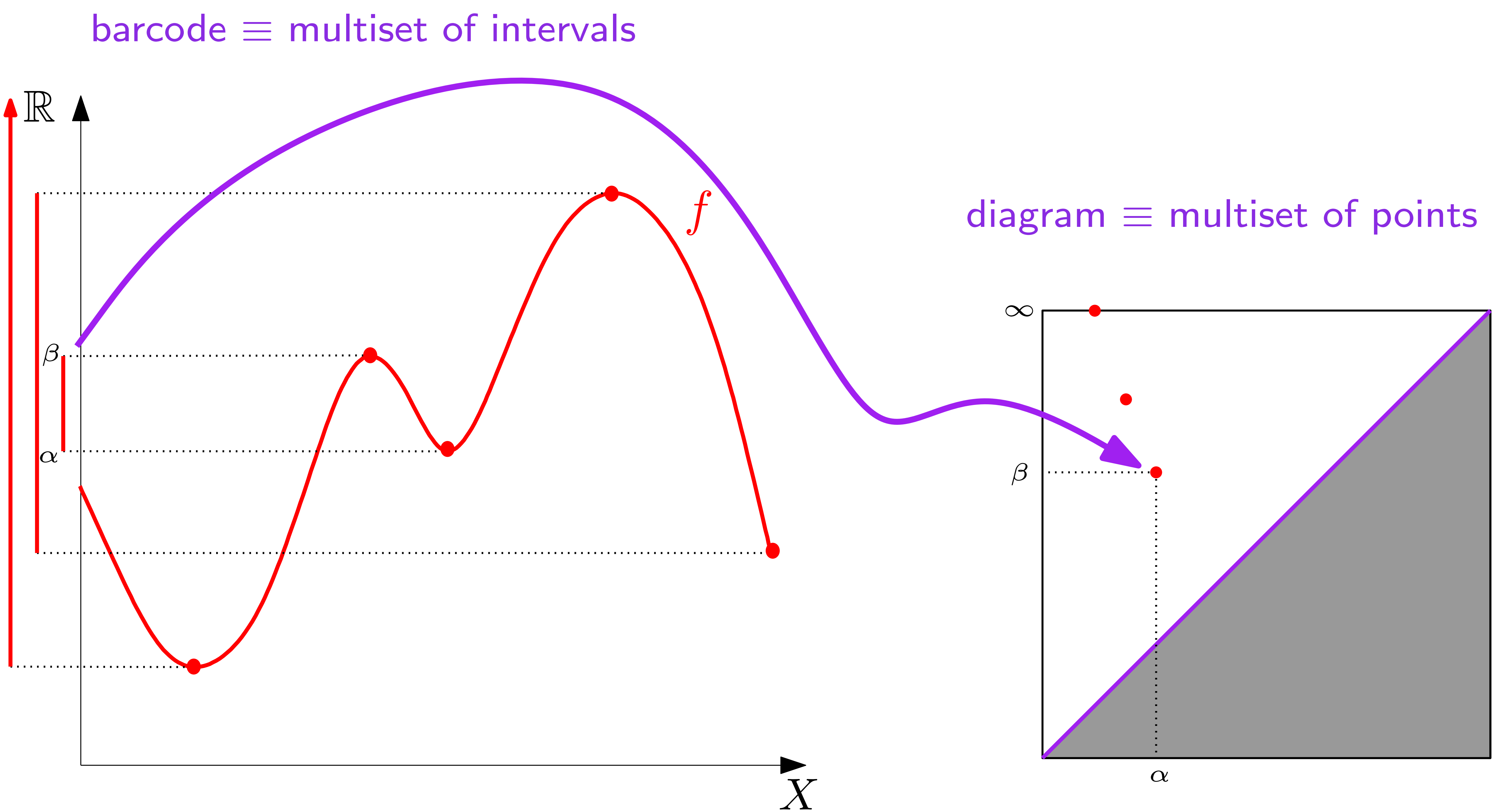
# Stability of persistence barcodes/diagrams

$X$ topological space, $f : X \to \mathbb{R}$ function

sublevel-sets filtration $\to$ barcode / diagram



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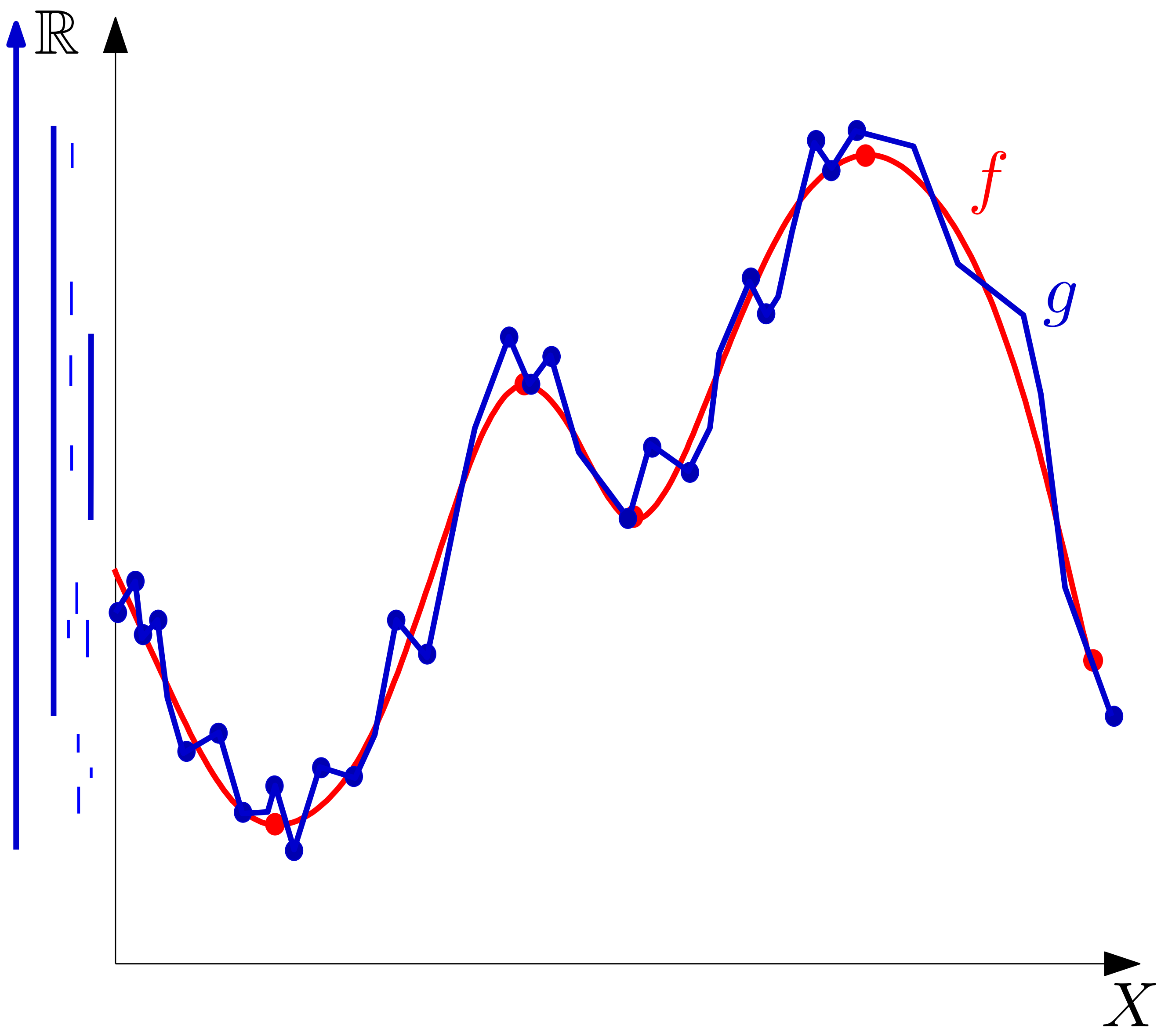
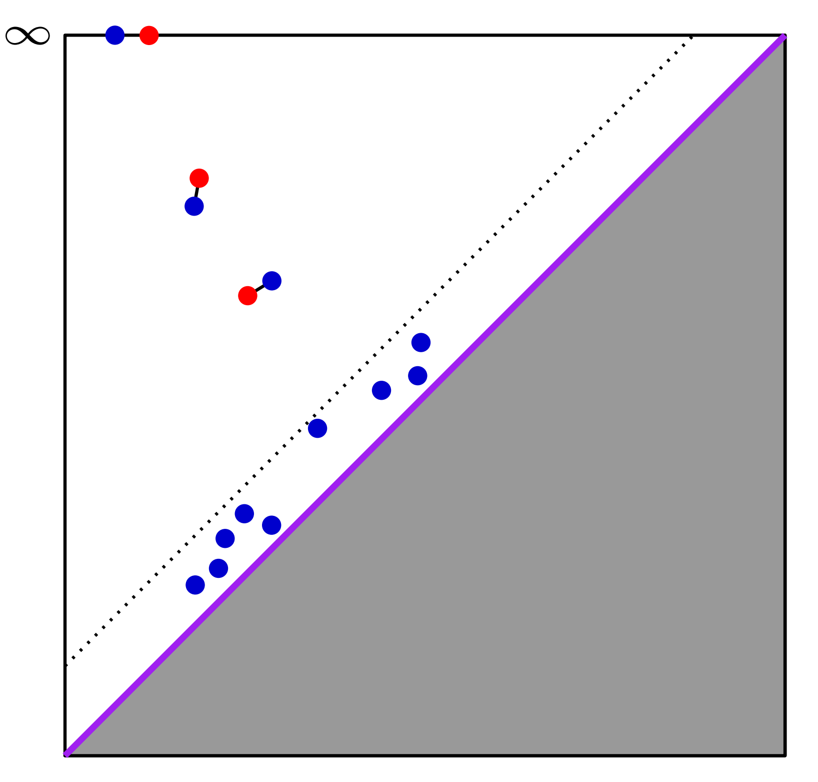


