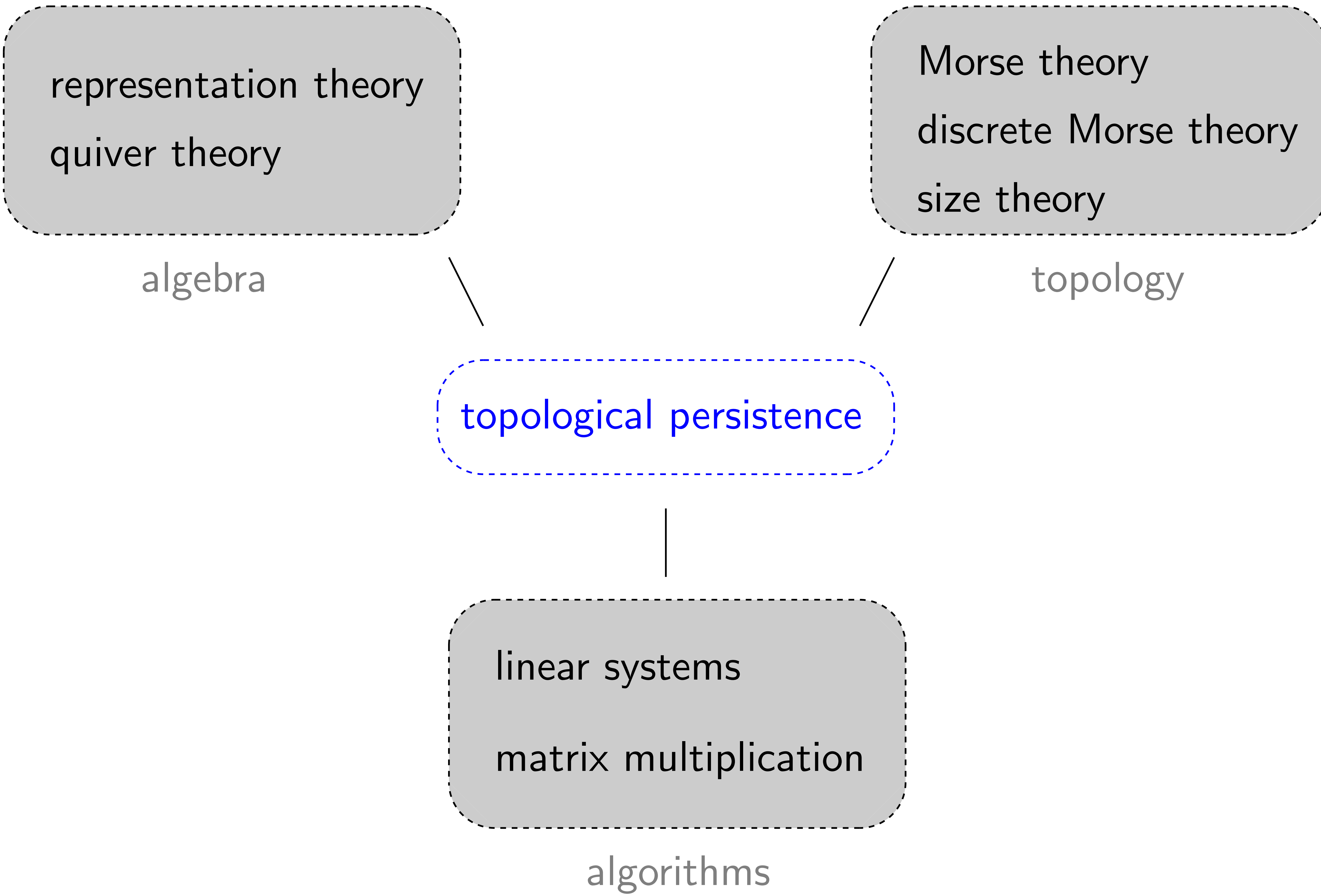
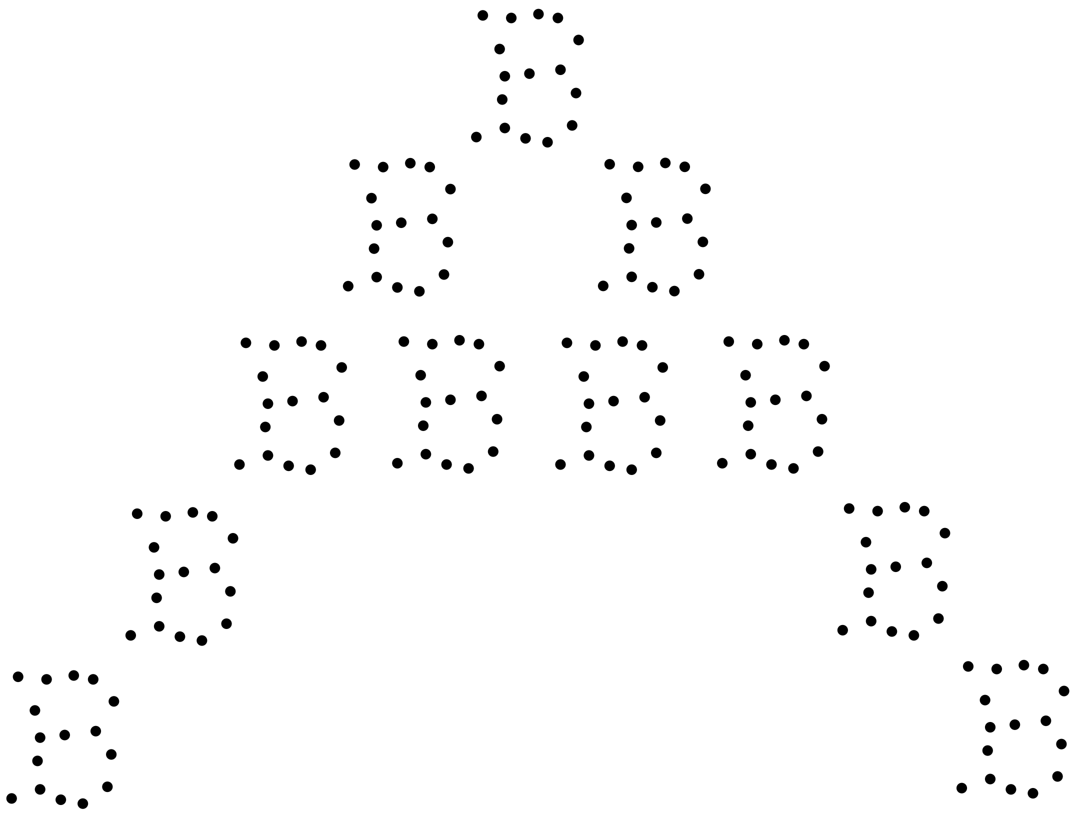
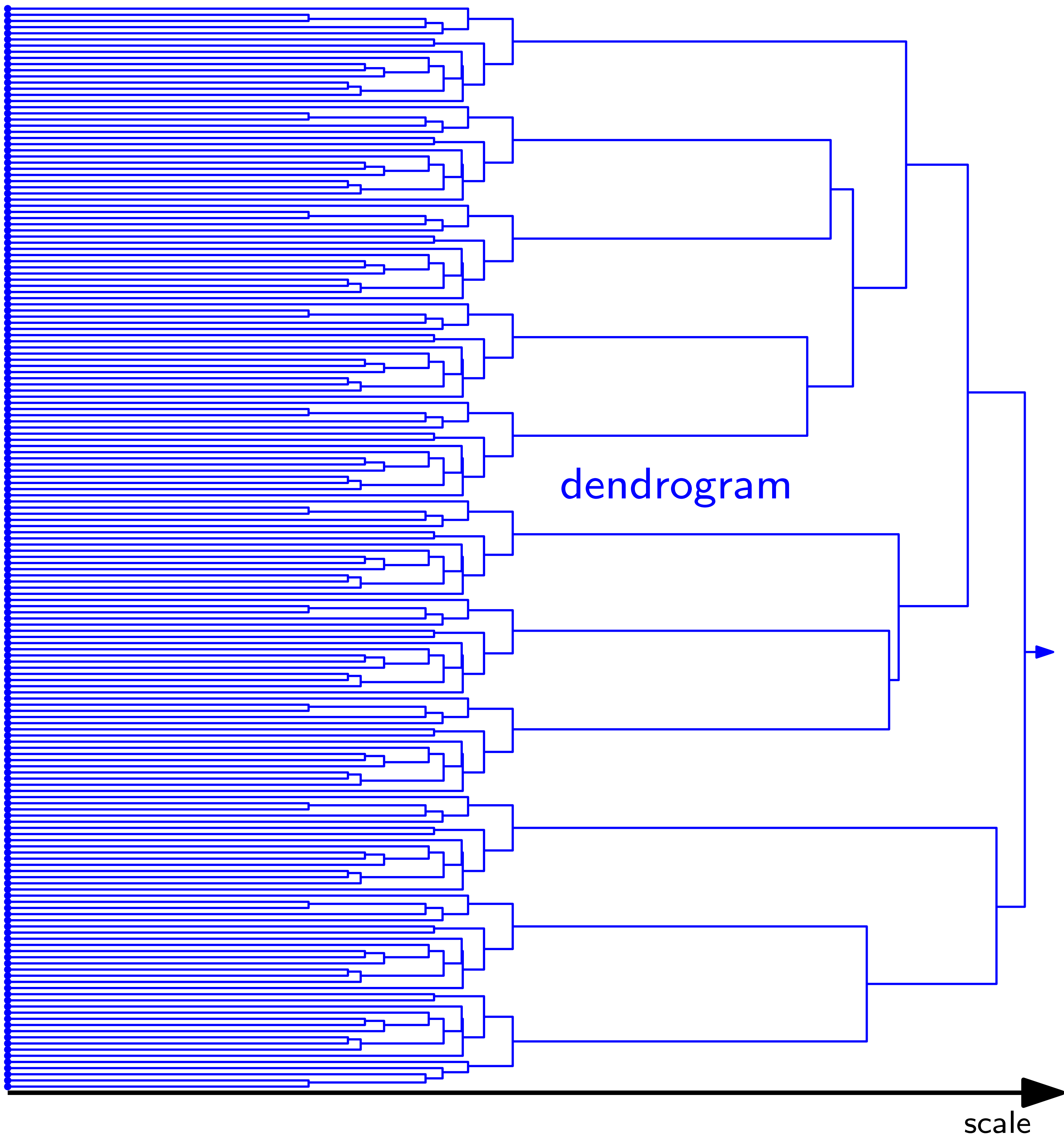
Connections



∃ barcodes: intuition (Agglomerative Hierarchical Clustering)





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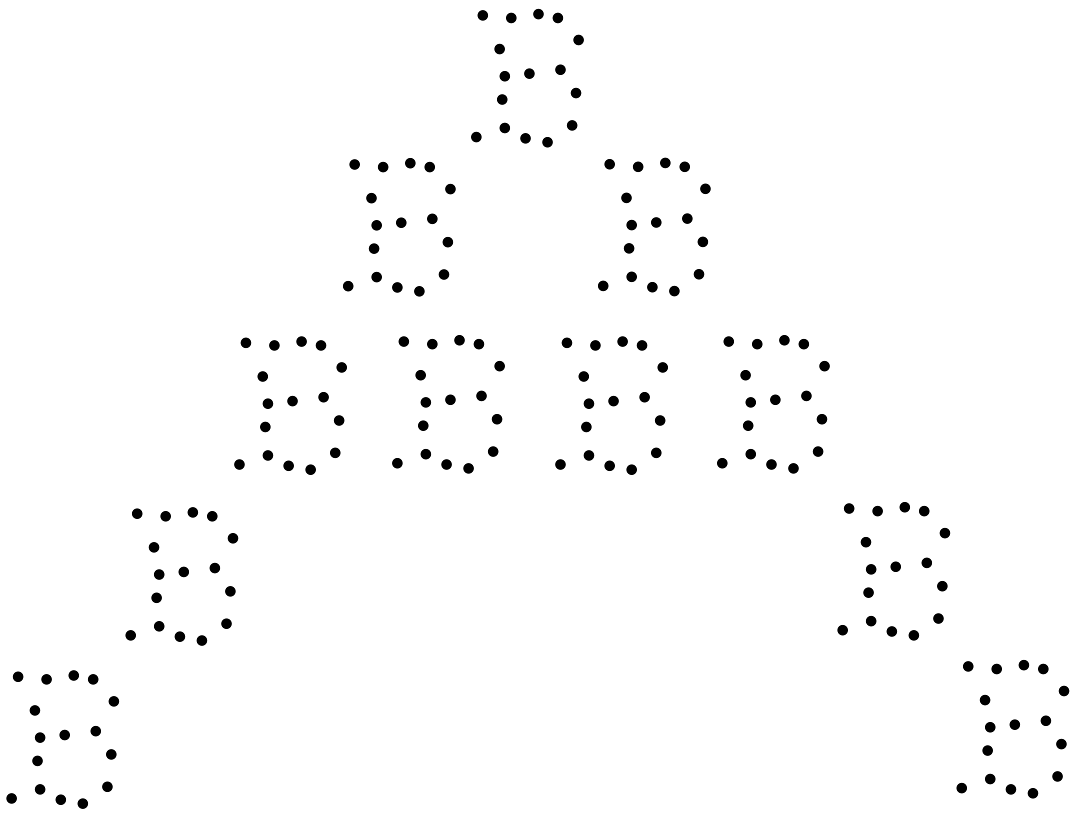

