Topological Descriptors for geometric data

Def: Hausdorff distance: $X, Y \subseteq (Z, d_Z)$.

$$d_H^Z(X,Y) = \max \left\{ \sup_{x\in X} \inf_{y\in Y} d_Z(x,y) \, ; \, \sup_{y\in Y} \inf_{x\in X} d_Z(x,y) \right\}$$

Def: Gromov-Hausdorff distance: $(X, d_X) \Leftrightarrow (Y, d_Y)$

$$d_{GH}(X,Y) = \inf_{\substack{(Z, d_Z) \\ \gamma_X : X \to Z \\ \gamma_Y : Y \to Z}} d_H^Z(\gamma_X(X), \gamma_Y(Y)) .$$

$\gamma_X : X \to Z$, $\gamma_Y : Y \to Z$ isometries

---

(1) Global signatures: (X, d_X) compact metric space.

Def: Čech / Nerve filtration:

$\forall t \geq 0$, $\sigma = \{x_0, \dots, x_k\} \in C_t(X, d_X) \Leftrightarrow \bigcap_{i=0}^{k} B_X(x_i, t) \neq \emptyset$

$\mathcal{C}(X, d_X) := \left(C_t(X, d_X) \right)_{t \geq 0}$.

→ interest:

Thm: (Nerve) let $X \subseteq (Z, d_Z)$. (equipped with the induced metric)

Suppose $\forall \sigma \subseteq X$ finite, $\bigcap_{x\in\sigma} B_Z(x,t)$ is either $\emptyset$ or contractible (homotopy equivalent to a pt) ↳ is some (as a pt).

Then, $C_t(X, d_Z) \simeq \bigcup_{x\in X} B_Z(x,t)$.

↑ homotopy equivalence (→ some "topological structure")

Optional: define homotopy equivalence:
- homotopy of maps
- pseudo-inverse

↳ $C_t(X, d_Z)$ acts as a combinatorial proxy for the union of t-balls around X.

Def: Vietoris-Rips filtration:

$\forall t \geq 0$, $\sigma = \{x_0, \dots, x_k\} \in R_t(X, d_X) \Leftrightarrow d_X(x_i, x_j) \leq t \;\; \forall i, j$

$\Leftrightarrow \operatorname{diam}(\sigma) \leq t$.

$\mathcal{R}(X, d_Z) = \left(R_t(X, d_X) \right)_{t \geq 0}$. → interest: easy to compute

Prop: (relationship between the two filtrations):

For any (X, d_X) compact:

$\forall t \geq 0, \quad R_t(X, d_X) \subseteq C_t(X, d_X) \subseteq R_{2t}(X, d_X)$.

$\to$ ~~proof: big double in~~

$\to$ proof:

- $\sigma = \{x_0, \dots, x_k\} \in R_t(X, d_X) \Rightarrow d_X(x_0, x_i) \leq t \quad \forall i$

  $\Rightarrow x_0 \in B_X(x_i, t) \quad \forall i$

  $\Rightarrow \bigcap_{i=0}^{k} B_X(x_i, t) \neq \emptyset \Rightarrow \sigma \in C_t(X, d_X)$.

- $\sigma = \{x_0, \dots, x_k\} \in C_t(X, d_X) \Rightarrow \cancel{\forall i,j,\ d_X(x_i, x_j)}$ $\bigcap_{i=0}^{k} B_X(x_i, t) \neq \emptyset$.

  let $x \in \bigcap_{i=0}^{k} B_X(x_i, t)$.

  $\hookrightarrow \forall i,j, \quad d_X(x_i, x_j) \leq d_X(x_i, x) + d_X(x, x_j)$

  $\leq 2t$.

  $\Rightarrow \sigma \in R_{2t}(X, d_X)$. ∎

Prop: ~~when~~ If $(X, d_X) = (\mathbb{R}^d, \ell^\infty)$, then $\mathcal{C}(X, d_X) = \mathcal{R}(X, d_X)$:

$\forall t \geq 0, \quad C_t(X, d_X) = R_t(X, d_X)$.

$\to$ proof:

⚠ Note 1: $B_{\ell^\infty}(x, t) = \prod_{r=1}^{d} [x_r - t, x_r + t]$.

$\Rightarrow$ it suffices to focus on each coordinate separately.

⚠ Note 2: in $\mathbb{R}$, any 3 or more pairwise intersecting intervals have a common intersection.

∎

Thm: (stability)

For (X, d_X) and (Y, d_Y) compact metric spaces:

$$d_b^\infty\left(D_g\,\mathcal{R}(X, d_X),\ D_g\,\mathcal{R}(Y, d_Y)\right) \leq 2\, d_{GH}(X, Y).$$

② **Local signatures**:

Def: ~~Compact~~ Intrinsic metric space (X, d_X):

$$\forall x, y \in X, \quad d_X(x,y) = \inf_{\substack{\gamma : [0,1] \to X \\ \gamma(0)=x,\ \gamma(1)=y}} |\gamma|$$

requires a length structure on X.