# Distances between persistence diagrams

**Input:** Two persistence diagrams $X, Y$

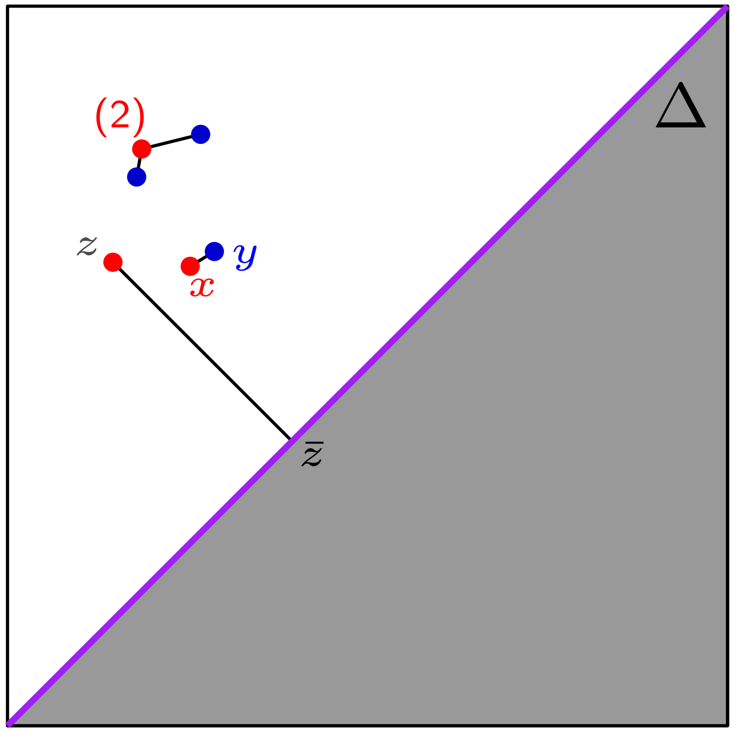
Given a partial matching $\Gamma : X \leftrightarrow Y$:

cost of a matched pair $(x, y) \in \Gamma$: $c_p(x, y) := \|x - y\|_\infty^p$

cost of an unmatched point $z \in X \sqcup Y$: $c_p(z) := \|z - \bar{z}\|_\infty^p$

cost of $\Gamma$:

$$c_p(\Gamma) := \left( \sum_{(x,\,y) \text{ matched}} c_p(x, y) \; + \sum_{z \text{ unmatched}} c_p(z) \right)^{1/p}$$



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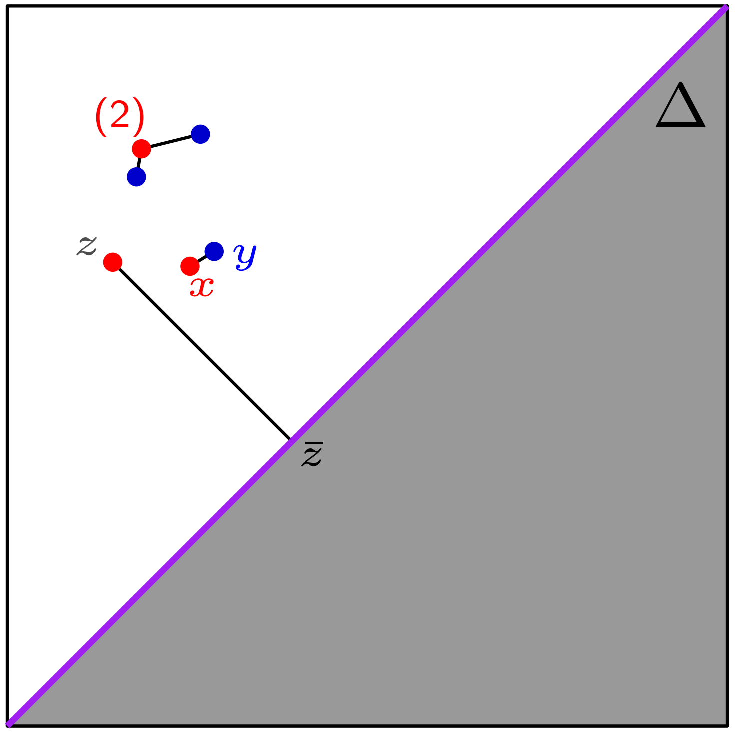



**Def:** $p$-th diagram distance (extended pseudometric):

$$\mathrm{d}_p(X, Y) := \inf_{\Gamma : X \leftrightarrow Y} c_p(\Gamma)$$

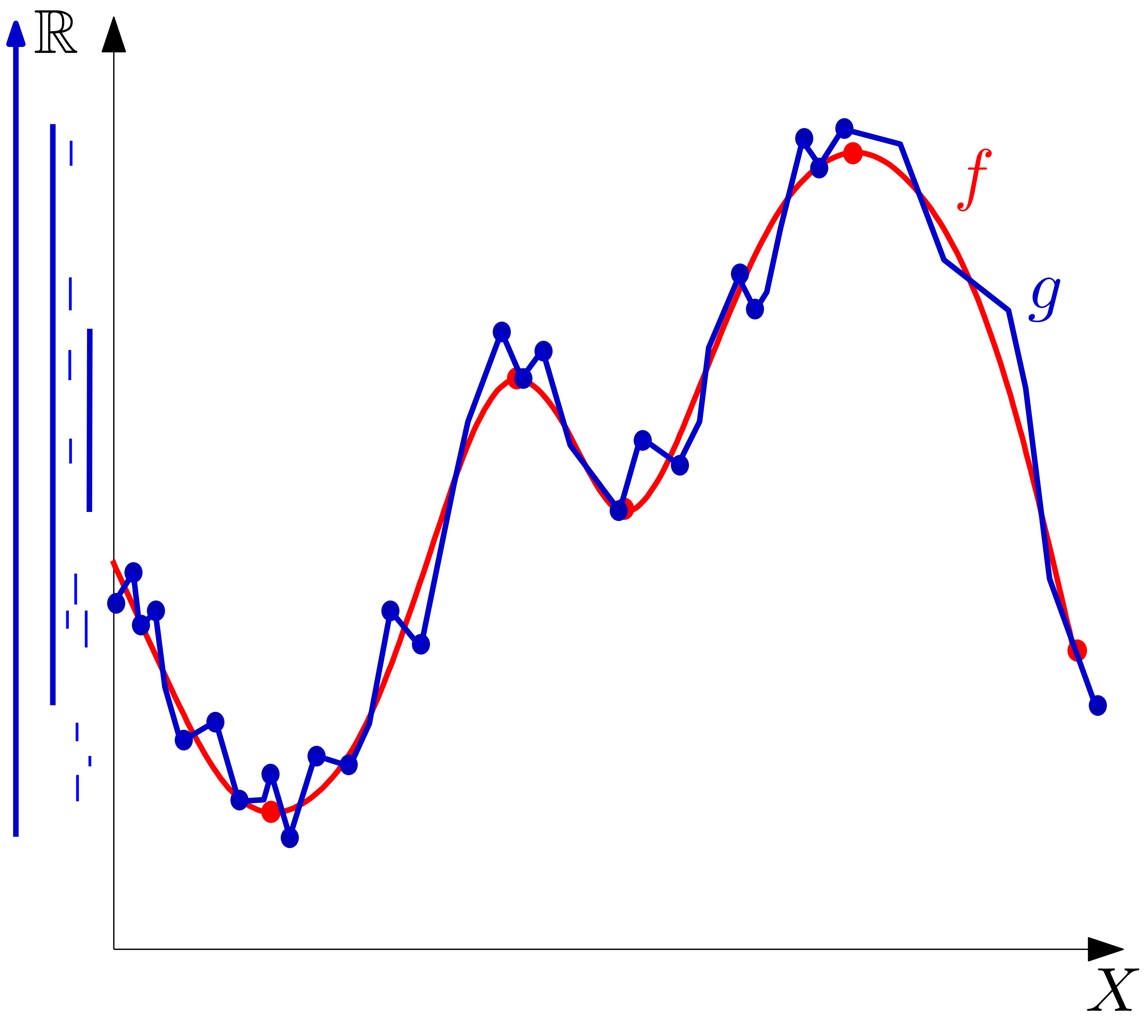
**Def:** bottleneck distance:

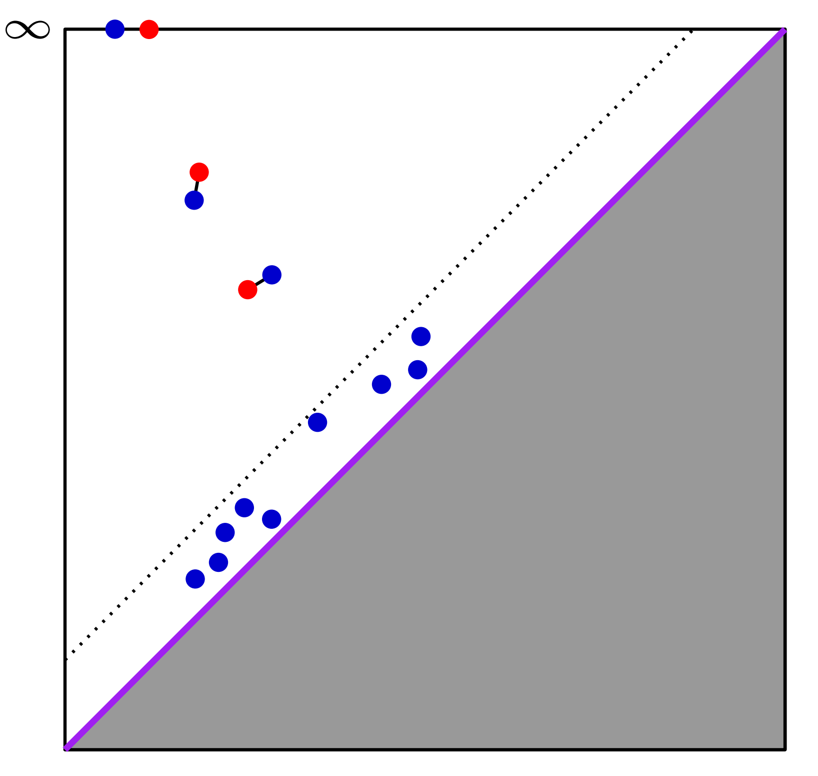
$$\mathrm{d}_\infty(X, Y) := \lim_{p \to \infty} \mathrm{d}_p(X, Y)$$

# Stability of persistence barcodes/diagrams

**Theorem:** For any pfd functions $f, g : X \to \mathbb{R}$,

$$d_\infty(\operatorname{dgm} f, \operatorname{dgm} g) \le \|f - g\|_\infty$$



# A simple (suboptimal) proof

Let $f, g : X \to \mathbb{R}$ be pfd, and let $\varepsilon = \|f - g\|_\infty$.