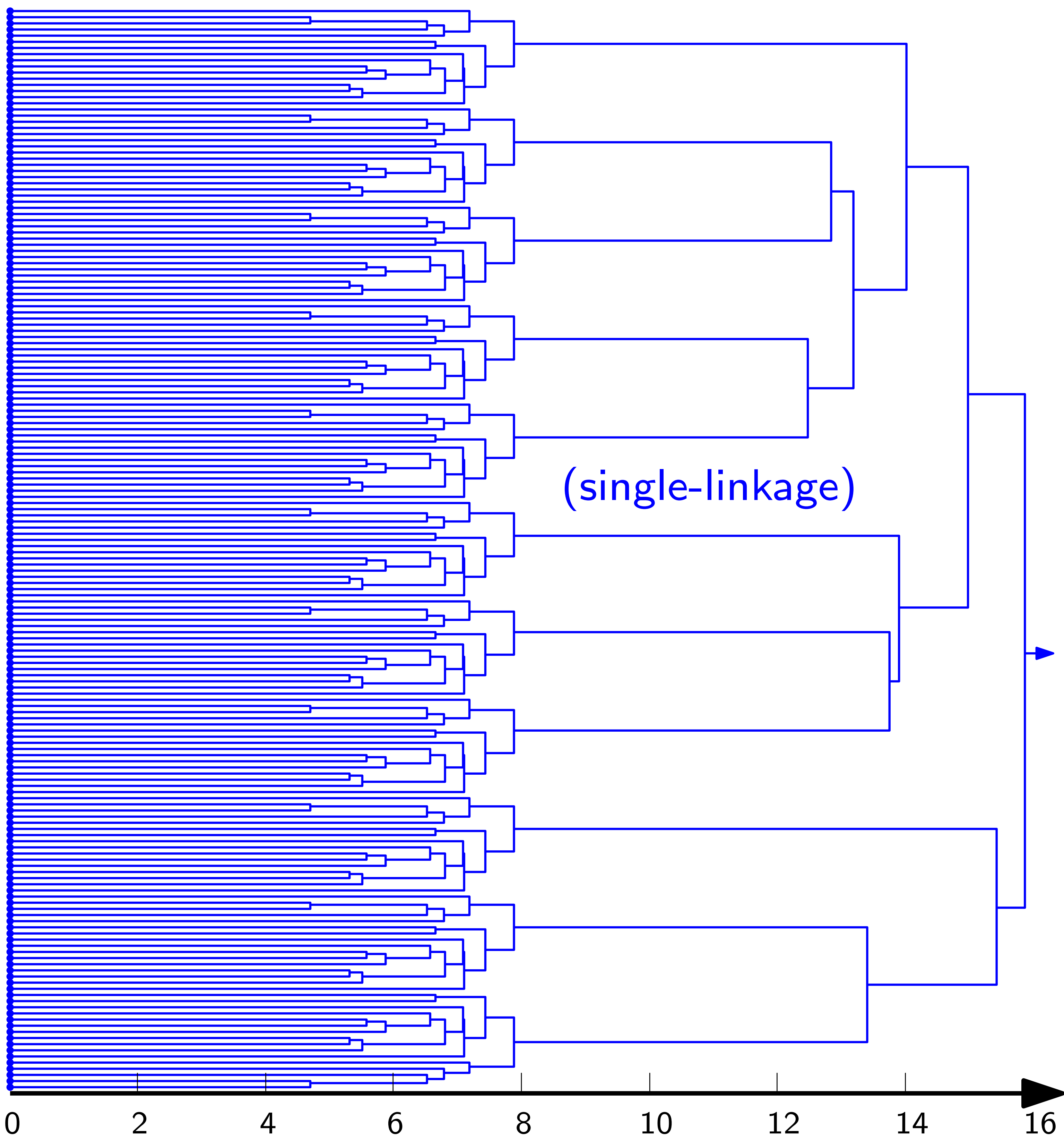

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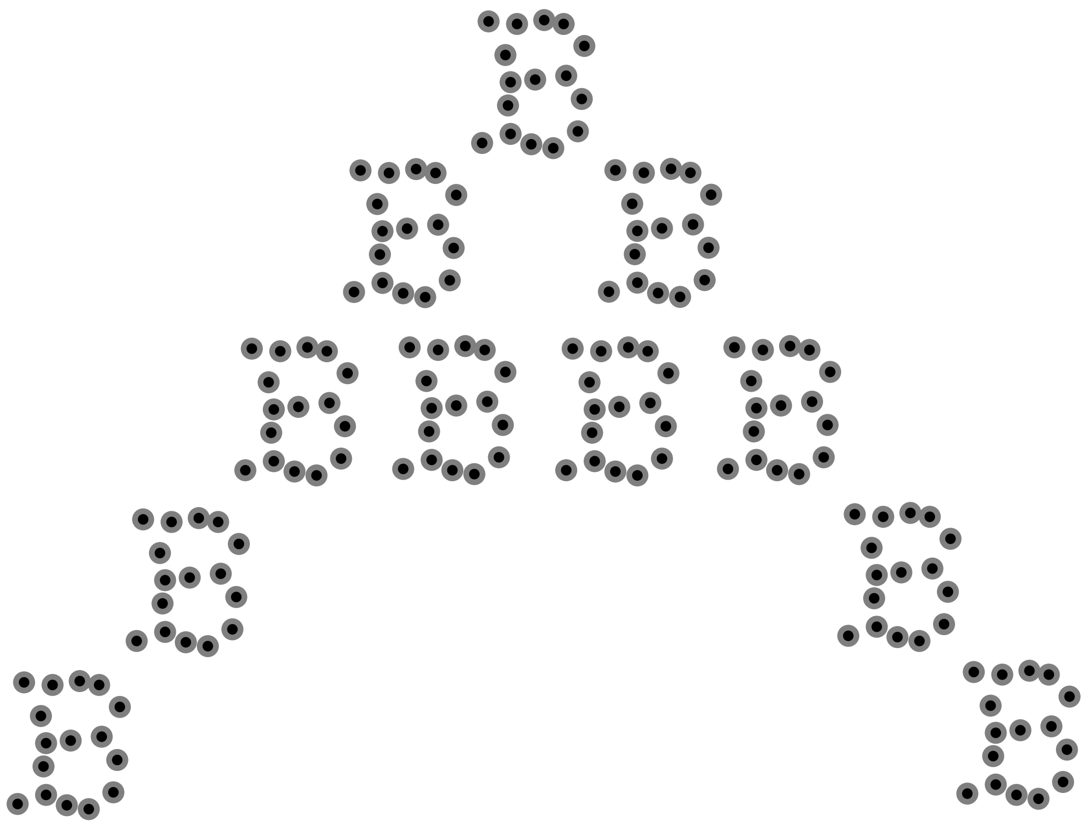

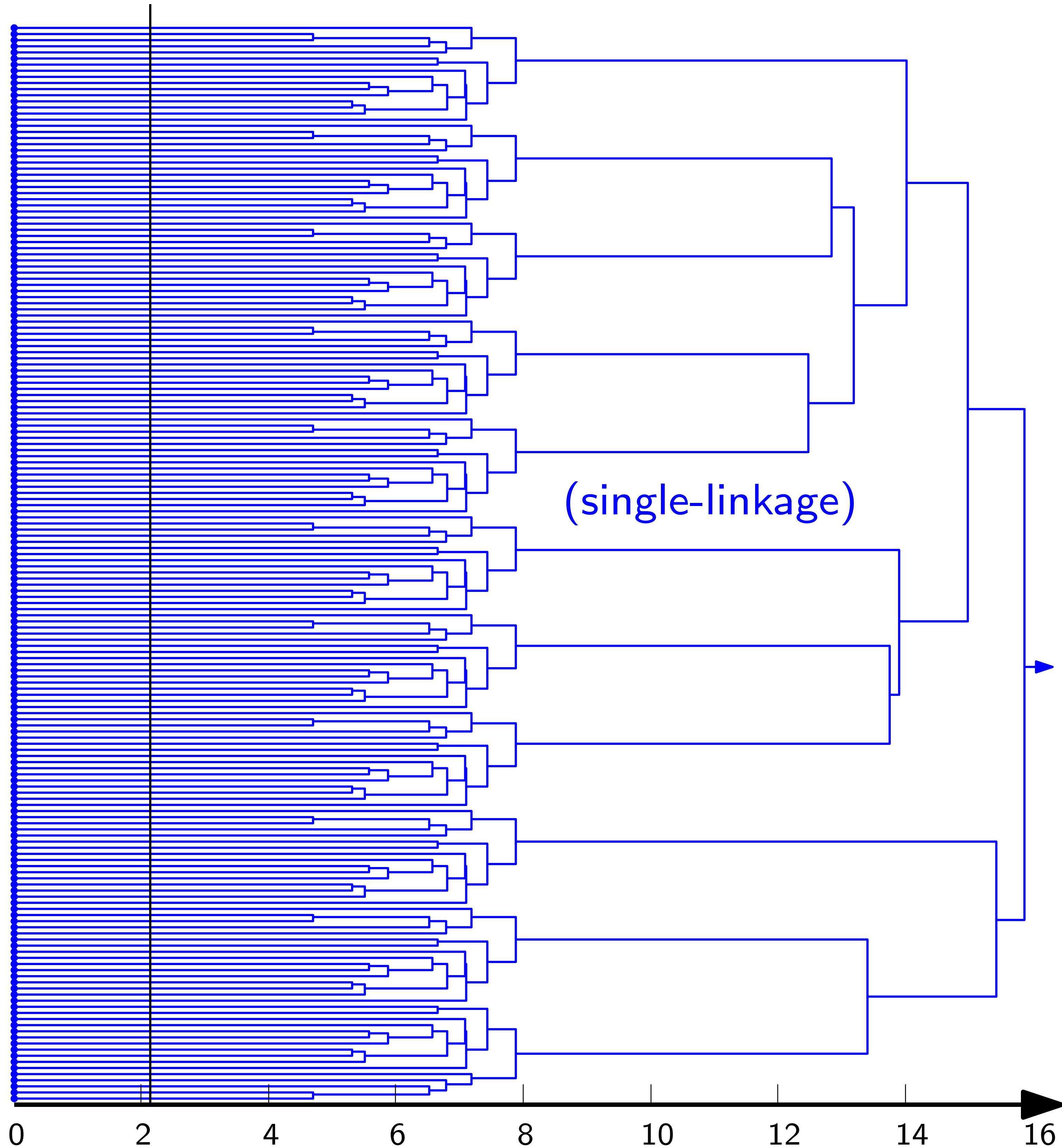

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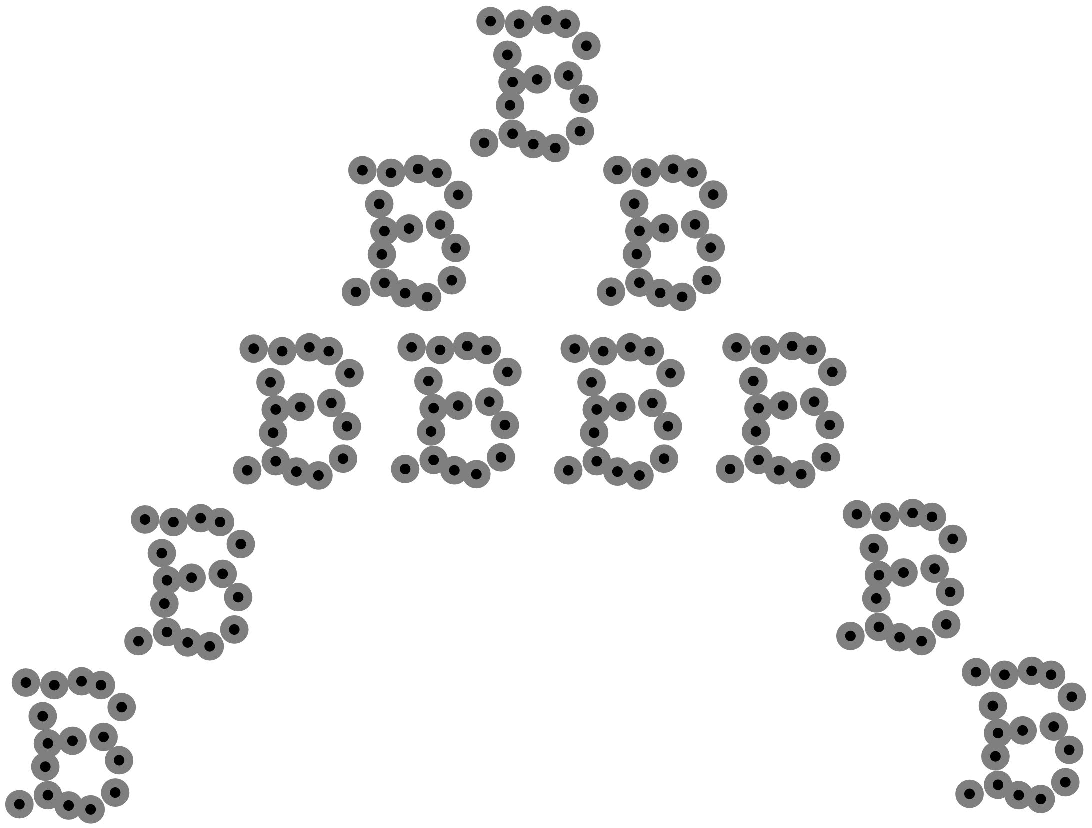

