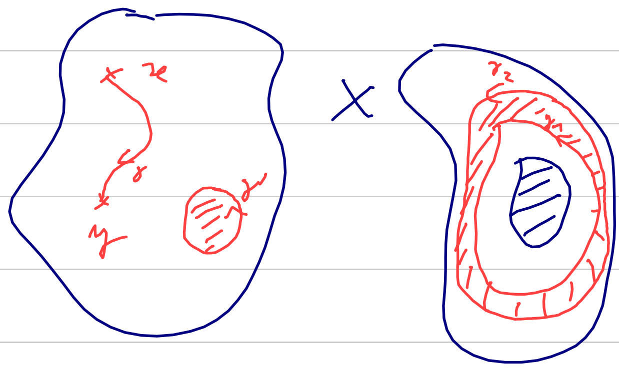
Singular and Simplicial Homology

① Intuition :

Invariant that captures the structure of a topological space X.

↳ typically :



$x \sim y \iff \exists\, \gamma : [0,1] \to X$ s.t. $\partial\gamma = \{x, y\}$

(belong to some path-c.c)

"contractible" $\gamma' \sim 0 \iff \exists\, \Sigma$ s.t. $\partial\Sigma = \gamma$

$\gamma_1 \sim \gamma_2 \iff \exists\, \Sigma$ s.t. $\partial\Sigma = \gamma_1 \cup \gamma_2$.

"deformable into each other"

↳ algebraic formulation :
- $x, y, \gamma, \gamma', \gamma_i, \Sigma \in$ module (vect. space)
- "$\cup$" → "$\pm$"
- ∂ : linear operator

cycles ⟷ kernel of ∂

boundaries ⟷ image of ∂

② Formalisms :

Singular homology : Category Top :
- obj. : topo. spaces
- mor. : continuous maps

Simplicial homology : Category Simp :
- obj : simplicial complexes
- mor : simplicial maps

③ Chains: "linear combinations of singular simplices"

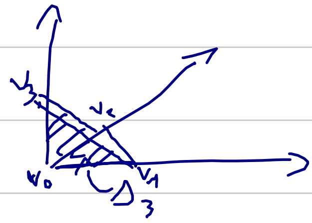
Def: Standard simplex: $\Delta_n := \mathrm{Conv}\{v_0, \dots, v_n\} \subset \mathbb{R}^n$

$$= \Big\{ \sum_{i=0}^{n} \lambda_i v_i \;\Big|\; \lambda_i \geq 0 \ \forall i, \ \sum_{i=0}^{n} \lambda_i = 1 \Big\}.$$

where: $v_0 = 0 \in \mathbb{R}^n$

$\forall i, \ v_i = (0 \_ 0, 1, 0 \_ 0)$

$\uparrow$ i^{th} position



Def, (singular) n-simplex: map $\sigma : \Delta_n \to X$.

$\sigma \in \mathcal{C}(\Delta_n, X)$

k: ring (field)

(singular) n-chain: formal k-linear combination of n-simplices:

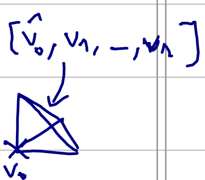
$$c = \sum_{\substack{i \in I \\ |I| < \infty}} \alpha_i \cdot \sigma_i$$

↳ space of n-chains: free module (vector space) on $\mathcal{C}(\Delta_n, X)$

$$C_n(X; k) := \bigoplus_{\sigma \in \mathcal{C}(\Delta_n, X)} k = k^{(\mathcal{C}(\Delta_n, X))}$$

④ Boundary operator:

Def: Boundary operator $\partial_n : C_n(X; k) \to C_{n-1}(X; k)$



$(*)$

↳ on basis elements (simplices): $\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$

$(= \mathrm{Conv}\{v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n\})$

↳ on the whole of $C_n(X; k)$ by linearity:

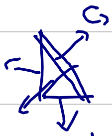
$$\partial_n \Big(\sum_{\substack{i \in I \\ |I| < \infty}} \alpha_i \sigma_i \Big) := \sum_{i \in I} \alpha_i \, \partial_n(\sigma_i).$$

(*) <u>caveat</u>: in the formula, implicit identification of the facet $[v_0, \dots, \hat{v}_i, \dots, v_n]$ with Δ_{n-1}.