$F_t := f^{-1}((-\infty, t])$

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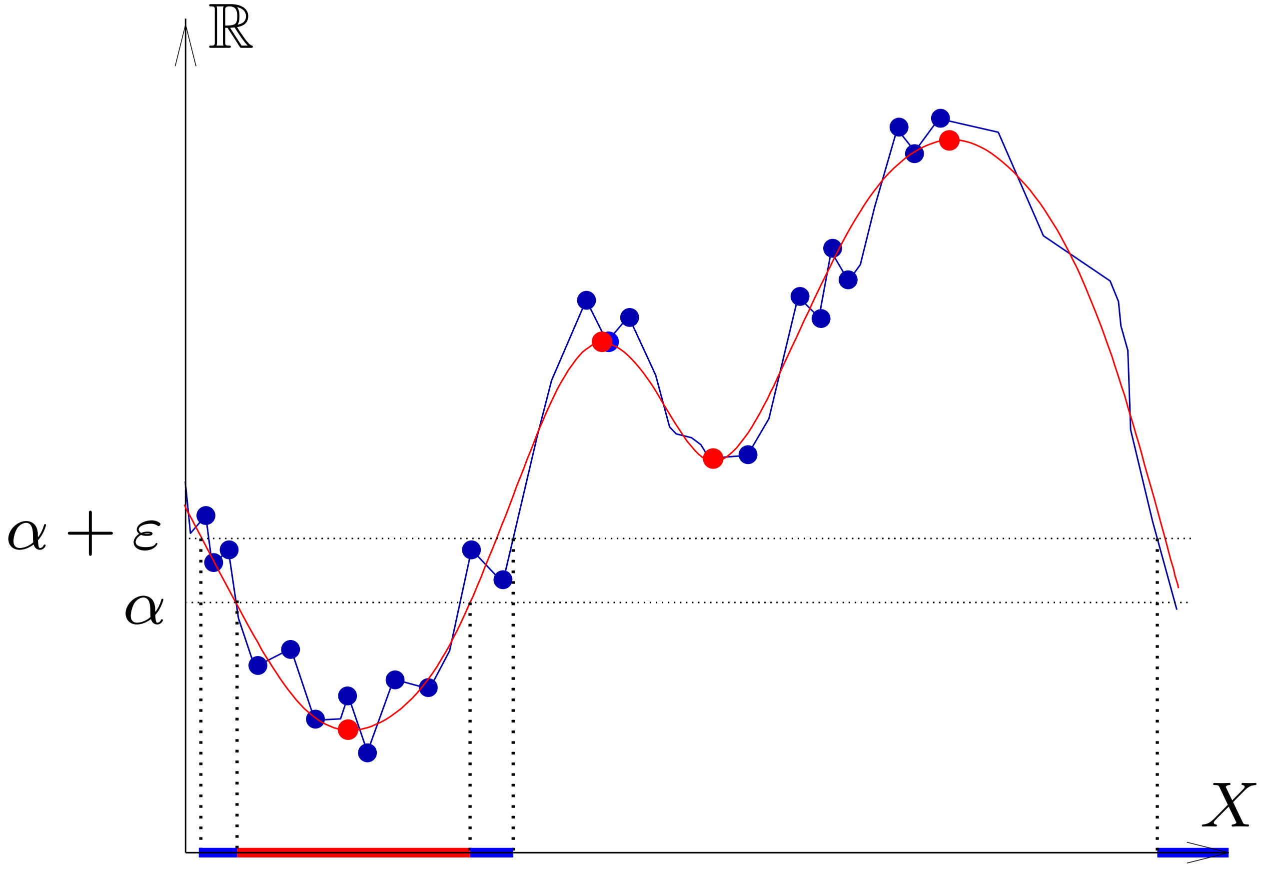
- Key observation: $\{F_t\}_t$ and $\{G_t\}_t$ are $\varepsilon$-**interleaved** w.r.t. inclusion:

$$\forall t \in \mathbb{R},\ G_{t-\varepsilon} \subseteq F_t \subseteq G_{t+\varepsilon}$$



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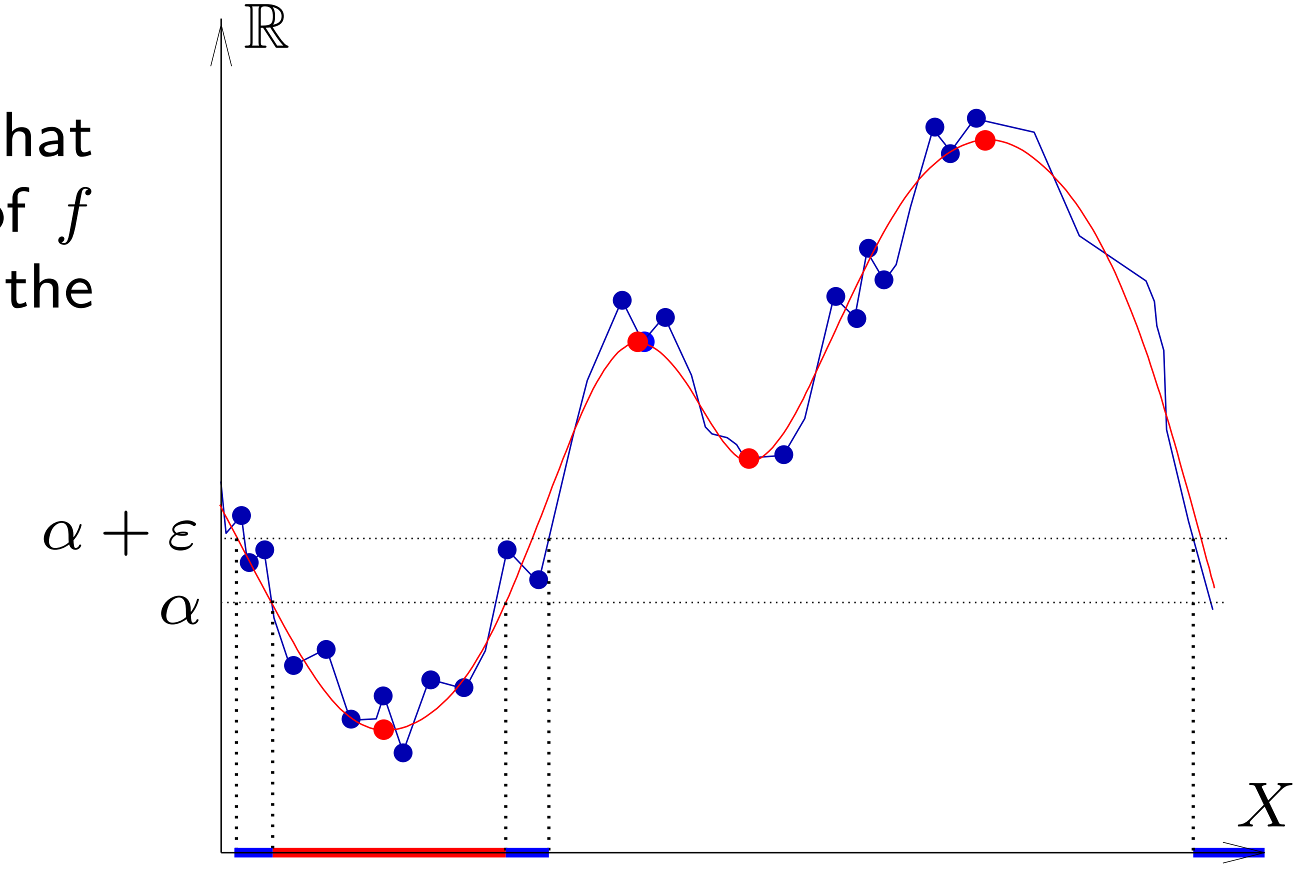
→ Intuition: every homological feature that appears/dies at time $\alpha$ in the filtration of $f$ appears/dies at time $\alpha + \varepsilon$ at the latest in the filtration of $g$, and vice versa.