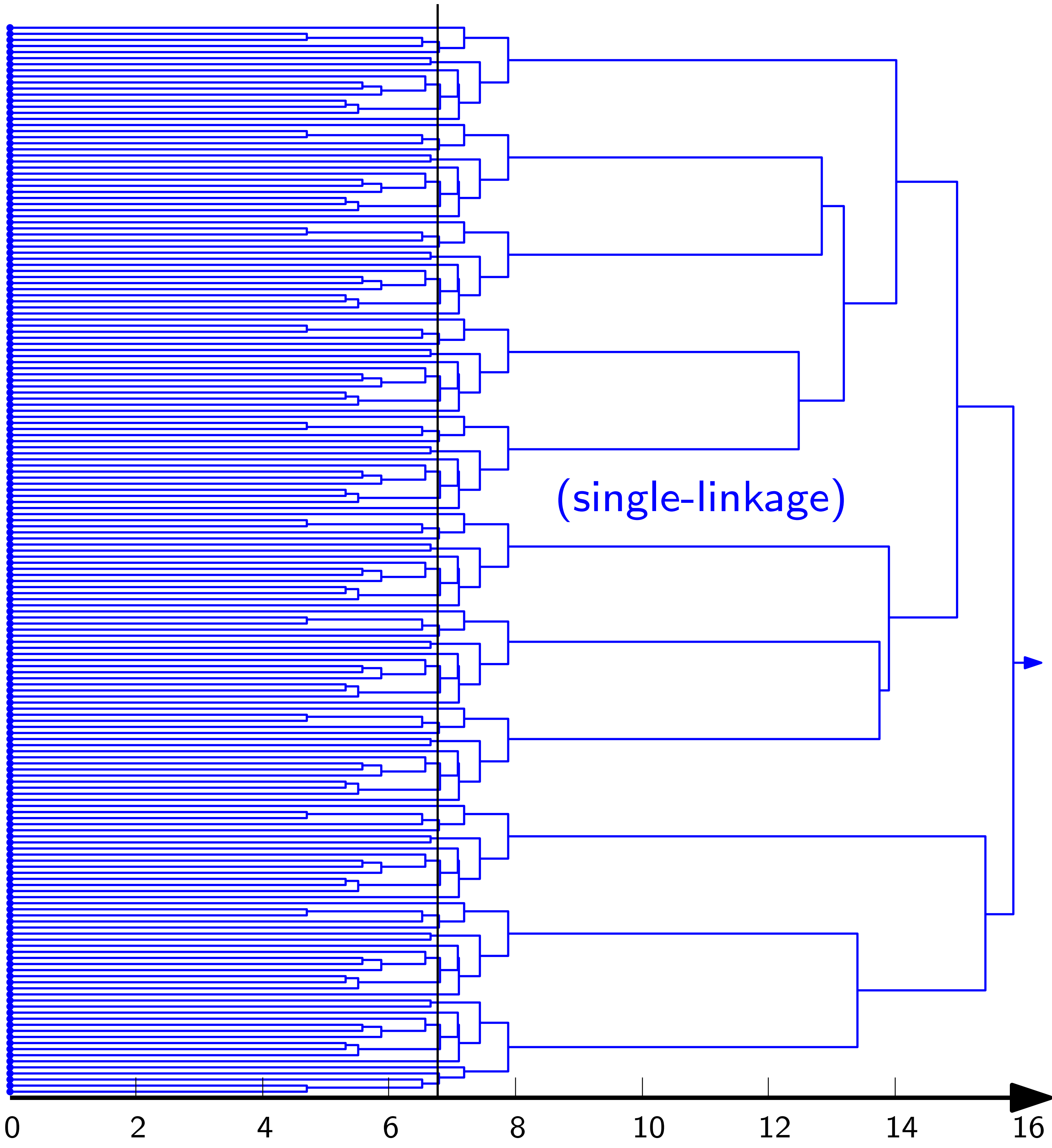

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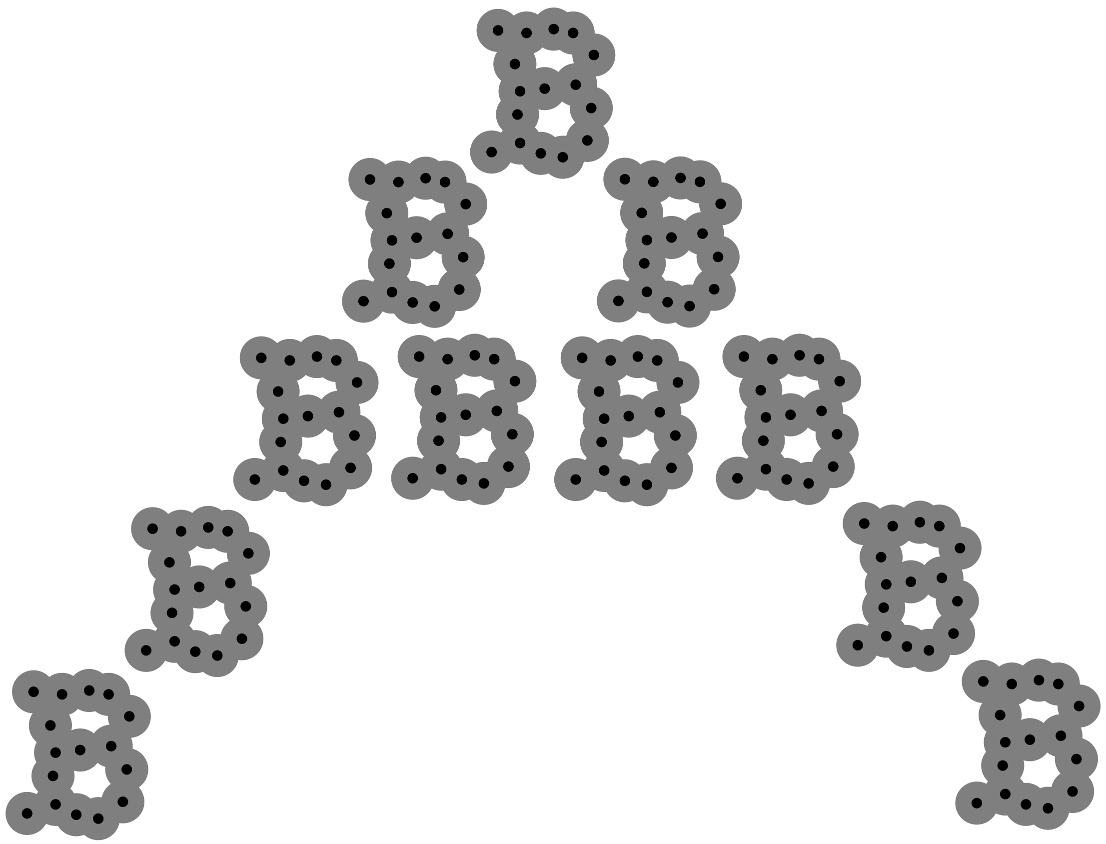

