.

Proof of Gabriel's Theorem (general case)

Theorem: [Gabriel I]
Assuming Q is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$ iff Q is Dynkin.

$\Longleftarrow$ same process as before (proves Gabriel II as well):

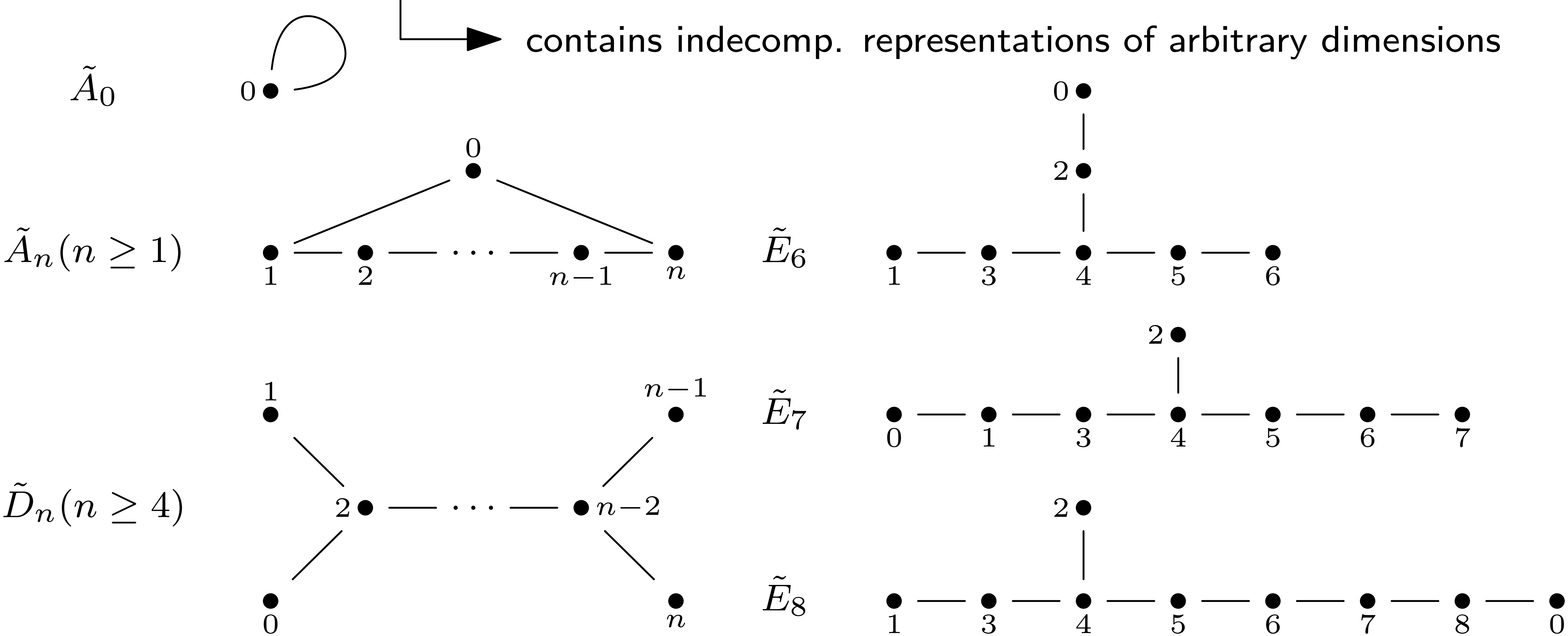
- Define Coxeter functors for arbitrary (finite, connected, loop-free) quivers
- iterate Coxeter functor to eventually send every indecomposable to zero
- derive bijection between isom. classes of indecomposables to positive roots of $q_{\mathtt{Q}}$ via simple representations

Proof of Gabriel's Theorem (general case)

Theorem: [Gabriel I]
Assuming Q is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$ iff Q is Dynkin.

$\Longrightarrow$ every connected quiver that is not Dynkin contains one of these:

→ contains indecomp. representations of arbitrary dimensions



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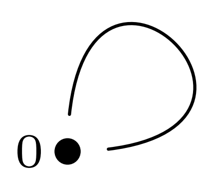
$\tilde{A}_0$ 

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$$\begin{array}{ccc} \boldsymbol{k}^r & \xrightarrow{f} & \boldsymbol{k}^r \\ \phi \downarrow & & \downarrow \psi \\ \boldsymbol{k}^s & \xrightarrow{g} & \boldsymbol{k}^s \end{array}$$

$\mathbb{V} = (\boldsymbol{k}^r, f)$ isomorphic to $\mathbb{W} = (\boldsymbol{k}^s, g)$

$\Leftrightarrow$

$r = s$ and $\exists \phi, \psi \in \mathrm{Aut}(\boldsymbol{k}^r)$ s.t. $f = \psi^{-1} \circ g \circ \phi$

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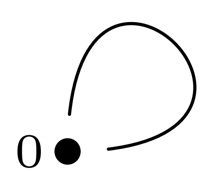

→ contains indecomp. representations of arbitrary dimensions

$\to$ injection from conjugacy classes of Jordan block matrices to isomorphism classes of indecomposables

(injection becomes bijection when $\boldsymbol{k}$ is algebraically closed)

$$\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

Beyond Gabriel's Theorem

Gabriel's theorem is about:

- Dynkin quivers
- finite-dimensional representations

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Finite connected quivers:

Theorem: [Kac]
The set of dimension vectors of finite-dimensional indecomposable representations of a finite connected quiver $\mathtt{Q}$ is precisely the set of positive roots of its Tits form. In particular, this set is independent of the arrow orientations in $\mathtt{Q}$ and of the base field.