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- both filtrations are $2\varepsilon$-discretizations of $\{H_{n\varepsilon}\}_{n\in\mathbb{Z}}$,where $H_{n\varepsilon} = \begin{cases} F_{n\varepsilon} \text{ if } n \text{ is even} \\ G_{n\varepsilon} \text{ if } n \text{ is odd} \end{cases}$

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$\to$ **goal**: bound distances between diagrams of filtrations and discretizations

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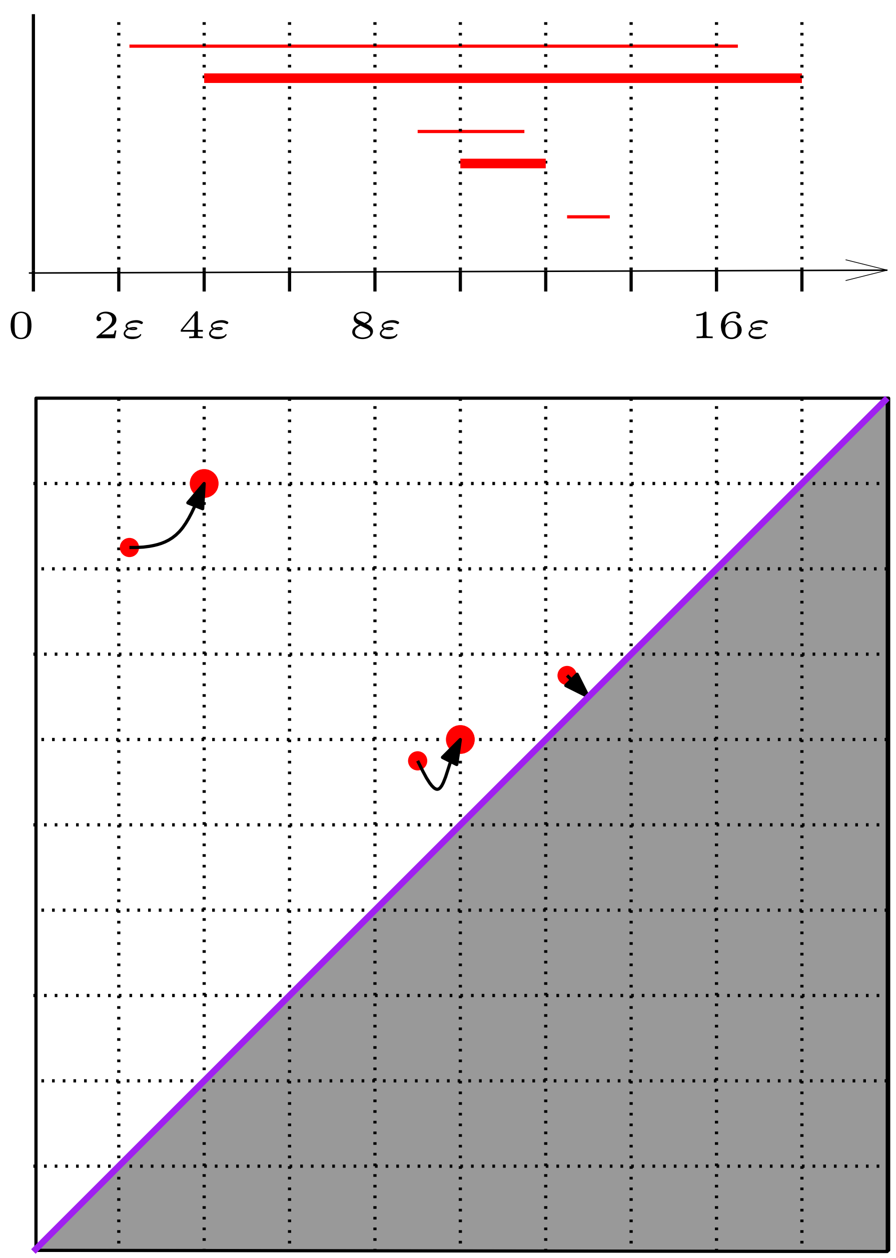


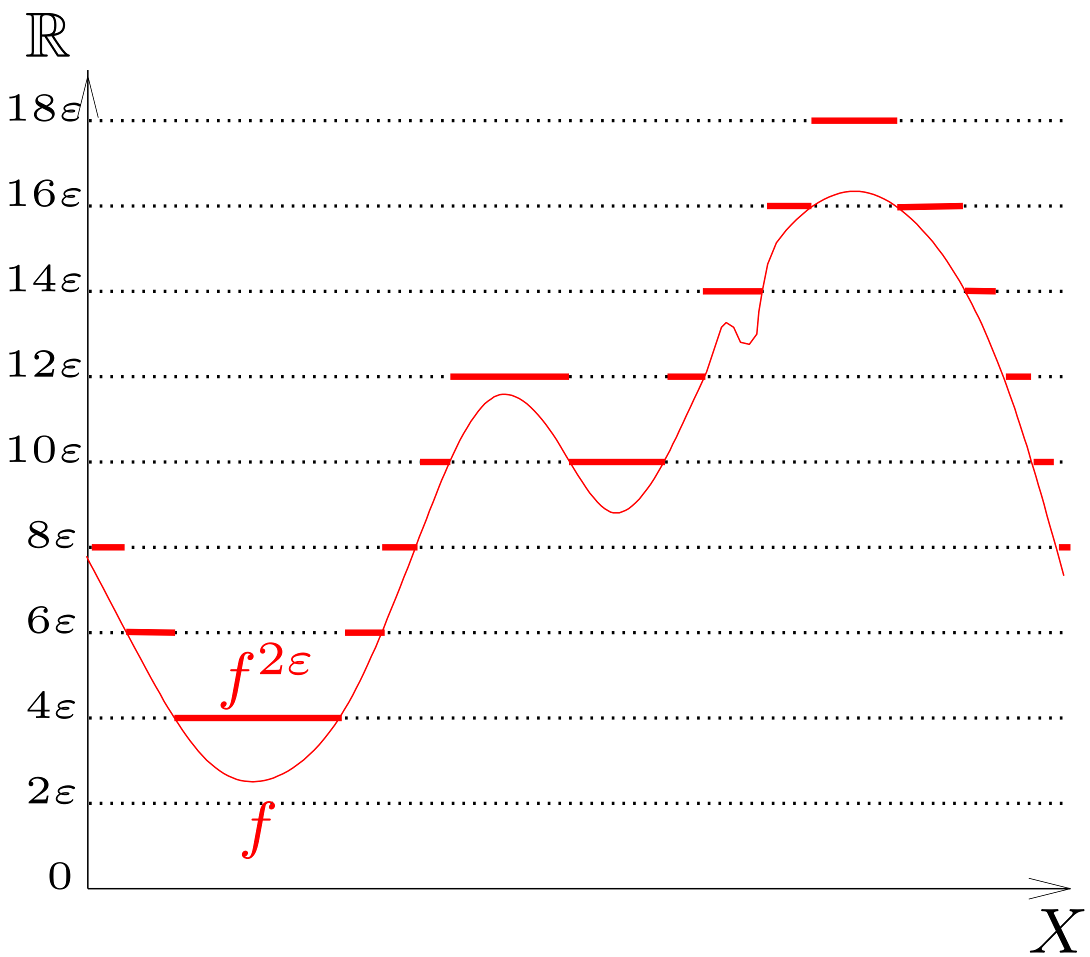

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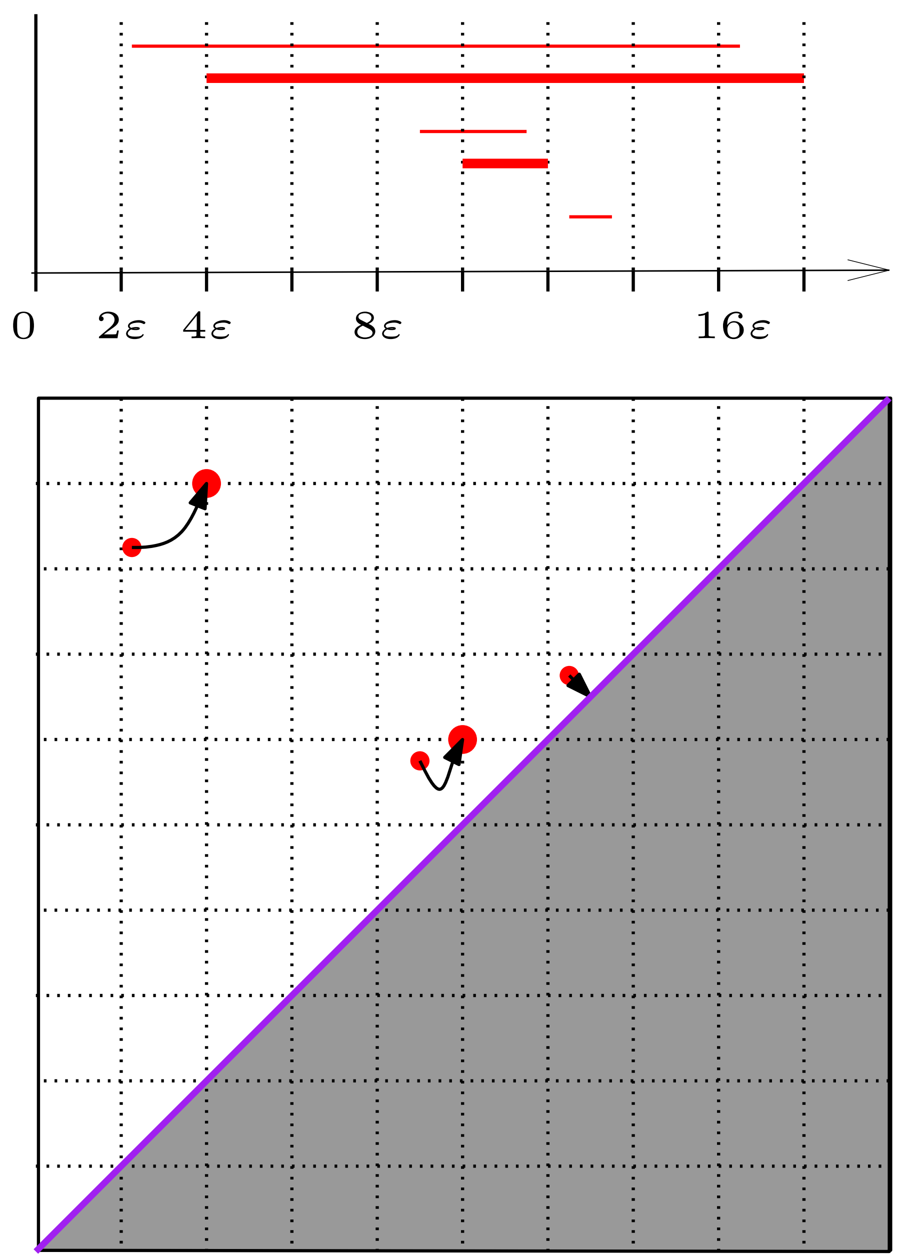