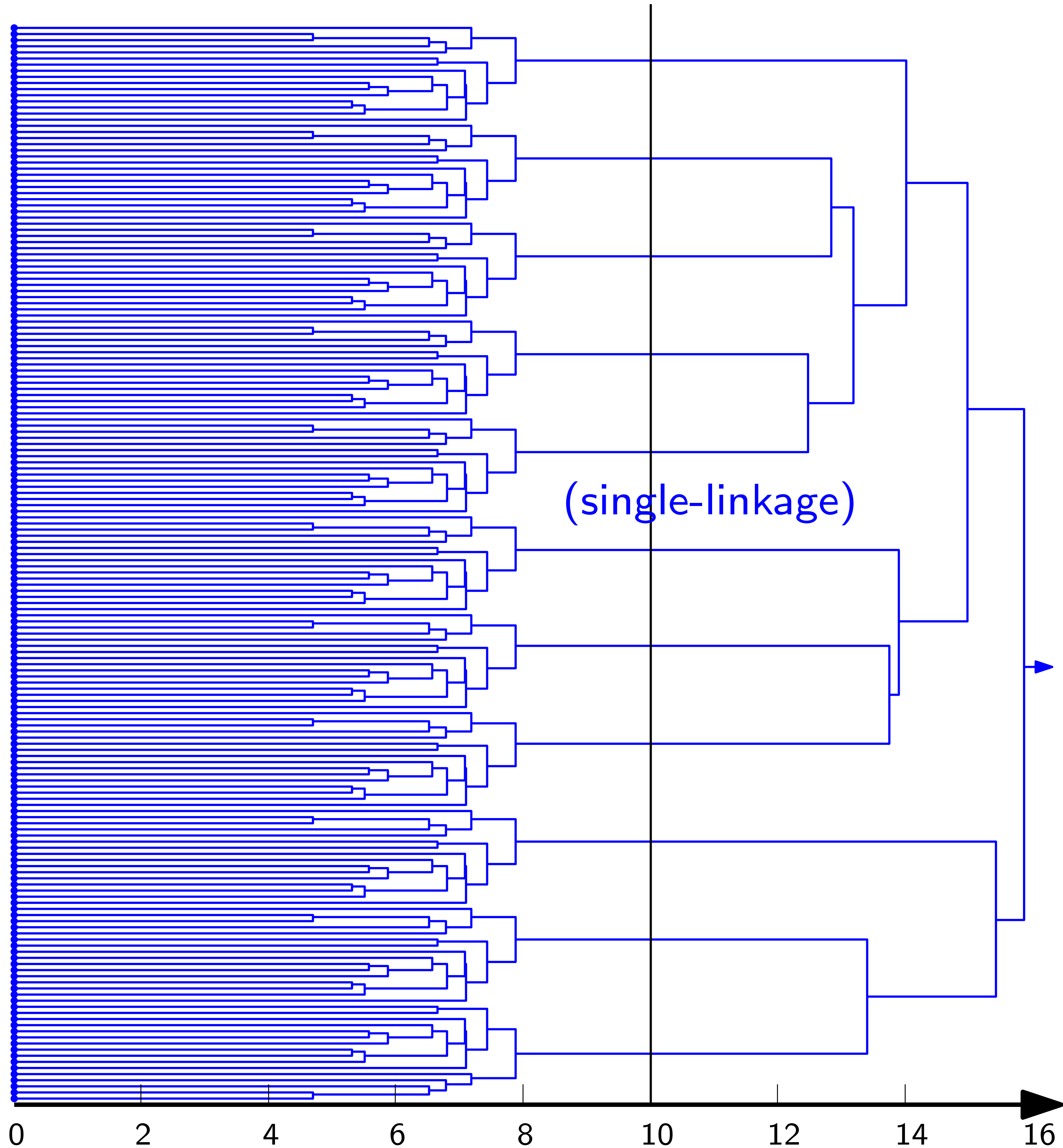

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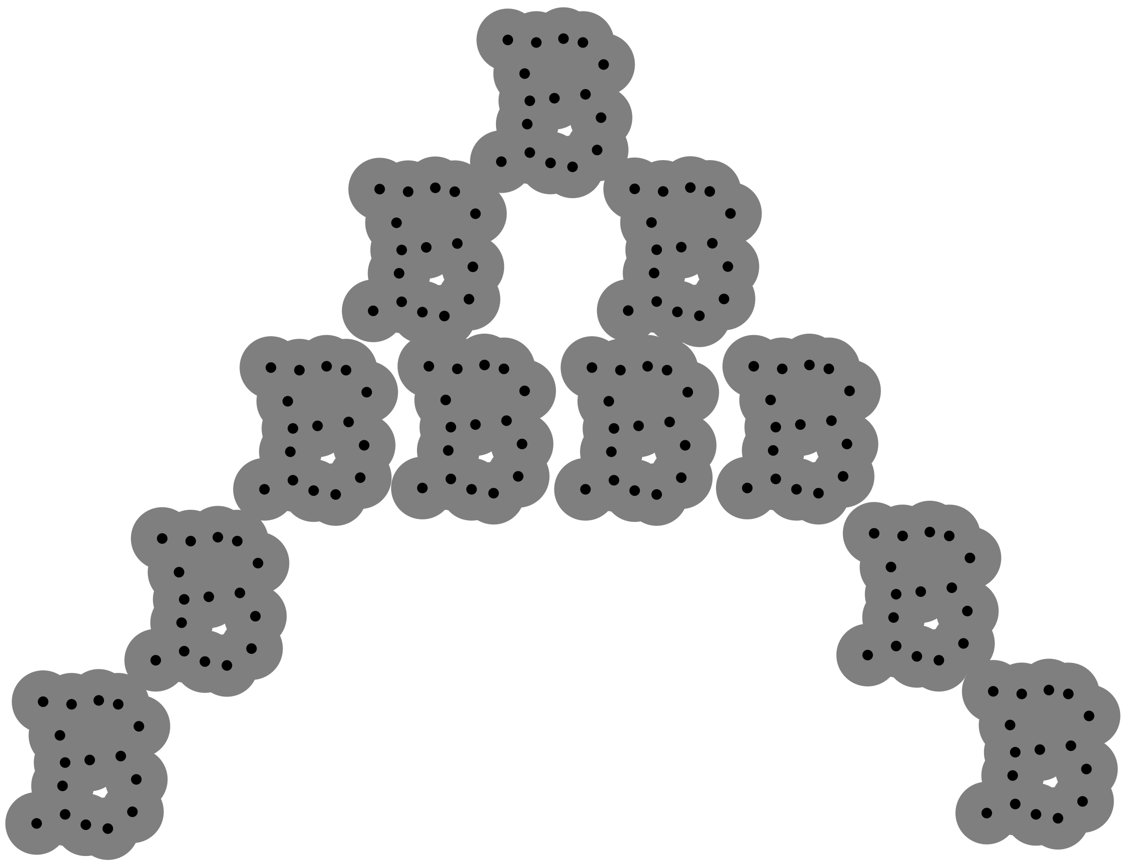

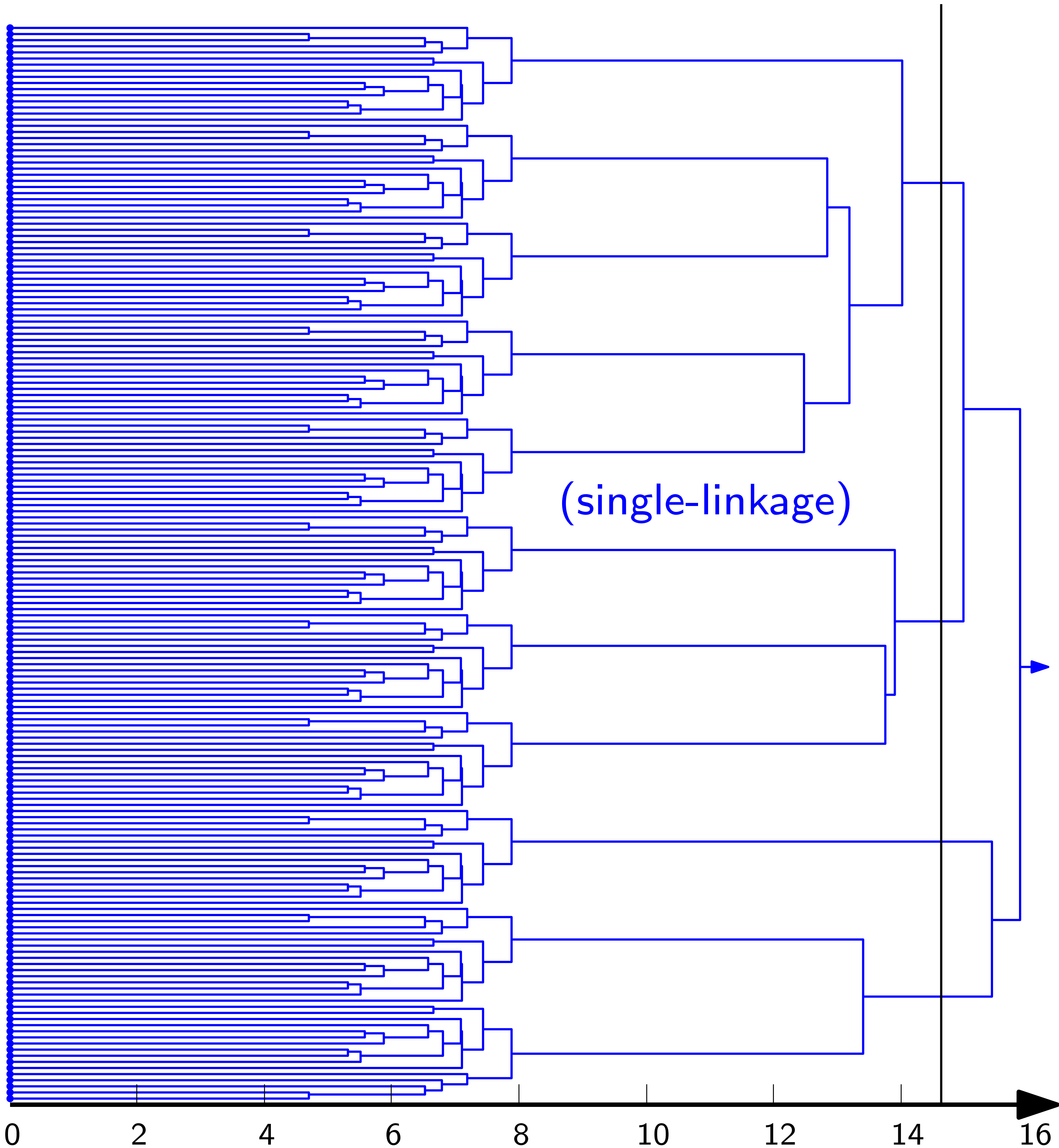

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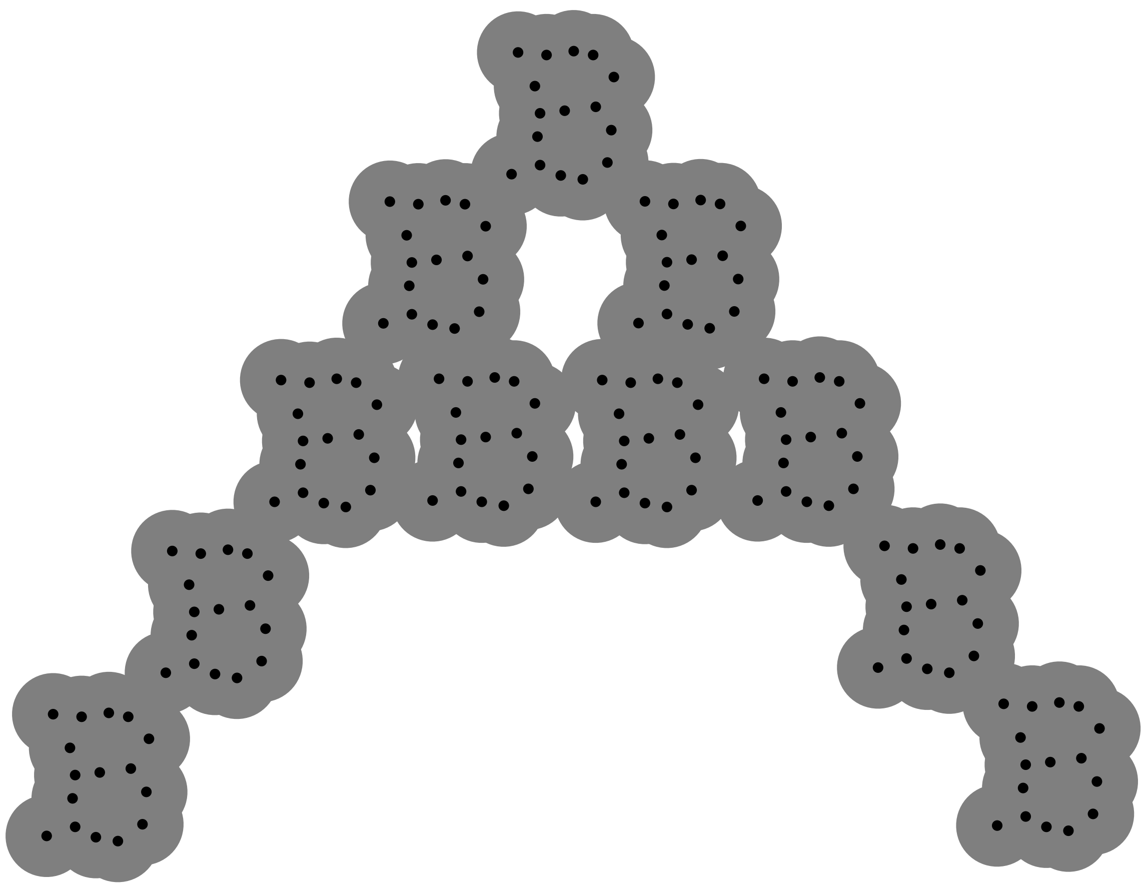