(catch: the map $\mathbb{V} \mapsto \underline{\dim}\, \mathbb{V}$ may not be injective)

Beyond Gabriel's Theorem

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Finite disconnected quivers:

$$\mathtt{Q} = \mathtt{Q}_1 \sqcup \mathtt{Q}_2 \Longrightarrow \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q}) \cong \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q}_1) \times \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q}_2)$$

Beyond Gabriel's Theorem

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Finite quivers, infinite-dimensional representations:

$\rightarrow$ path algebras, modules, Auslander-Reiten theory

Theorem: [Auslander+Gabriel]
For a Dynkin quiver $\mathtt{Q}$, every indecomposable representation in $\mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$ has finite dimension, and every representation in $\mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$ is a direct sum of indecomposable representations. In particular, $\mathtt{Q}$ has finitely many isomorphism classes of indecomposable representations, and all of them are finite-dimensional.

Beyond Gabriel's Theorem

Gabriel's theorem is about:

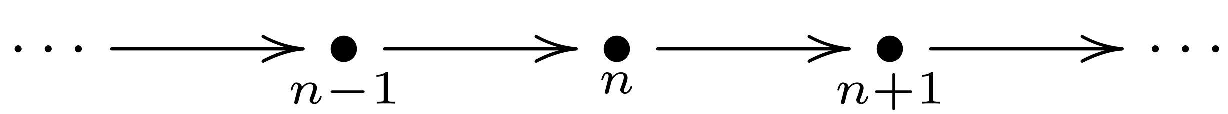
- Dynkin quivers
- finite-dimensional representations

Infinite quivers:

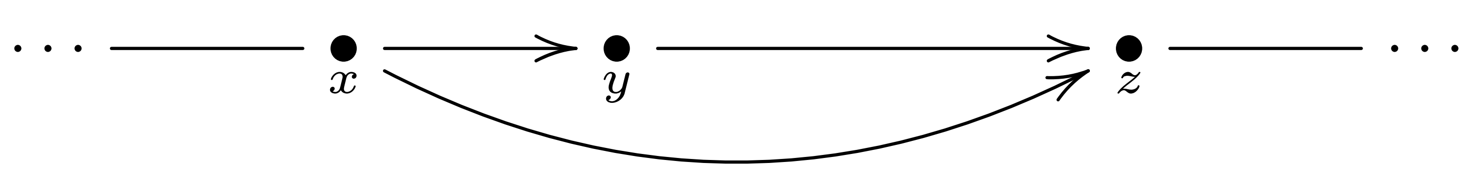
$\rightarrow$ research in progress

(*pointwise finite dimensional*)

Theorem: [Webb] Decomposition of *pfd* rep. of the $\mathbb{Z}$ quiver



Theorem: [Crawley-Boevey] Decomposition of *pfd* rep. of the $\mathbb{R}$ quiver



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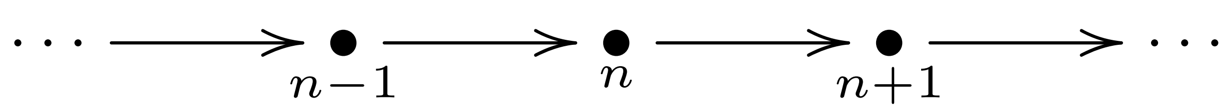
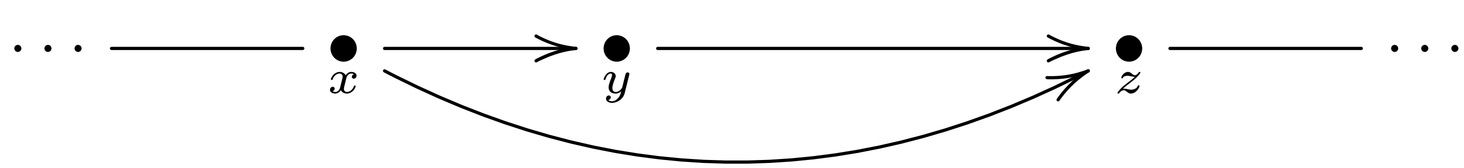


Theorem: [Botnan 2015]
Arrow orientations don't matter

Theorem: [Crawley-Boevey] Decomposition of *pfd* rep. of the $\mathbb{R}$ quiver



Theorem: [Cochoy, O. 2016]
Arrow orientations don't matter*

* under exactness conditions