Pixelization map: $\forall \alpha \leq \beta$,

$$\pi_{2\varepsilon}(\alpha, \beta) = \begin{cases} (\lceil \frac{\alpha}{2\varepsilon} \rceil 2\varepsilon, \lceil \frac{\beta}{2\varepsilon} \rceil 2\varepsilon) \text{ if } \lceil \frac{\beta}{2\varepsilon} \rceil > \lceil \frac{\alpha}{2\varepsilon} \rceil \\ (\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}) \text{ if } \lceil \frac{\beta}{2\varepsilon} \rceil = \lceil \frac{\alpha}{2\varepsilon} \rceil \end{cases}$$

**Proposition:** If $f : X \to \mathbb{R}$ is pfd, then $\pi_{2\varepsilon}$ induces a bijection $\mathrm{dgm}\, f \to \mathrm{dgm}\, f^{2\varepsilon}$. Moreover, $\mathrm{d}_\infty(\mathrm{dgm}\, f, \mathrm{dgm}\, f^{2\varepsilon}) \leq 2\varepsilon$.

proof: discretization is an **additive functor**, whose effect on each summand (taken independently) is to discretize its support.

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Proposition + triangle inequality $\Rightarrow \mathrm{d}_\infty(\mathrm{dgm}\, f, \mathrm{dgm}\, g) \leq 8\varepsilon$

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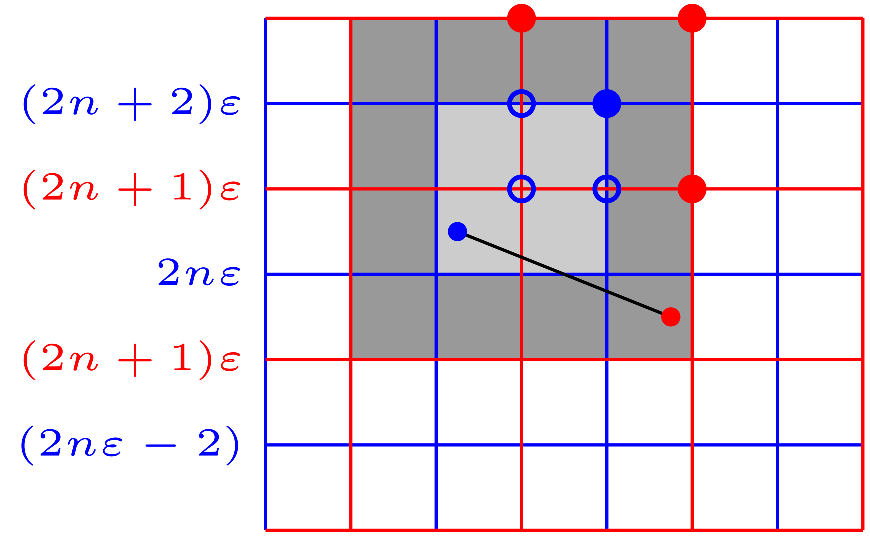
Improvement: $\boxed{\mathrm{d}_\infty(\mathrm{dgm}\, f, \mathrm{dgm}\, g) \leq 3\varepsilon}$



# From sup-norm perturbation to interleaving

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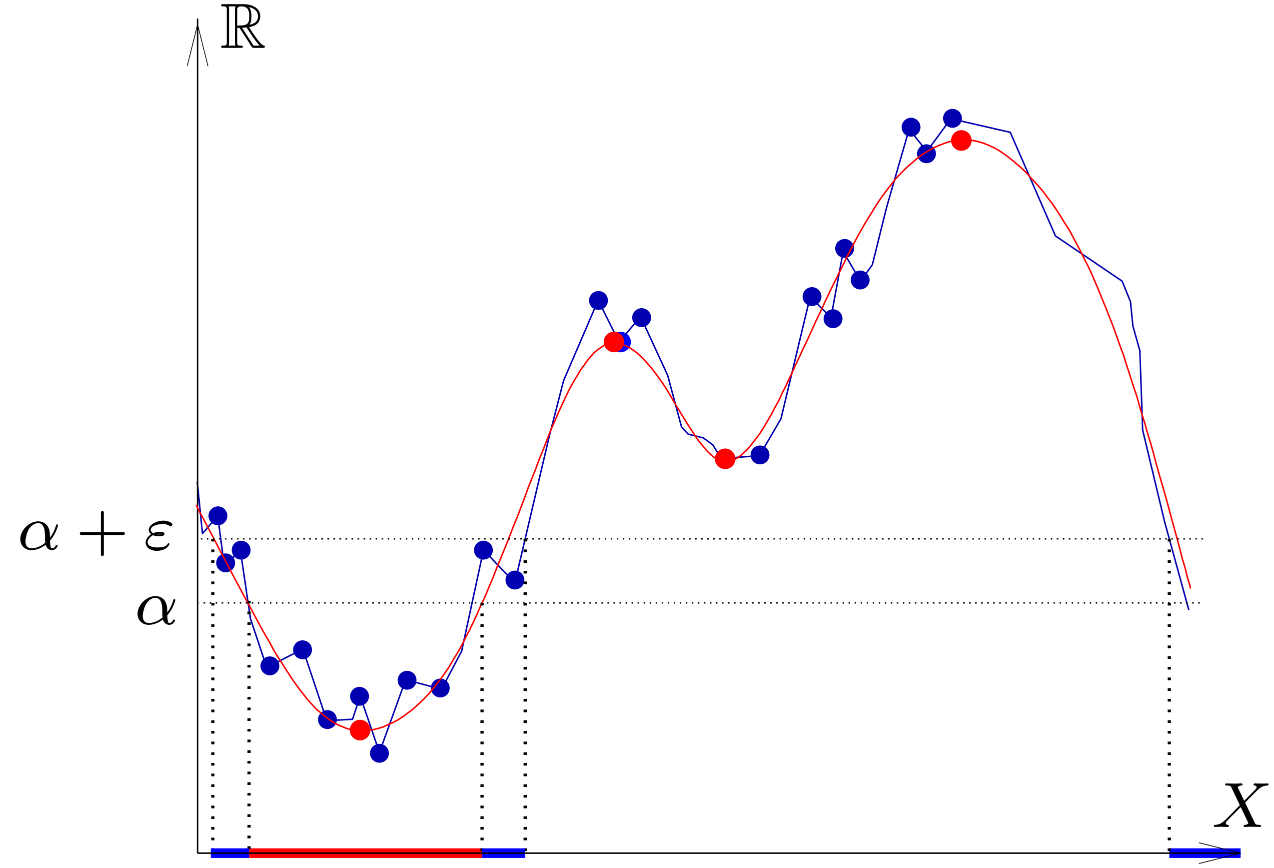

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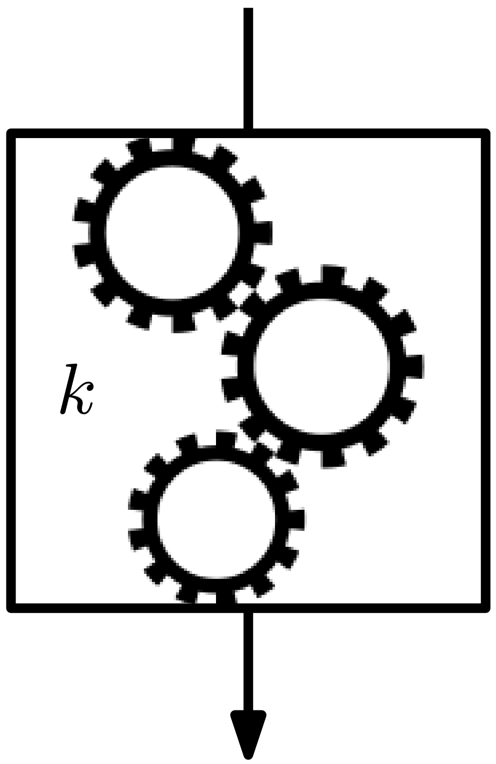


(homology functor)

$$\cdots \to \mathrm{H}_\bullet(F_0) \to \mathrm{H}_\bullet(G_\varepsilon) \to \mathrm{H}_\bullet(F_{2\varepsilon}) \to \cdots \to \mathrm{H}_\bullet(F_{2n\varepsilon}) \to \mathrm{H}_\bullet(G_{(2n+1)\varepsilon}) \to \cdots$$