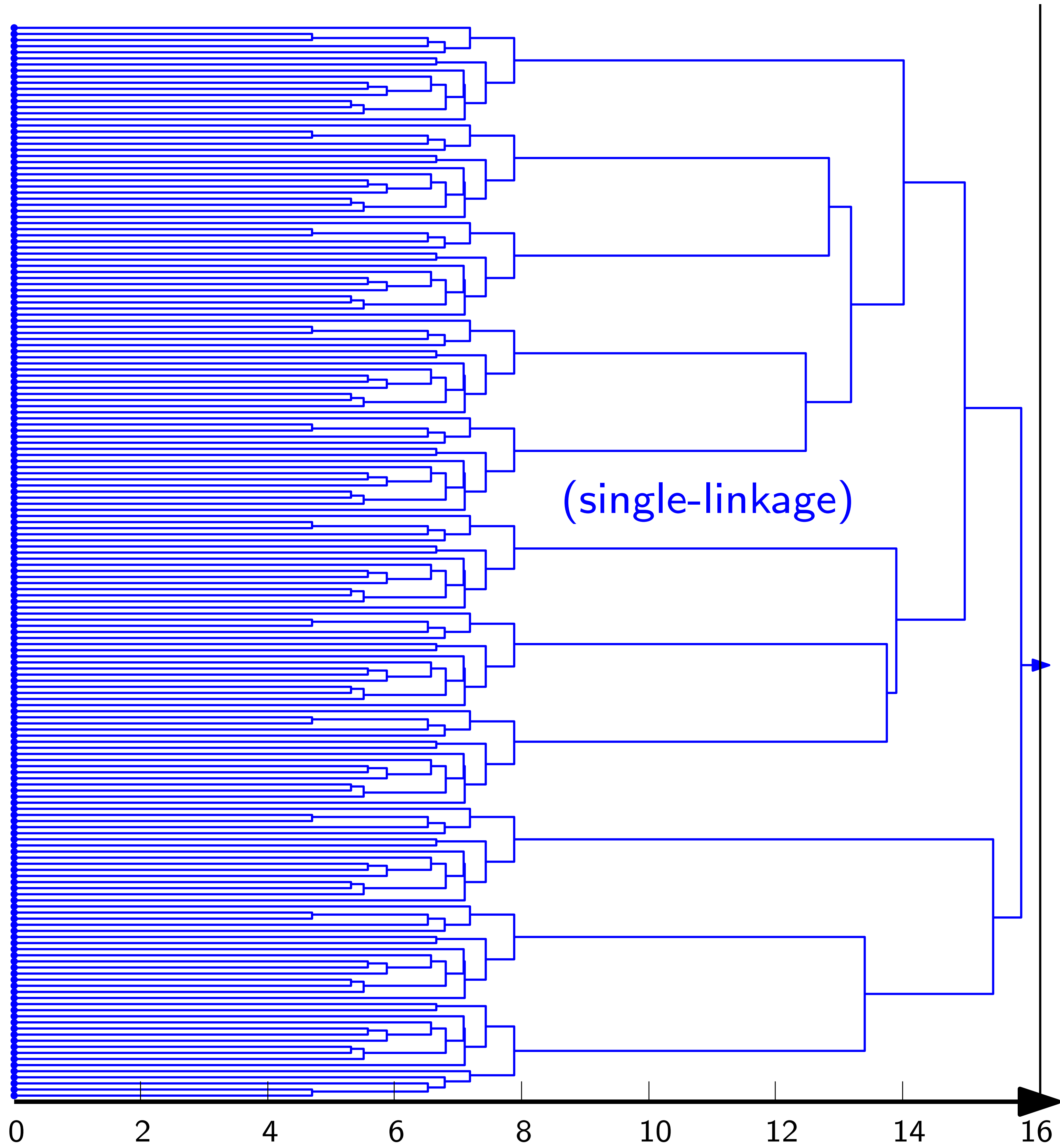

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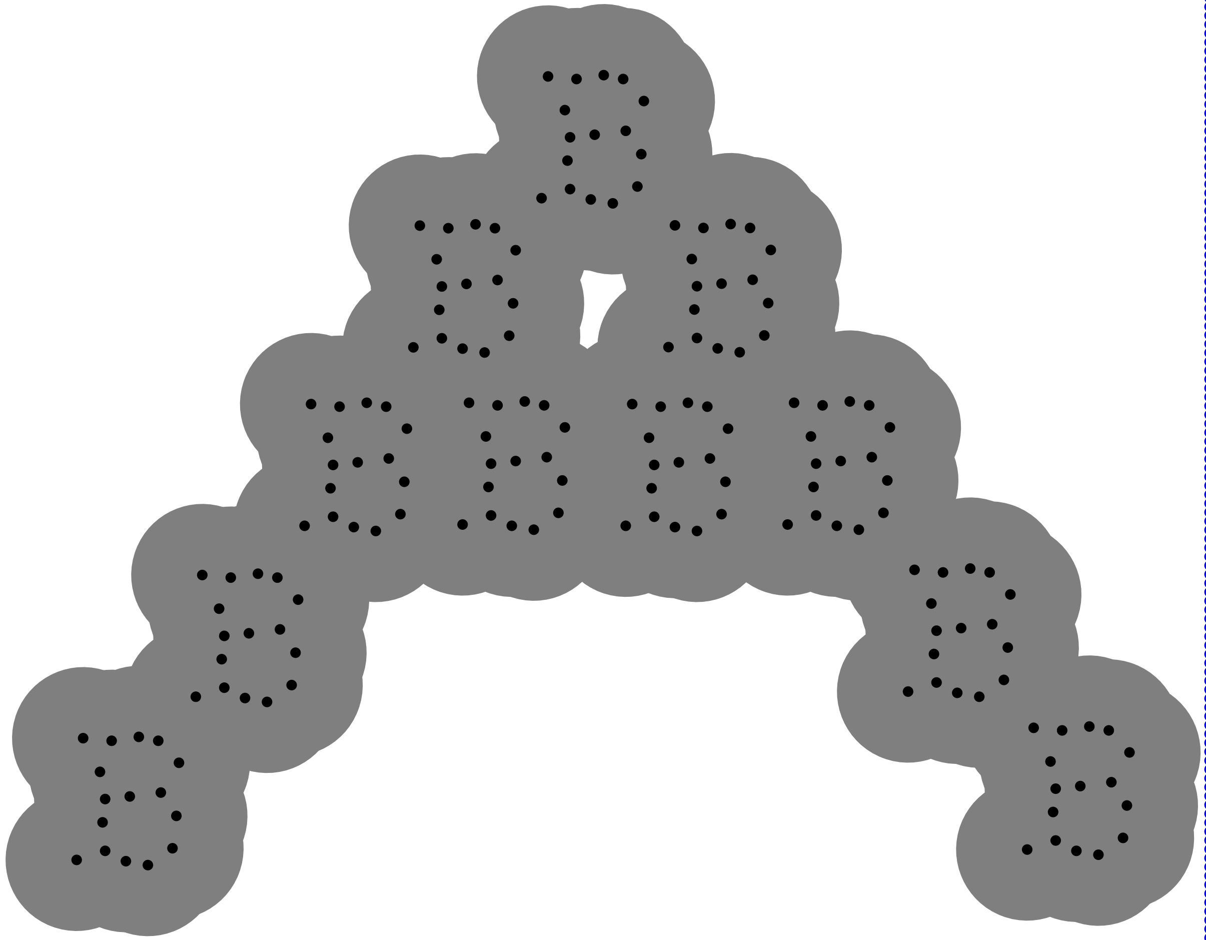

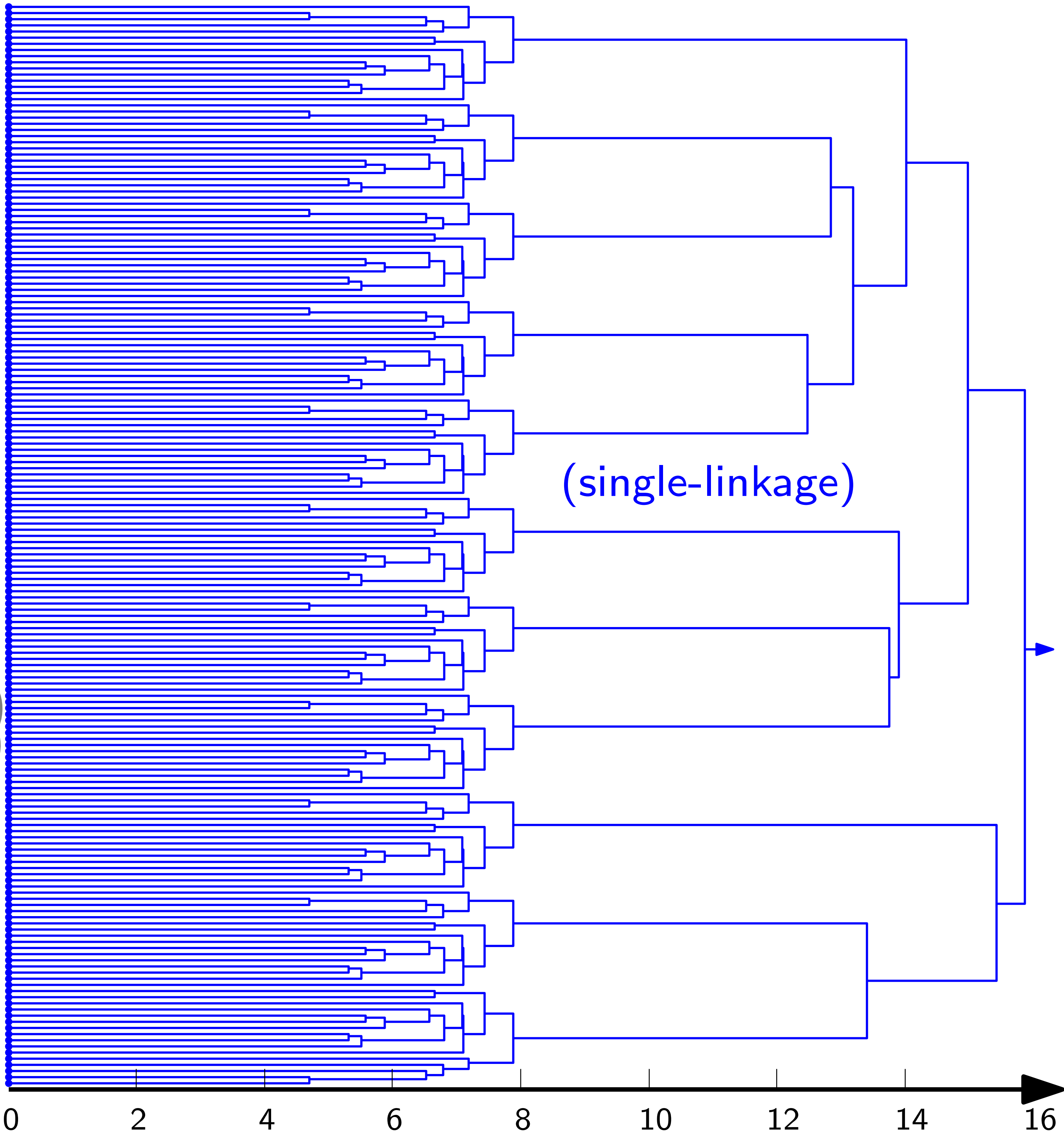

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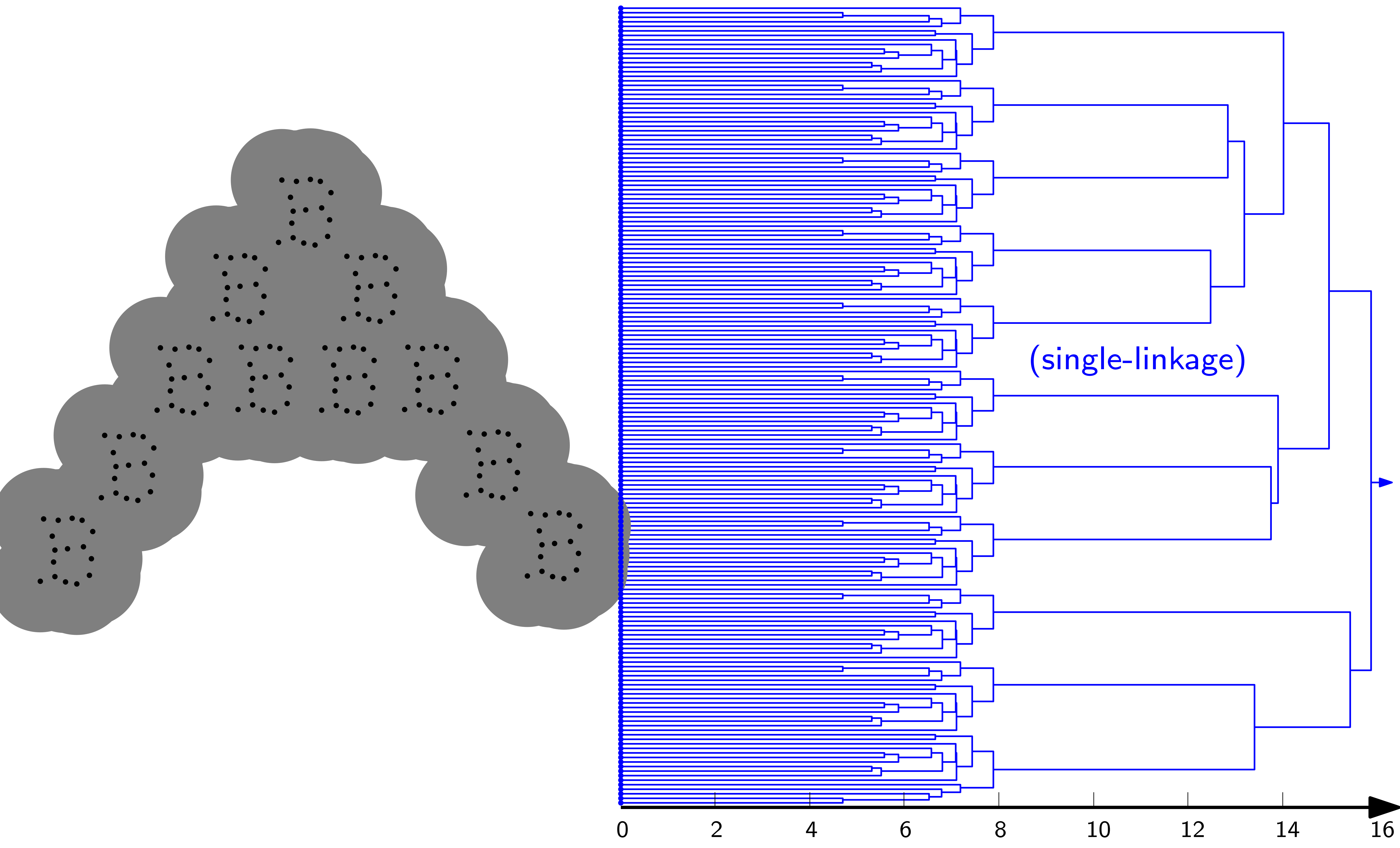

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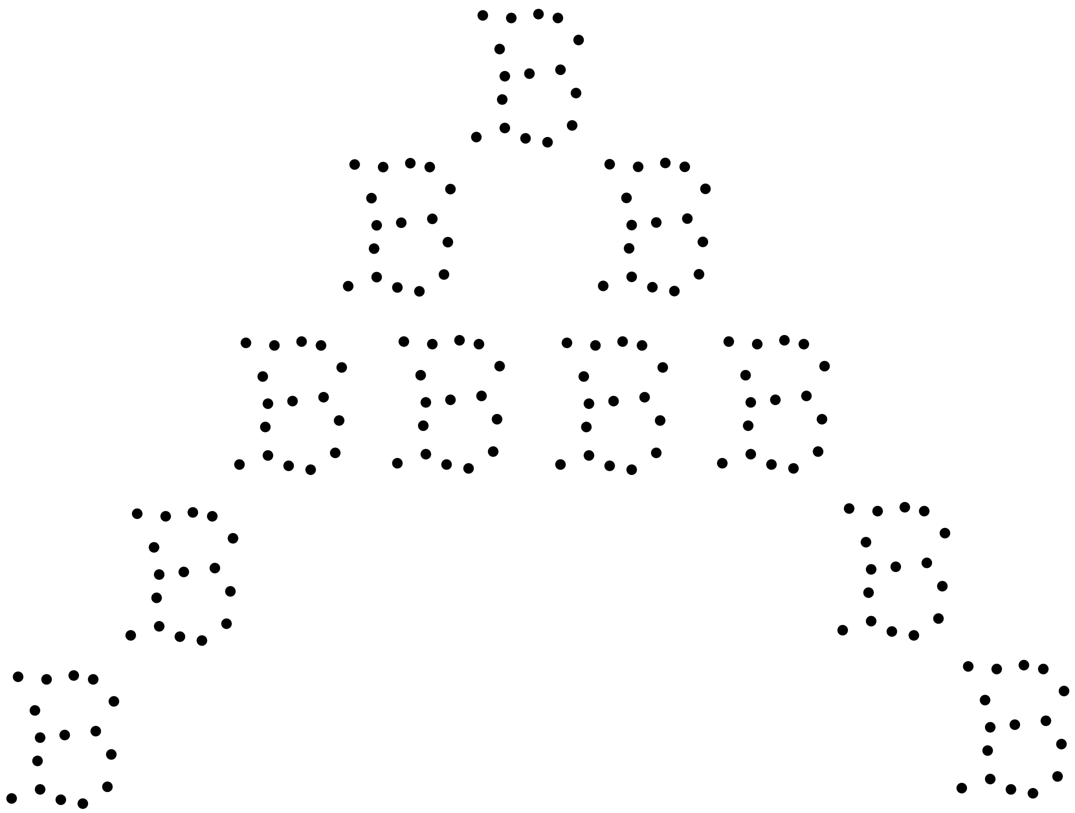