$\hookrightarrow$ $\exists$ $(2)^{\wedge}$ possible such identifications.

<u>but</u> only 2 classes of equivalence, depending on sign of permutation

(same sign $\Rightarrow$ mere rotation of simplex
opposite signs $\Rightarrow$ flip of the map)



(orientations of facets in the formula of ∂_n)

Def: Orientation of Δ_n $\Longleftrightarrow$ order on $v_0, \dots, v_n$. $(\pi \in \Sigma_{n+1})$

$\pi \sim \pi' \Longleftrightarrow \operatorname{sgn} \pi = \operatorname{sgn} \pi'$.

Prop: $\forall n \geq 0$, $\partial_n \circ \partial_{n+1} = 0$.

<u>Pf</u>: (exercise) $\rightarrow$ show nullity on basis elements (simplices).

For $\sigma \in C_{n+1}(X;k)$, $\partial_n \circ \partial_{n+1}(\sigma) = \partial_n\left(\sum_{i=0}^{n+1} (-1)^i \sigma|_{[-,\hat{v}_i,-]}\right)$

$$= \sum_{i=0}^{n+1} (-1)^i \partial_n \left(\sigma|_{[-,\hat{v}_i,-]}\right)$$

$$= \sum_{i=0}^{n+1} (-1)^i \left(\sum_{0 \leq j < i} (-1)^j \sigma|_{[-,\hat{v}_j,-,\hat{v}_i,-]} + \sum_{i<j\leq n+1} (-1)^{j+1} \sigma|_{[-,\hat{v}_i,-,\hat{v}_j,-]} \right).$$

$$= \sum_{j<i} (-1)^{i+j} \sigma|_{[-,\hat{v}_j,-,\hat{v}_i,-]} - \sum_{i<j} (-1)^{i+j} \sigma|_{[-,\hat{v}_i,-,\hat{v}_j,-]}$$

(exchange roles of i,j in 2^{nd} term)

$$= \sum_{j<i} (-1)^{i+j} \sigma|_{[-,\hat{v}_j,-,\hat{v}_i,-]} - \sum_{j<i} (-1)^{i+j} \sigma|_{[-,\hat{v}_j,-,\hat{v}_i,-]}$$

$= 0$. $\boxed{\Xi}$

5 Homology groups: (singular)

$$\dots \xrightarrow{\partial_{n+1}} C_n(X;k) \xrightarrow{\partial_n} C_{n-1}(X;k) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0(X;k) \xrightarrow{\partial_0} 0 .$$

By the proposition, this is a chain complex.

Def: $(C_\bullet, \partial_\bullet) := \dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$ (possibly infinite both ways)

s.t. C_n module $\forall n$

∂_n linear map $\forall n$

$\partial_n \circ \partial_{n+1} = 0 \quad \forall n$

↳ homology: $H_n(C_\bullet, \partial_\bullet) := \ker \partial_n / \operatorname{Im} \partial_{n+1}$.

(quotient of modules → note: $\operatorname{Im} \partial_{n+1} \subseteq \ker \partial_n$)

→ here: $H_n(X;k) := \ker \partial_n / \operatorname{Im} \partial_{n+1}$ in $C_\bullet(X;k)$.
(singular homology)

6 Morphisms: let $f : X \to Y$ continuous.

$$\begin{tikzcd} \Delta_n \arrow[d, "\sigma"] \\ X \arrow[d, "f"] \\ Y \end{tikzcd}$$

$f_n : C_n(X;k) \to C_n(Y;k)$

$\Psi(\Delta_n, X) \ni \sigma \longmapsto f \circ \sigma \in \Psi(\Delta_n, Y)$

$\sum_{\substack{i \in I \\ |I| < \infty}} \lambda_i \sigma_i \longmapsto \sum_{i \in I} \lambda_i f_n(\sigma_i)$