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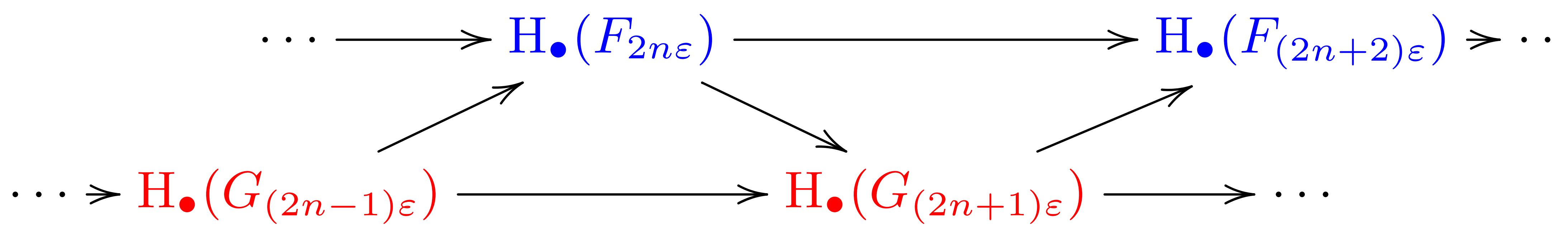
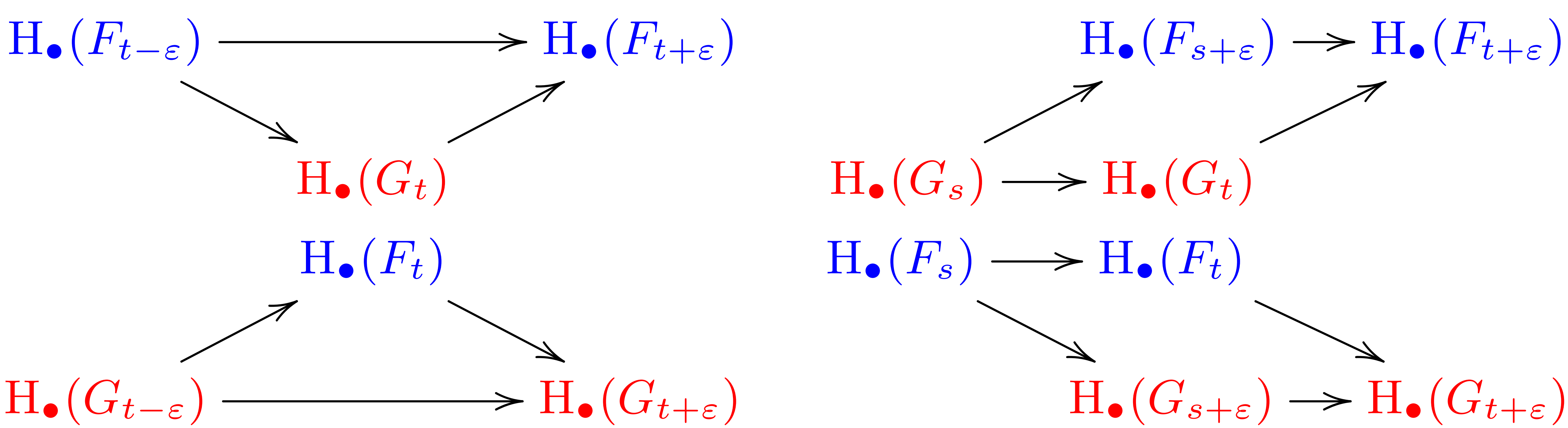
**Weak version of interleaving:** (used in previous proof)




**Interleaving:**



# From sup-norm perturbation to interleaving

**Categorical viewpoint:** For $M, N : (\mathbb{R}, \leq) \to \mathsf{vect}_k$

$$\begin{array}{ccc} M(s) & \longrightarrow & M(t) \\ \downarrow & \Downarrow & \downarrow \\ N(s) & \longrightarrow & N(t) \end{array}$$

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  $-[\varepsilon] : M \longmapsto M[\varepsilon]$ s.t. $M[\varepsilon](t) = M(t + \varepsilon)$

  (obvious effect on internal maps and morphisms)

  $\exists\ \xi : M \Rightarrow M[\varepsilon]$ given by $\xi(t) := M(t \to t + \varepsilon)$

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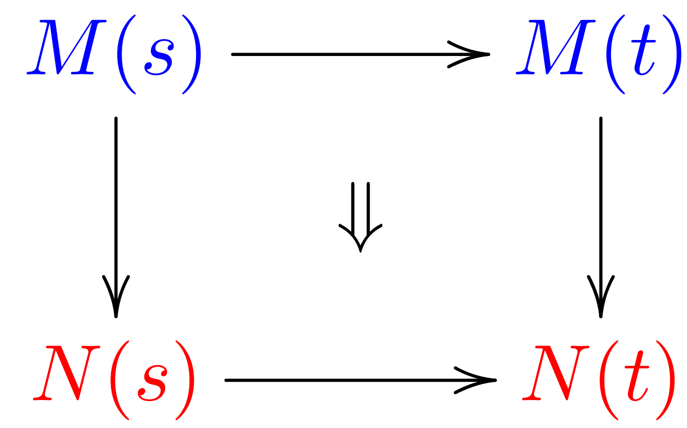


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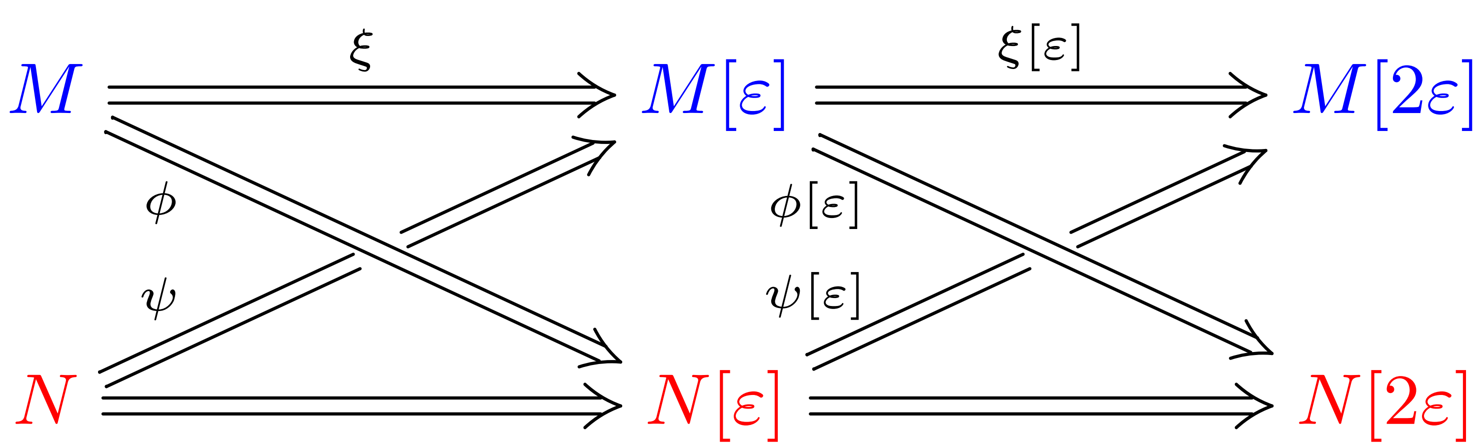

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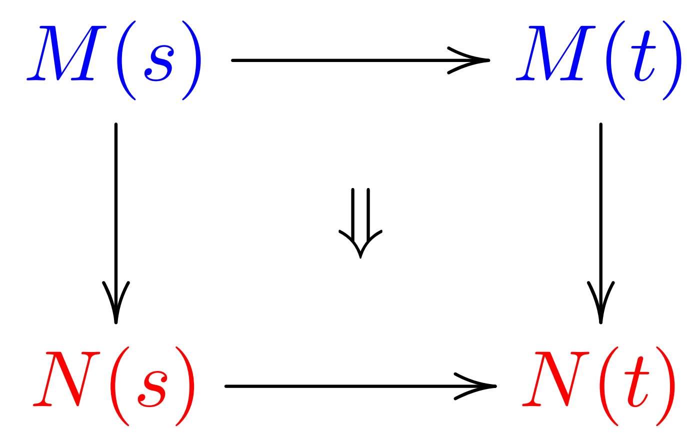


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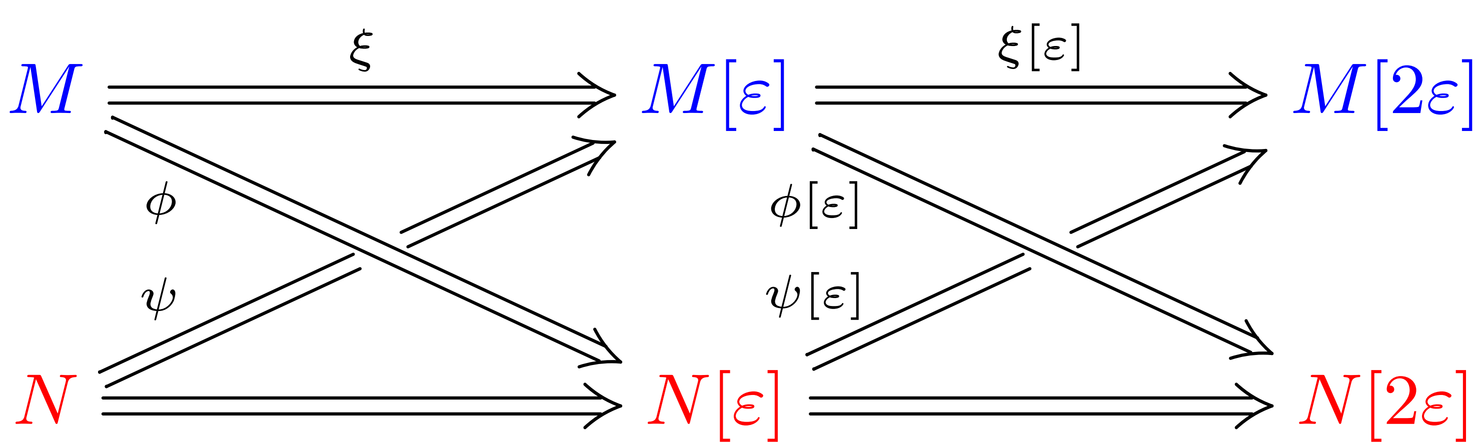