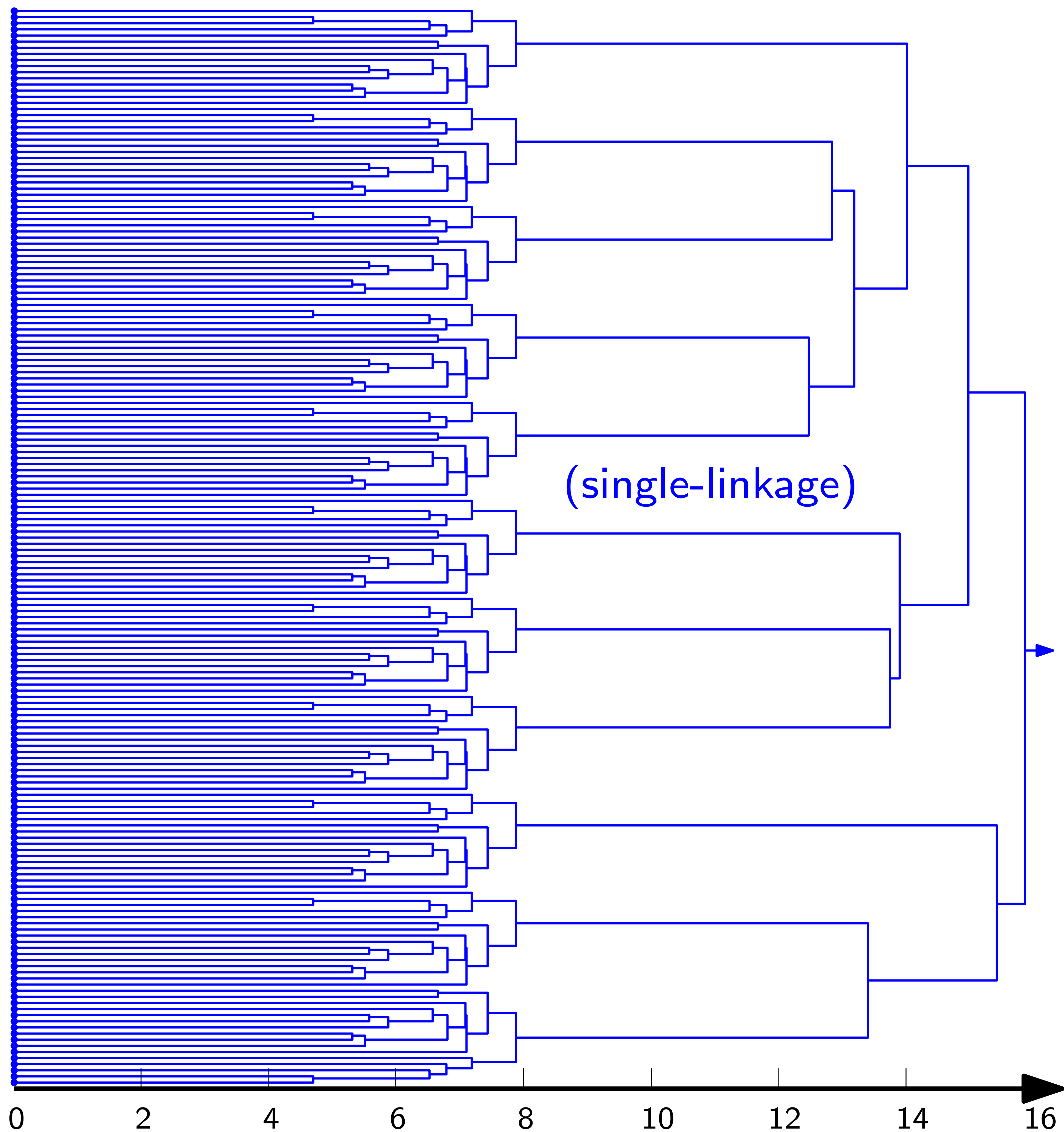

dendrogram is:

- informative
- unstable

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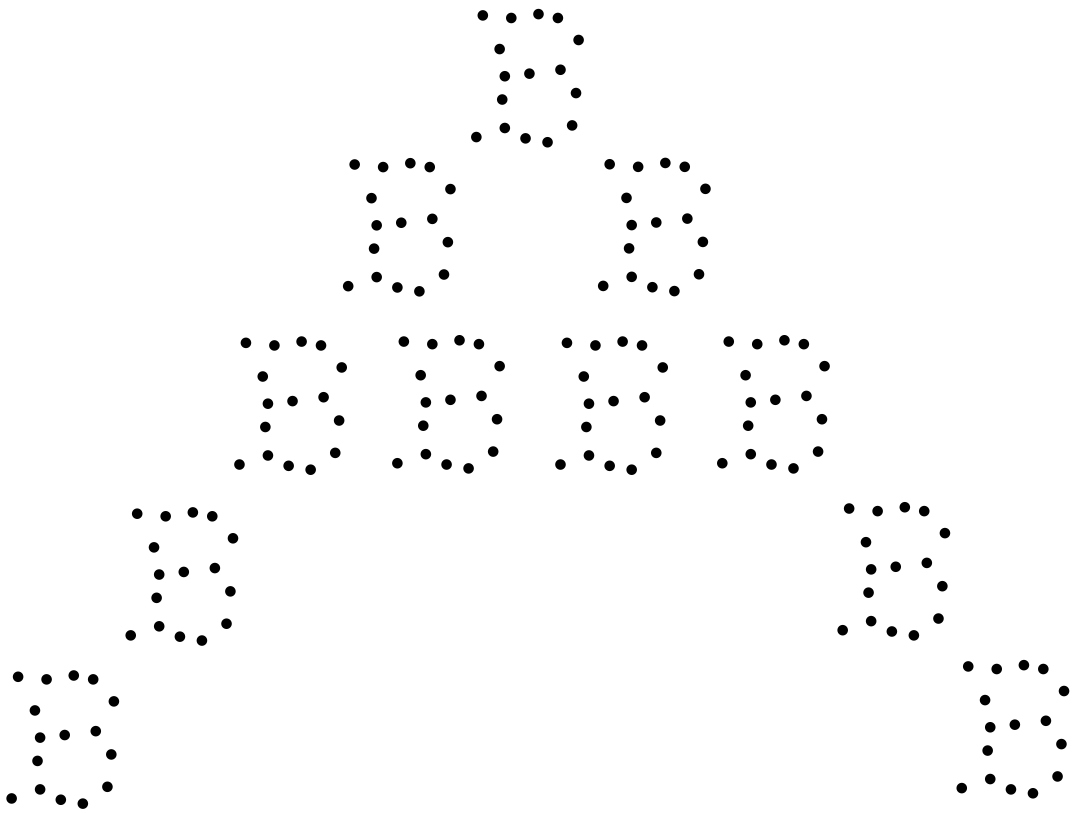
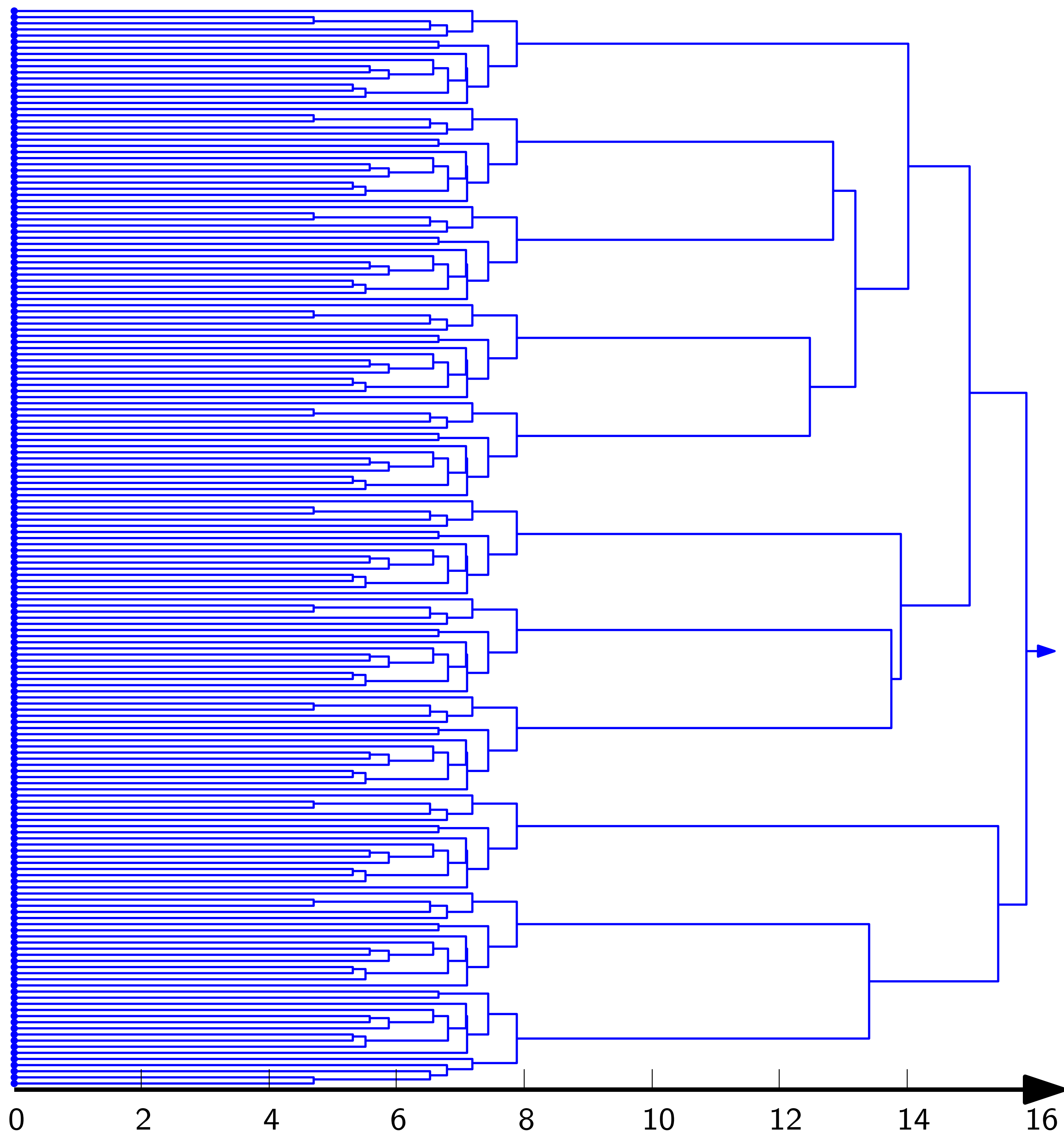


dendrogram → barcode



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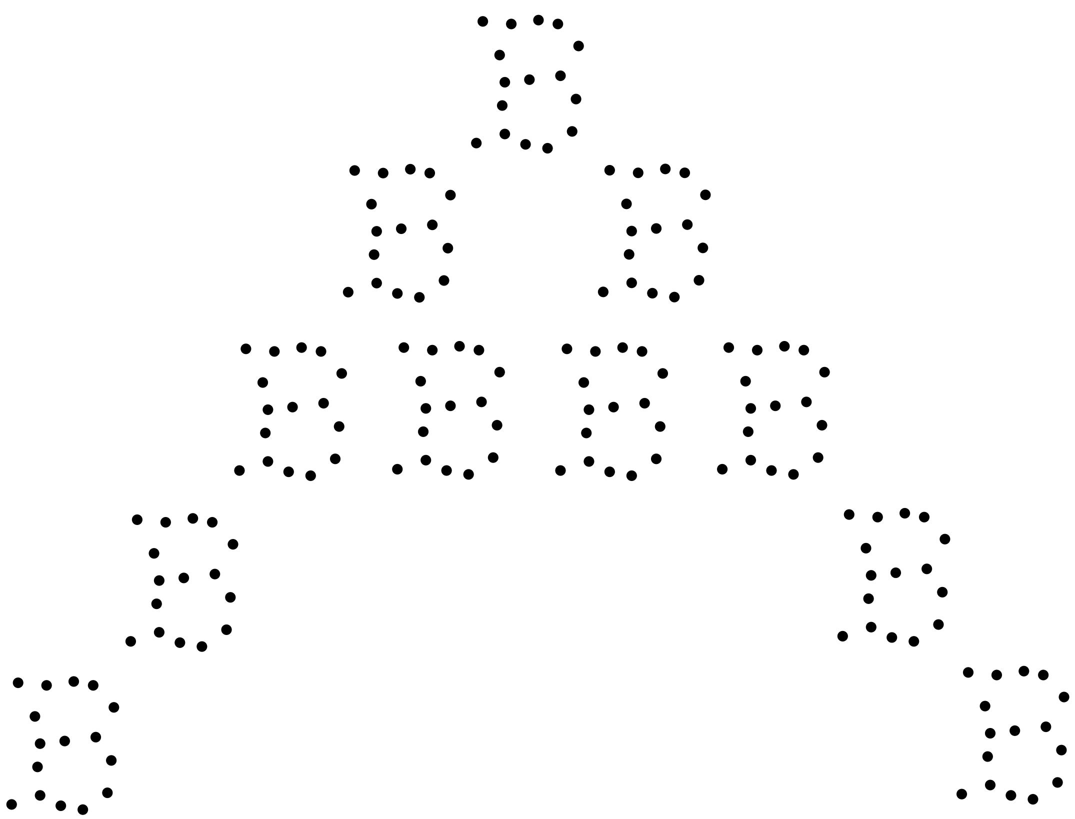