Def: Given (X, d_X) intrinsic metric space, $x \in X$:

$$\rho(X, x) = \sup\{ r > 0 \mid \forall r' < r,\ B_X(x, r') \text{ is convex} \}$$

$$\rho(X) = \inf\{ \rho(X, x) \mid x \in X \}.$$

ie: $\forall y, z \in B_X(x, r')$, $\exists!$ shortest path $y \to z$ and that path lies in $B_X(x, r')$.

examples:

- in $\mathbb{R}^d$: $\rho(X) = +\infty$
- ~~on~~ S^d: $\rho(X) = \pi/2$ (radius of hemisphere)
- in hyperbolic space: $\rho(X) = +\infty$

Def: (Gromov-Hausdorff distance between pointed spaces):

~~(X, x_0, d_X)~~

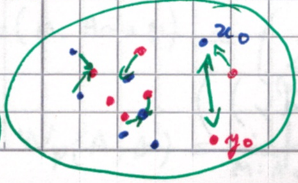
$$d_H^Z((X, x_0), (Y, y_0)) = \max\{ d_Z(x_0, y_0),\ d_H^Z(X, Y) \}.$$

$$d_{GH}((X, x_0, d_X), (Y, y_0, d_Y)) = \inf_{(Z, d_Z)} d_H^Z\left((\gamma_X(X), \gamma_X(x_0)), (\gamma_Y(Y), \gamma_Y(y_0))\right).$$

enforce x_0, y_0 to be matched:

$\left.\begin{array}{l}\gamma_X : X \to Z \\ \gamma_Y : Y \to Z\end{array}\right\}$ isometries

(Z, d_Z)



Addendum: proof of stability:

Lemma: (preliminary)

(X, d_X) finite metric space $\Rightarrow \exists$ isometry $\gamma : X \to (\mathbb{R}^d, \ell^\infty)$ where $d = |X|$.

$\to$ proof: • take distance matrix $D_{ij} = d_X(x_i, x_j)$.

↳ the columns give the embedding (coordinates)

ie: • $\forall i$, $\gamma(x_i) := \left(d_X(x_i, x_1), \dots, d_X(x_i, x_d) \right)$.

then • $\forall i, j$, $\max_k \{ | d_X(x_i, x_k) - d_X(x_j, x_k) | \} = d_X(x_i, x_j)$

by triangle inequality. ∎

$\to$ proof of the stability thm:

$$\mathcal{R}(X, d_X) \leftarrow (X, d_X) \xrightarrow{\gamma_X} (Z, d_Z) \xrightarrow{\gamma} (\mathbb{R}^d, \ell^\infty) \quad \text{where } d = |X| + |Y|$$

$$\mathcal{R}(Y, d_Y) \leftarrow (Y, d_Y) \xrightarrow{\gamma_Y} (Z, d_Z)$$

$(\mathbb{R}^d, \ell^\infty) \to \mathcal{R}(\gamma \circ \gamma_X(X), \ell^\infty)$, $\quad (\mathbb{R}^d, \ell^\infty) \to \mathcal{R}(\gamma \circ \gamma_Y(Y), \ell^\infty)$

a) isometries $\Rightarrow \begin{cases} \mathcal{R}(X, d_X) \simeq \mathcal{R}(\gamma \circ \gamma_X(X), \ell^\infty) \\ \mathcal{R}(Y, d_Y) \simeq \mathcal{R}(\gamma \circ \gamma_Y(Y), \ell^\infty) \end{cases}$ (as simplicial complexes / filtrations)

b) $\mathcal{R}(\gamma \circ \gamma_X(X), \ell^\infty) \underset{\text{(previous lemma)}}{\overset{\boxed{(\times 2)}}{=}} \mathcal{C}(\gamma \circ \gamma_X(X), \ell^\infty) \underset{\text{(Nerve thm.)}}{=} \ell^\infty\text{-offsets filt. of } \gamma \circ \gamma_X(X)$.

$\|$ sublevel-sets filt. of distance to $\gamma \circ \gamma_X(X)$.

same for Y

c) $d_{GH}(X, Y) \leq \varepsilon \Rightarrow d_H^{\ell^\infty}(\gamma \circ \gamma_X(X), \gamma \circ \gamma_Y(Y)) \leq \varepsilon$.

$\Rightarrow \| d^{\ell^\infty}_{\gamma \circ \gamma_X(X)} - d^{\ell^\infty}_{\gamma \circ \gamma_Y(Y)} \|_\infty \leq \varepsilon$.

(stability thm.) $\Rightarrow \underbrace{d_b^\infty \left(Dg\, d^{\ell^\infty}_{\gamma \circ \gamma_X(X)}, Dg\, d^{\ell^\infty}_{\gamma \circ \gamma_Y(Y)} \right)}_{} \leq \varepsilon$.

$\|$ (by b))

$\frac{1}{2} d_b^\infty \left(Dg\, \mathcal{R}(X, d_X), Dg\, \mathcal{R}(Y, d_Y) \right)$. ∎