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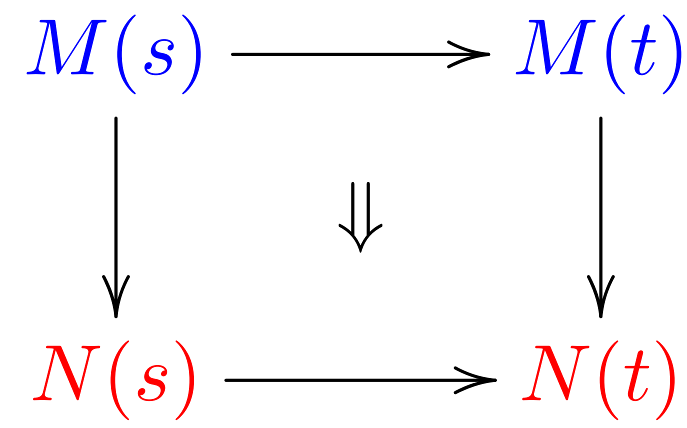
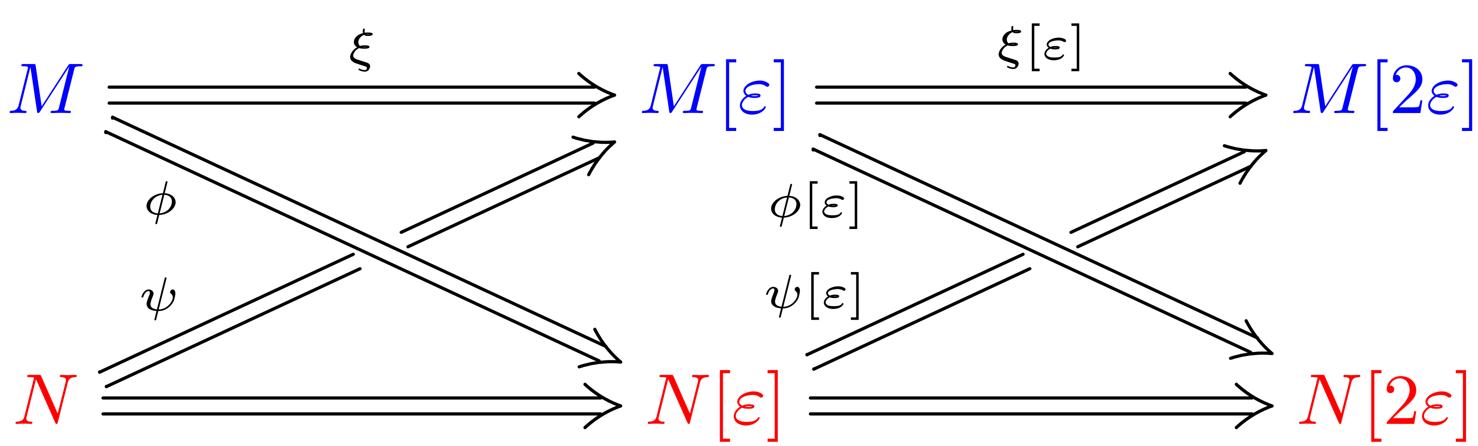



- $\varepsilon$**-shift/smoothing endofunctor:**

**Theorem:** (follows from functoriality of $\mathrm{H}_\bullet$)
For $f, g : X \to \mathbb{R}$ pfd, $\mathrm{d}_i(\mathrm{H}_\bullet(\{F_t\}_t),\, \mathrm{H}_\bullet(\{G_t\}_t)) \leq \|f - g\|_\infty$

**Theorem:** (isometry)
For $M, N : (\mathbb{R}, \leq) \to \mathsf{vect}_k$, $\mathrm{d}_\infty(\mathrm{dgm}\, M,\, \mathrm{dgm}\, N) = \mathrm{d}_i(M,\, N)$

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# From matching to interleaving

[Lesnick 2011] [Chazal et al. 2016]

Given barcodes $B, B'$ (locally finite sets of intervals of $\mathbb{R}$), and $\Gamma : B \leftrightarrow B'$:

- for $(b, b')$ matched, the **constant modules** $k_b, k_{b'}$ are $c_\infty(\Gamma)$-interleaved

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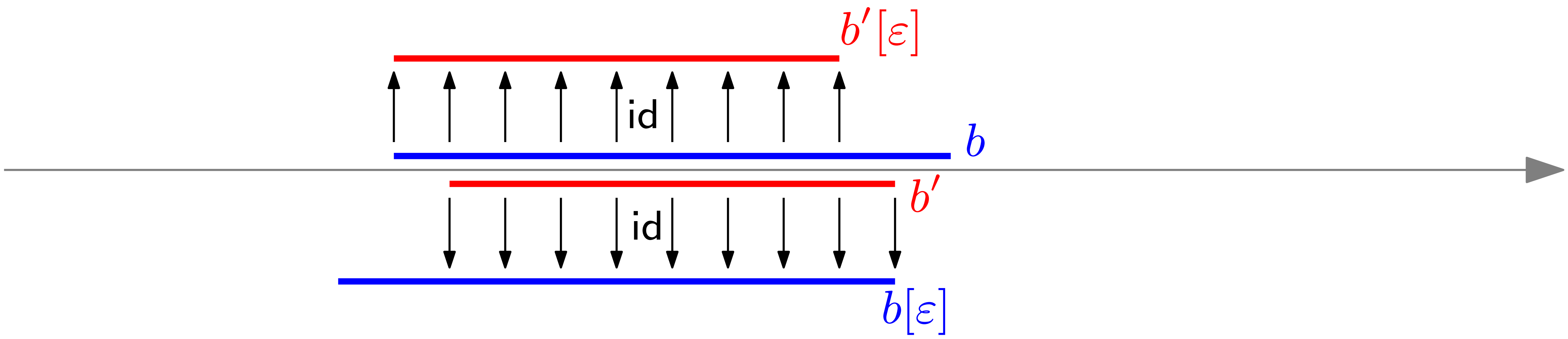

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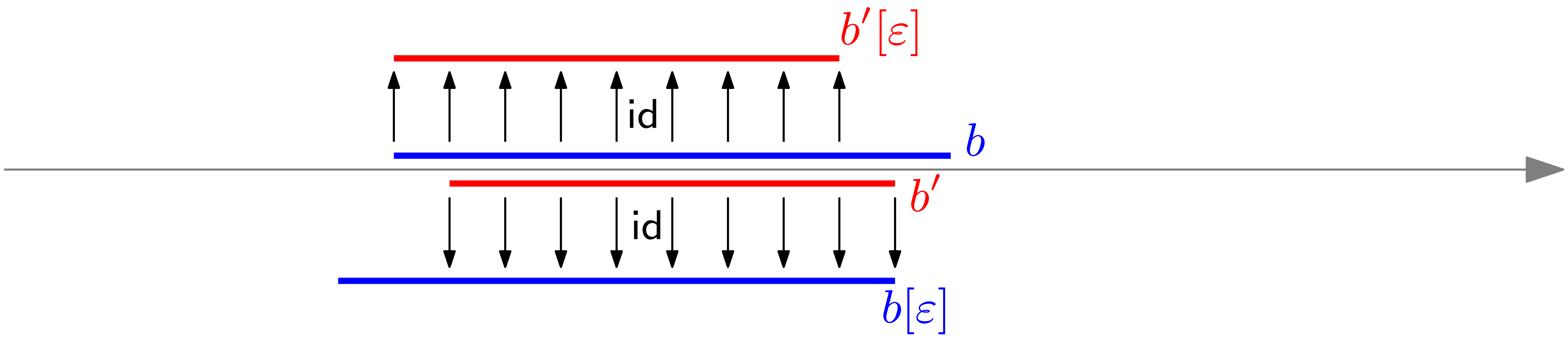

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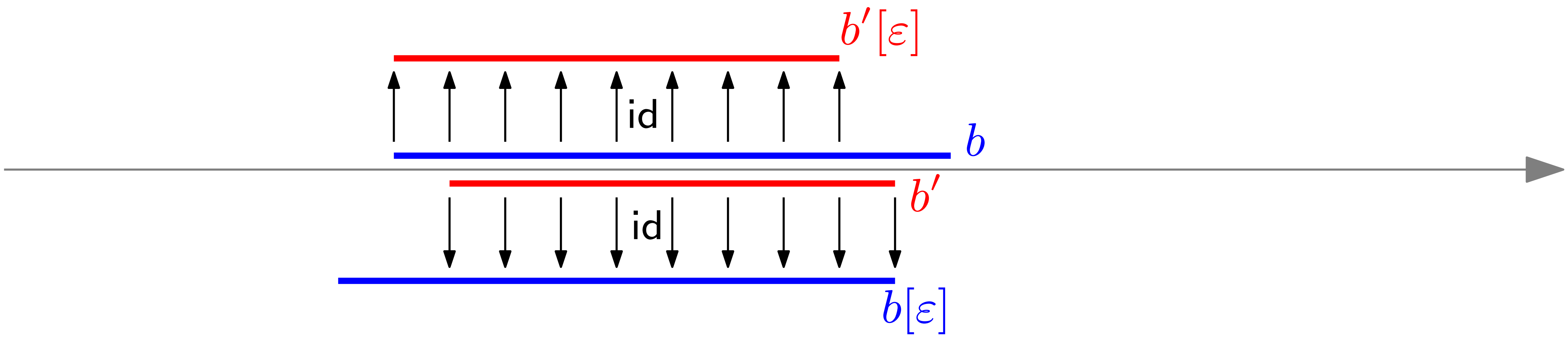