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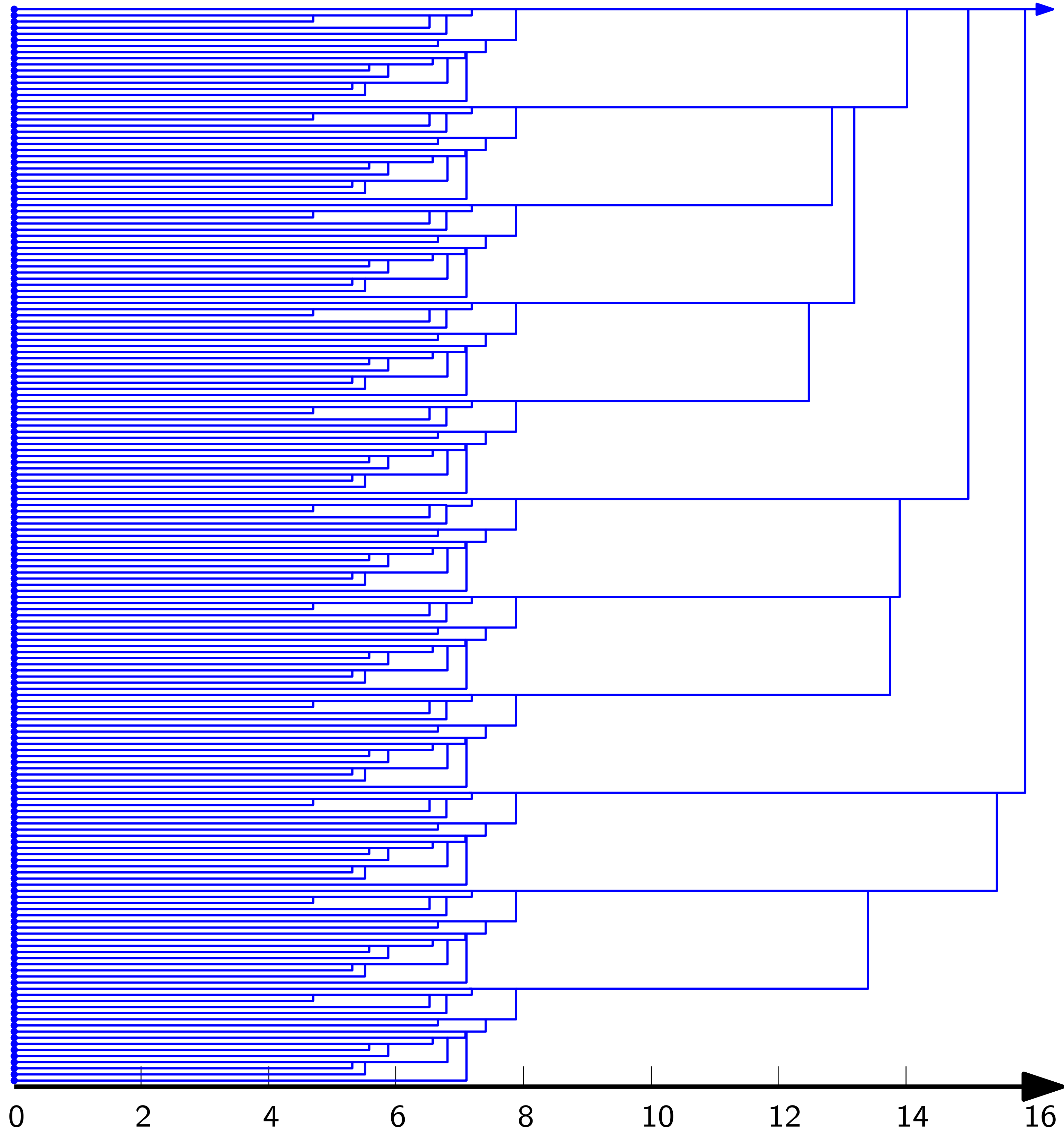

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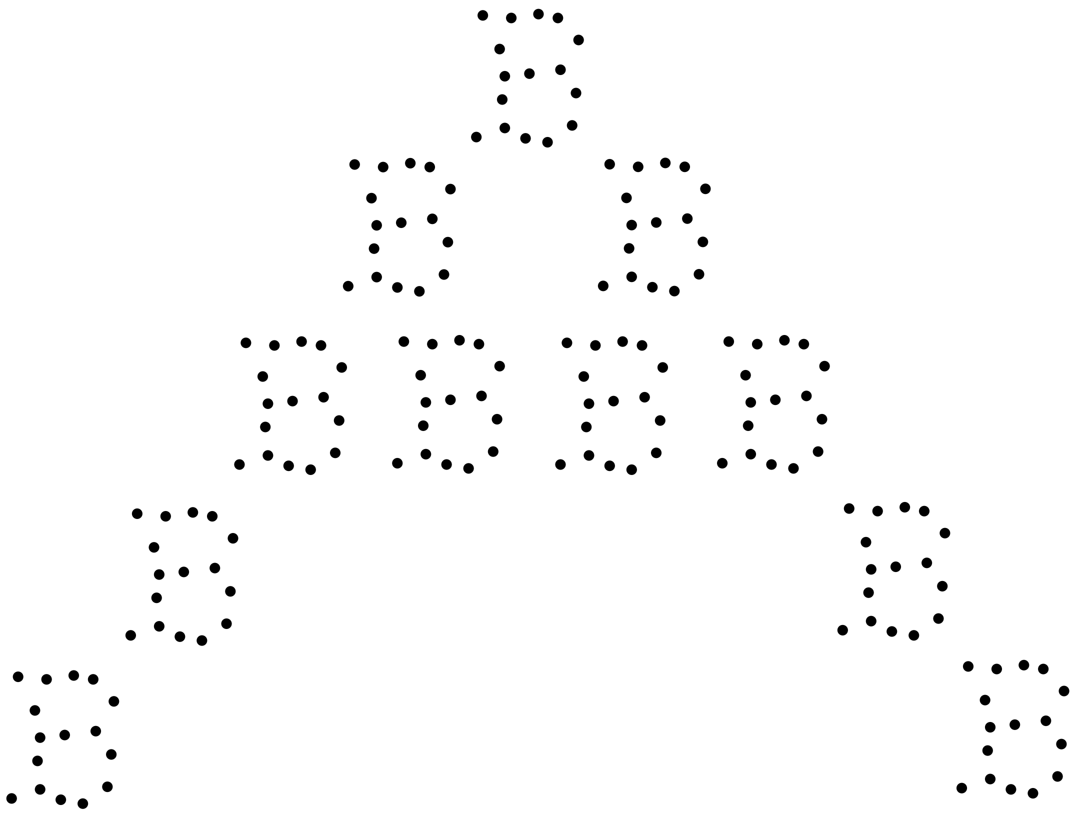

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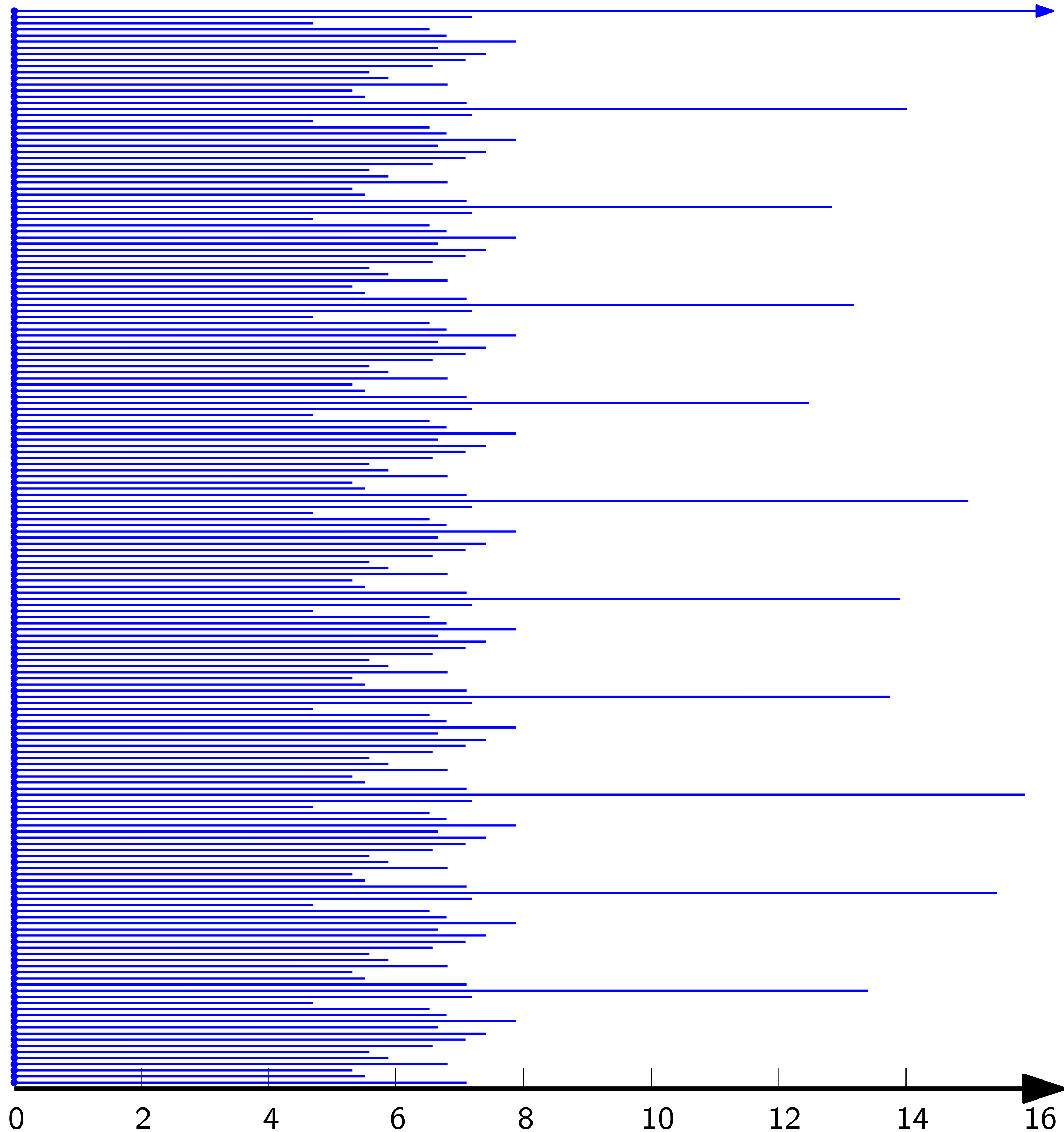