↳ $f_\bullet : C_\bullet(X;k) \to C_\bullet(Y;k)$ (collection of degree-wise maps)

$$\begin{tikzcd} \dots \arrow[r] & C_n(X;k) \arrow[r] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1}(X;k) \arrow[r] \arrow[d, "f_{n-1}"] & \dots \arrow[r] & C_0(X;k) \arrow[r] \arrow[d, "f_0"] & 0 \\ \dots \arrow[r] & C_n(Y;k) \arrow[r] & C_{n-1}(Y;k) \arrow[r] & \dots \arrow[r] & C_0(Y;k) \arrow[r] & 0 \end{tikzcd}$$

$\boxed{\text{Prop:}}$ $f_\bullet$ is a chain map, i.e.:

$$\forall n, \quad f_{n-1} \circ \partial_n = \partial_n \circ f_n. \qquad \text{(each square commutes)}$$

Pf: (exercise) $\to$ show equality on basis el^{ts} (simplices)

For $\sigma \in \mathcal{C}(\Delta_n, X)$: $\quad \Delta_n \xrightarrow{\sigma} X \xrightarrow{f} Y$

$$f_{n-1} \circ \partial_n(\sigma) = f_{n-1}\left(\sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \right)$$

$$= \sum_{i=0}^{n} (-1)^i f_{n-1}\left(\sigma|_{[\_, \hat{v}_i, \_]}\right) = \sum_{i=0}^{n} (-1)^i (f \circ \sigma)|_{[\_, \hat{v}_i, \_]}$$

$$= \partial_n (f \circ \sigma) = \partial_n (f_n(\sigma)) = (\partial_n \circ f_n)(\sigma). \qquad \boxed{\Xi}$$

$\mapsto$ homology:

$$H_n(f) := \text{induced map } f_*: H_n(X;k) \to H_n(Y;k)$$

(induced from $f_\bullet$ since $f_\bullet$ sends kernels to kernels and images to images)

$\boxed{\text{Prop:}}$ (functoriality)

(i) $H_n(g \circ f) = H_n(g) \circ H_n(f)$ for any $X \xrightarrow{f} Y \xrightarrow{g} Z$

(ii) $H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$

Pf: (i) follows immediately from associativity of composition of chain maps, which follows from that of continuous maps in $\Delta_n \xrightarrow{\sigma} X \xrightarrow{f} Y \xrightarrow{g} Z$

(ii) follows from (i)

$\boxed{\Xi}$

$\boxed{\text{Prop:}}$ (Homotopy invariance)

Corollary: $X \sim Y \Rightarrow H_n(X) \simeq H_n(Y)$.

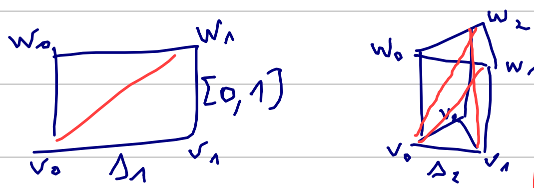
If $f \underset{\Gamma}{\sim} g : X \to Y$ then $f_* = g_*$.

i.e. $\exists\, \Gamma : [0,1] \times X \to Y$ s.t. $\Gamma(0,.) = f$ and $\Gamma(1,.) = g$.

Pf: (exercise, see [Hatcher] p. 112)

$\to$ Step 1: subdivide $[0,1] \times \Delta_n$ into simplices

(i.e. triangulate it)



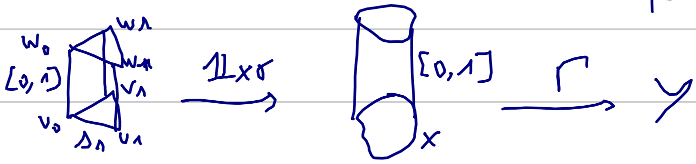
(note: always create edges with increasing vertex indices)

etc.