Note: this construction defines a (not fully) faithful functor Barcodes $\rightarrow$ vect$_k^{(\mathbb{R},\leq)}$

# From interleaving to matching

Approaches:

1. interpolation at functional level (soft+hard) [Cohen-Steiner et al. 2005]
2. discretizations (loose bound) [Chazal et al. 2009]
3. interpolation between modules [Chazal et al. 2009-2016]
4. matchings induced from morphisms [Bauer, Lesnick 2014]
5. Hall's marriage theorem [Bjerkevik 2016]

# From interleaving to matching

**Interpolation between modules**:

Given $M, N : (\mathbb{R}, \leq) \to \mathsf{Vect}_k$ with $\mathrm{d}_i(M, N) = \varepsilon$, find $(U_\alpha)_{0 \leq \alpha \leq \varepsilon}$ such that:

- $U_0 \simeq M$, $U_\varepsilon \simeq N$
- $\forall 0 \leq \alpha \leq \beta \leq \varepsilon$, $\mathrm{d}_i(U_\alpha, U_\beta) \leq \beta - \alpha$

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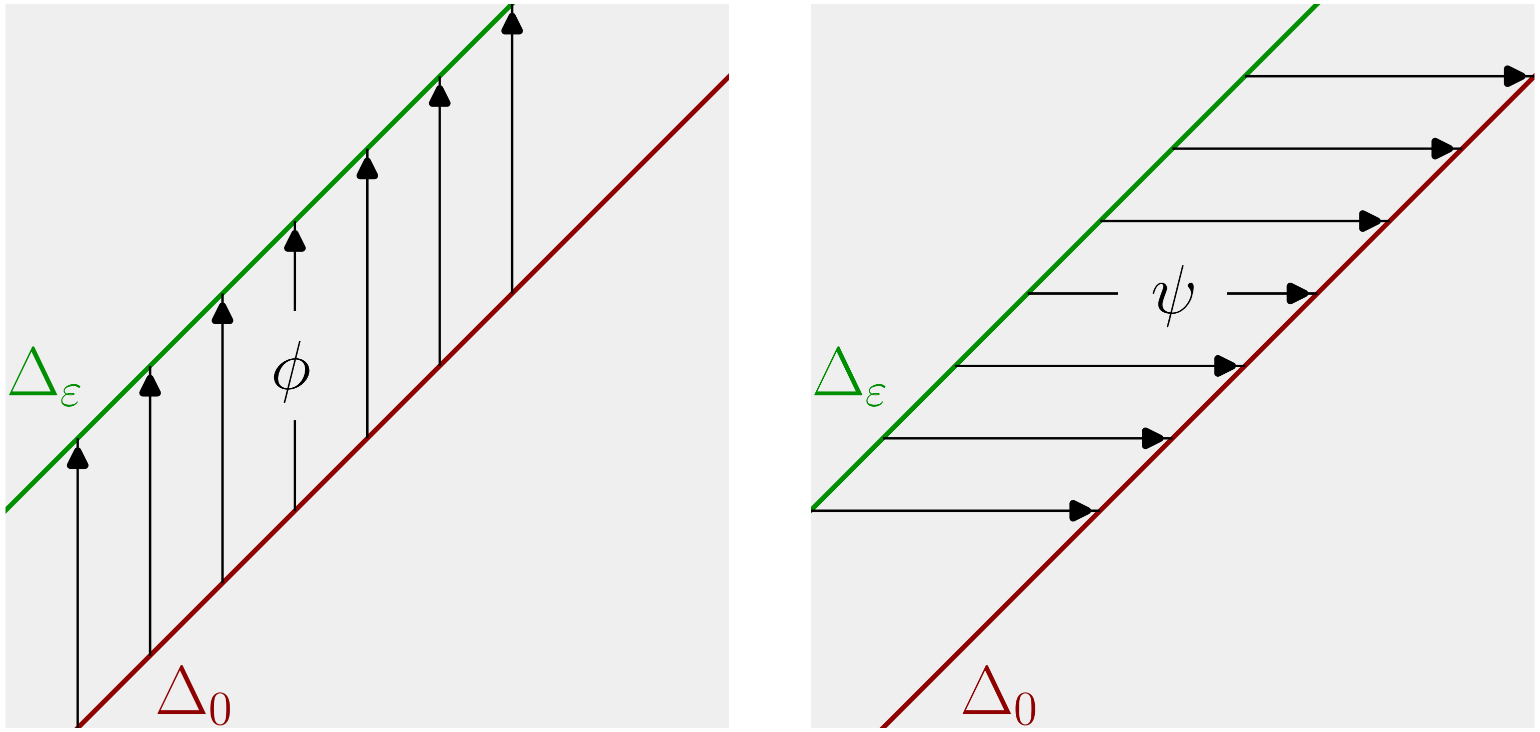
Embed $(\mathbb{R}, \leq)$ into $(\mathbb{R}^2, \leq)$ (w. product order) as $\Delta_t$ for an arbitrary $t \in \mathbb{R}_+$

$M : (\Delta_0, \leq) \to \mathsf{Vect}_k$ $\qquad$ $N : (\Delta_\varepsilon, \leq) \to \mathsf{Vect}_k$

$\varepsilon$-interleaving $(\phi, \psi)$ yields functor $F : (\Delta_0 \cup \Delta_\varepsilon, \leq) \to \mathsf{Vect}_k$



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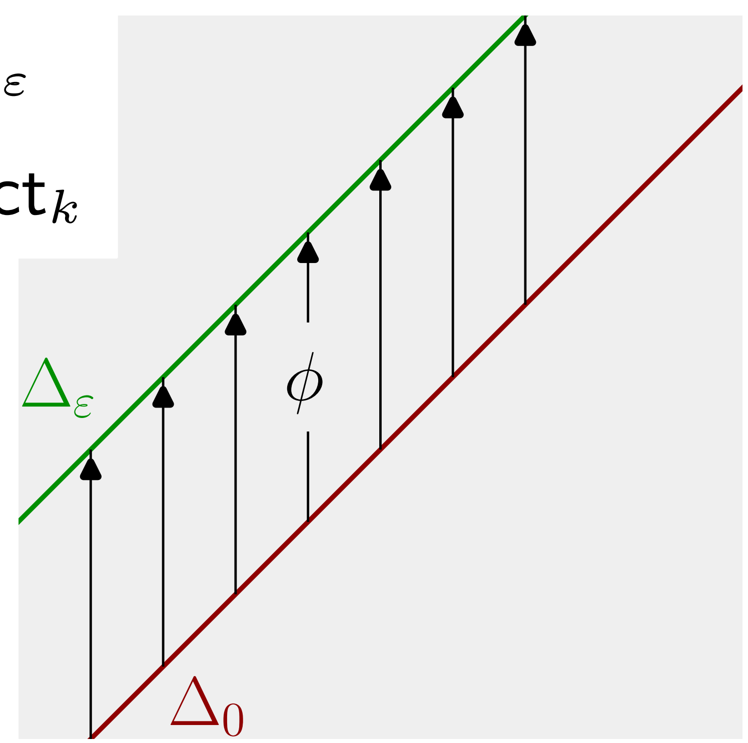
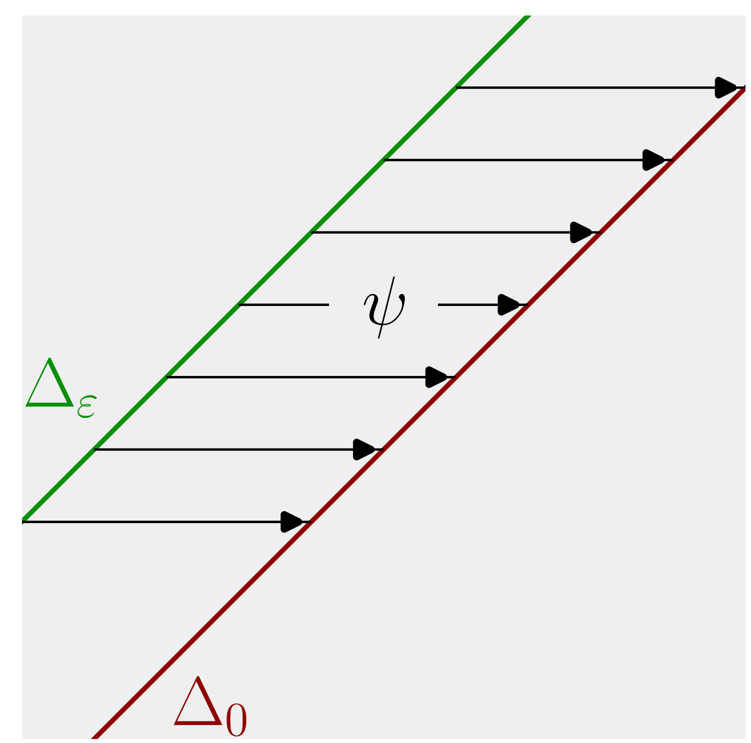
**interpolating family** $(U_\alpha)_{0\leq\alpha\leq\varepsilon}$

$\equiv$ functor $G : (\Delta_{[0,\varepsilon]}, \leq) \to \mathsf{Vect}_k$



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use **left Kan extension** of $\Delta_0 \cup \Delta_\varepsilon \hookrightarrow \Delta_{[0,\varepsilon]}$:

$G(t) := \varinjlim F|_{s \leq t \in \Delta_0 \cup \Delta_\varepsilon}$

$G(s \to t)$ given by universality of colimit