$\to$ step 2: define prism operators $p_n : C_n(X;k) \to C_{n+1}(Y;k)$

$$p_n(\sigma) := \sum_{i=0}^{n} (-1)^i \, \Gamma \circ (\mathbb{1} \times \sigma)\Big|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$$

where:



$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X;k) \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n(X;k) \arrow[r, "\partial_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "p_n"'] & C_{n-1}(X;k) \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "p_{n-1}"'] & \cdots \arrow[r] & 0 \\
\cdots \arrow[r] & C_{n+1}(Y;k) \arrow[r, "\partial_{n+1}"'] & C_n(Y;k) \arrow[r, "\partial_n"'] & C_{n-1}(Y;k) \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}$$

Claim: $p_\cdot$ is a chain homotopy between $f_\cdot$ and $g_\cdot$, i.e.:

$\forall n$, $\quad g_n - f_n = \partial_{n+1} \circ p_n - p_{n-1} \circ \partial_n$.

$\Rightarrow \quad g_* - f_* = 0$.

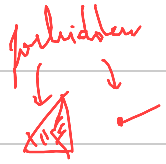
∎

(7) Simplicial homology: (for calculations)

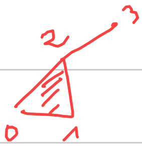
Reminder: category = Simp : $\begin{cases} \text{obj} = \text{simplicial complexes} \\ \text{mor} = \text{——— maps} \end{cases}$

(restriction to a smaller category whose objects and morphisms have a combinatorial description)

Formalism:



- simplicial complex : collection K of simplices over a fixed set of vertices in $\mathbb{R}^d$ (potentially uncountably many vertices)



s.t. $\begin{cases} \forall \sigma \in K, \ \forall \tau \text{ face of } \sigma, \ \tau \in K. \\ \forall \sigma, \tau \in \text{complex}, \ \sigma \cap \tau \text{ is a face of both } \sigma \text{ and } \tau \end{cases}$

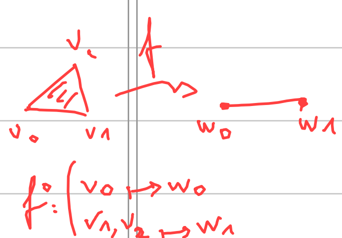
Note: σ is uniquely defined by its vertex set (as their convex hull)

- underlying topological space $|K|$:

$\begin{cases} \forall \sigma \in K, \text{ equip } \sigma \text{ with restriction of Eucl. topo.} \\ U \subseteq |K| \text{ is closed} \iff U \cap \sigma \text{ closed } \forall \sigma \in K. \end{cases}$

- simplicial map : $f : K \to L$

s.t. f is the barycentric extension of a map $f^\circ : K^\circ \to L^\circ$ (vertex sets)



Note: simplicial maps are continuous.

↳ Develop the same theory as above, restricting oneself to simplicial complexes and simplicial maps.

$\sigma : \Delta_n \to K$ simplicial $\equiv$ oriented simplex of K

$\sigma \equiv [v_0, \dots, v_n] \in K$

$$C^{\Delta}_n(K;k) = \bigoplus_{\substack{\{v_0,\dots,v_n\} \in K \\ (\text{non-oriented})}} k \quad \text{where it is understood that } [v_{\pi(0)}, \dots, v_{\pi(n)}] = \operatorname{sgn}(\pi) \cdot [v_0, \dots, v_n]$$

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \cdot [v_0, \dots, \hat{v}_i, \dots, v_n].$$

$$\partial_n \circ \partial_{n+1} = 0 \rightsquigarrow H^{\Delta}_{\bullet}(K;k).$$

(chain complexes)

⚠ Note: for K finite and k a field, $C_n(K;k)$ is a finite-dim. vector space and ∂_n a linear map $\Rightarrow$ computing $H^{\Delta}_{\bullet}(K;k)$ is a simple matter of linear algebra,

$\boxed{\text{Thm:}}$ (Equivalence)

$$H^{\Delta}_{\bullet}(K;k) \simeq H_{\bullet}(|K|;k).$$

calculation: (k field, K finite)

$$H^{\Delta}_n(K;k) \simeq k^{\dim H^{\Delta}_n(K;k)} \rightsquigarrow \text{just compute dimension}$$

$$\dim H^{\Delta}_n(K;k) = \operatorname{null} \partial_n - \operatorname{rank} \partial_{n+1}$$

$$= \underbrace{\dim C^{\Delta}_n(K;k)}_{= \# K_n} - \operatorname{rank} \partial_n - \operatorname{rank} \partial_{n+1}$$

only rank computations