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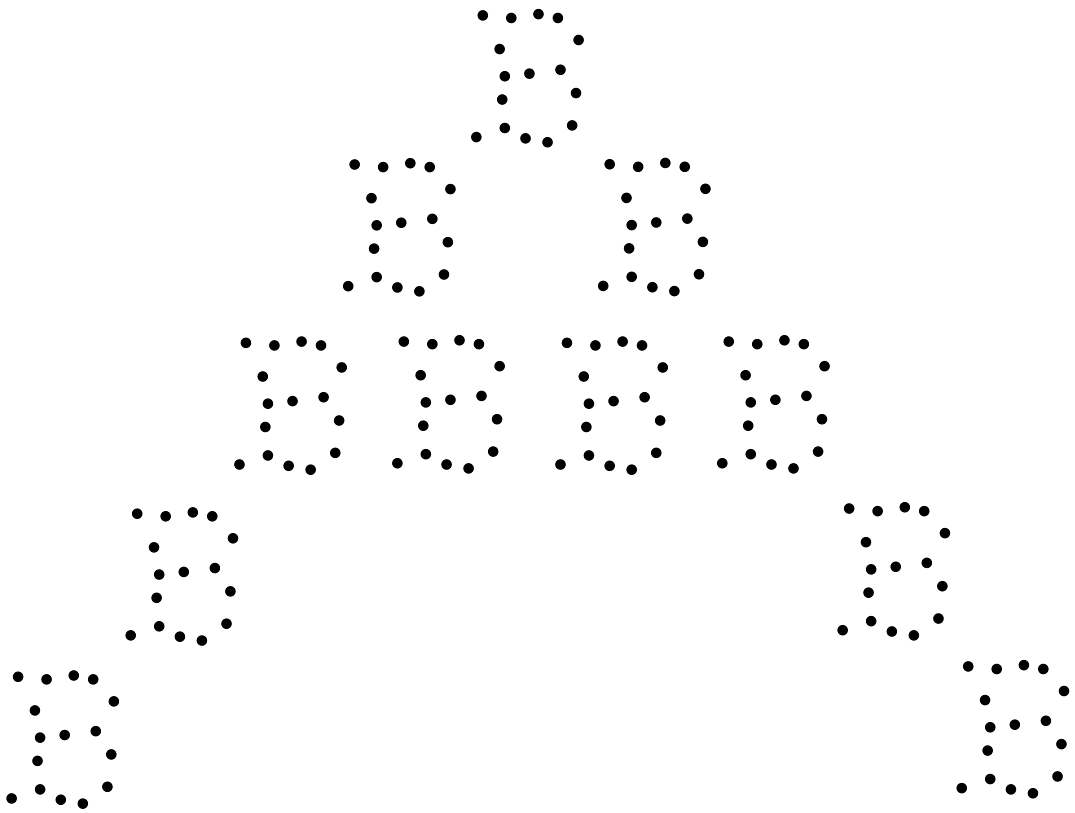
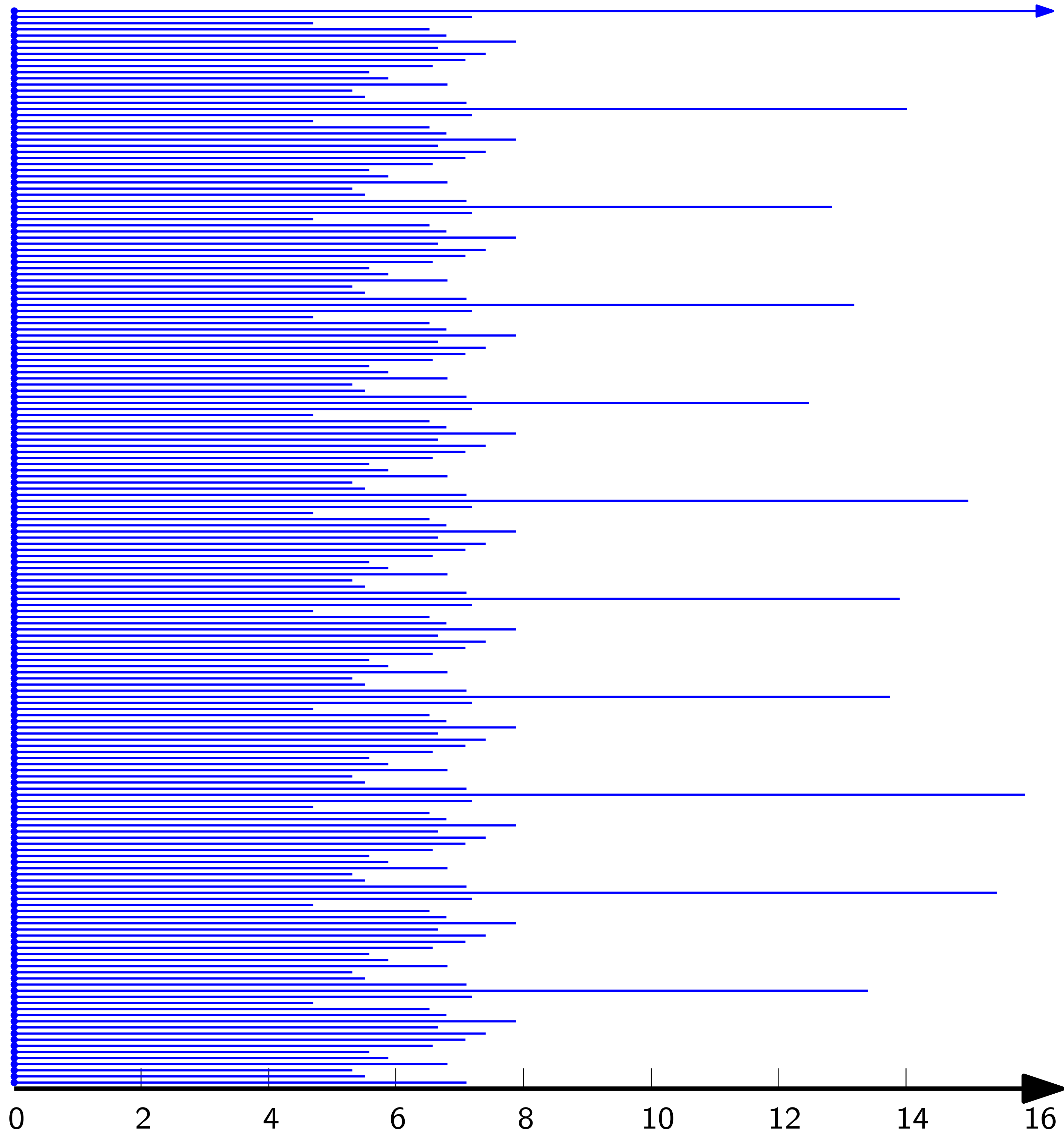


barcode is:

- less (but still) informative
- more stable



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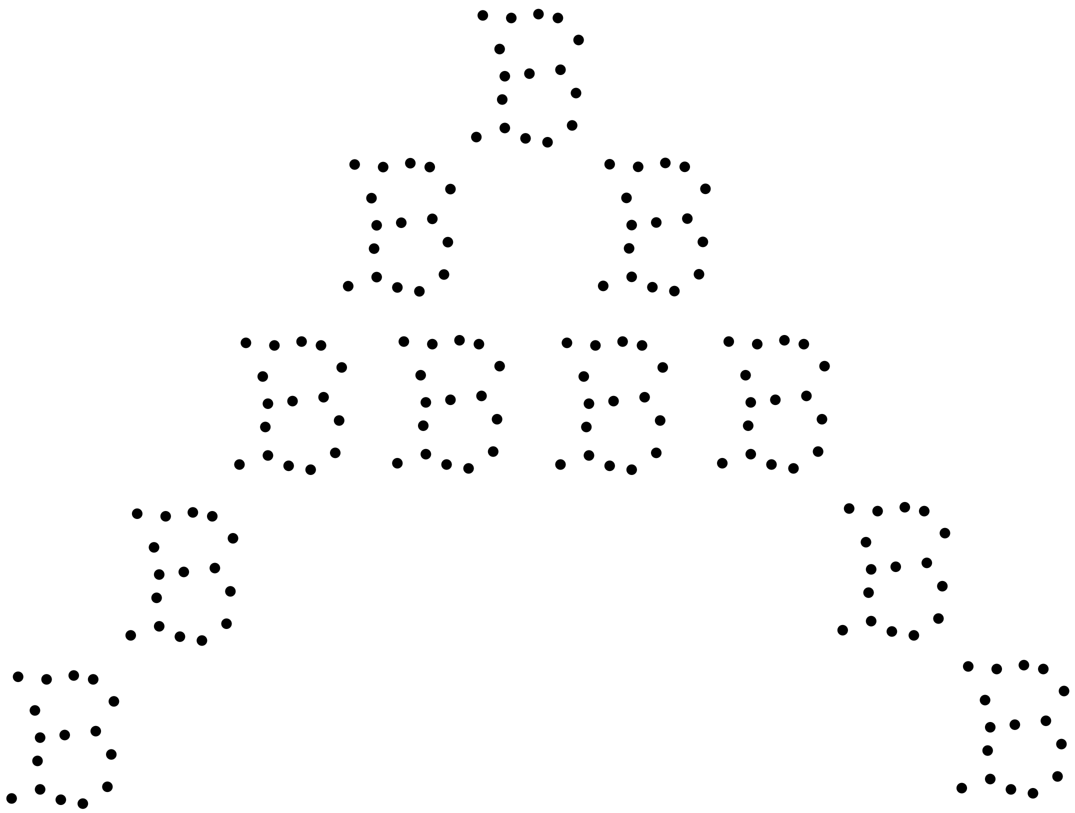

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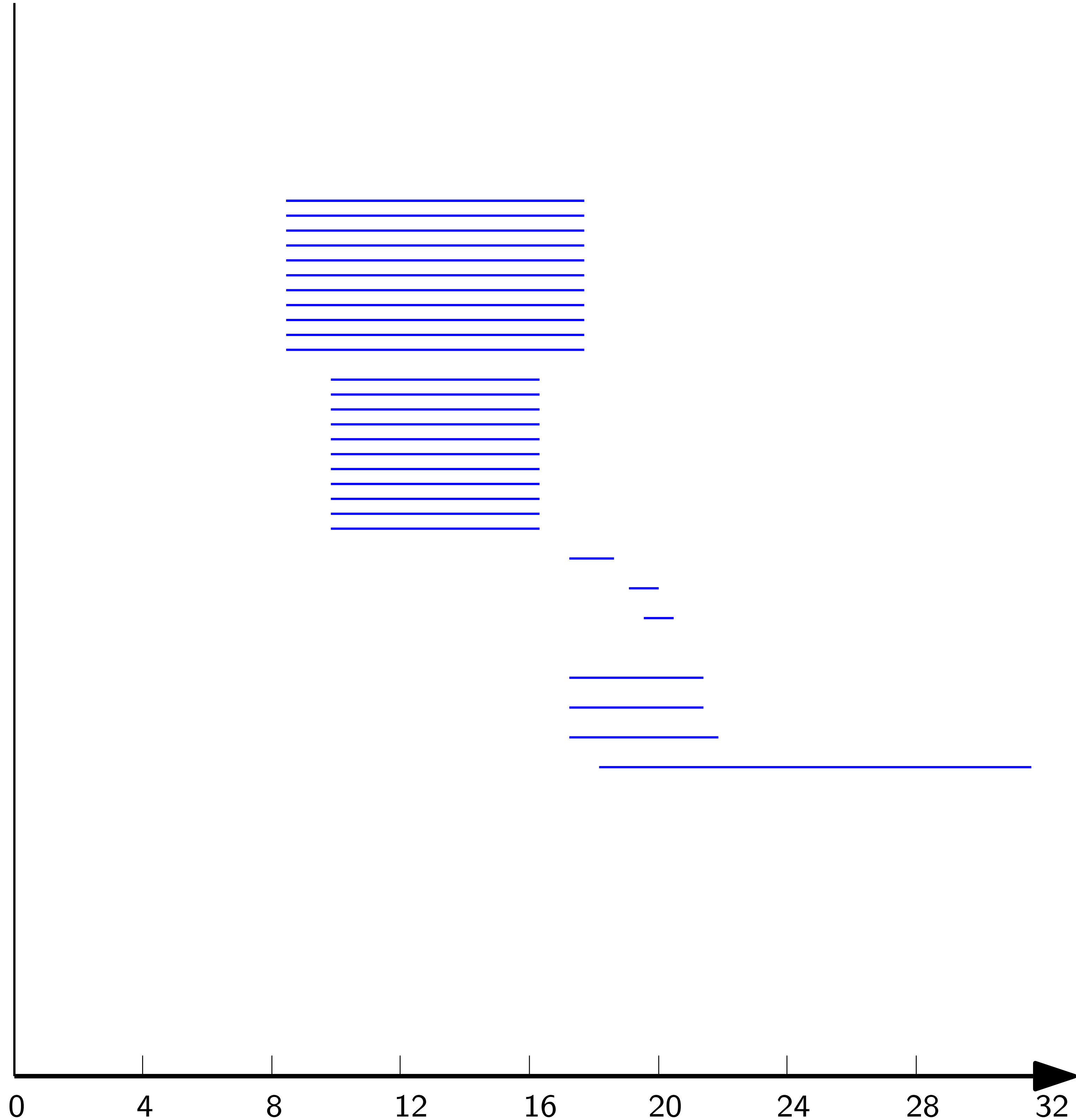
- generalizable



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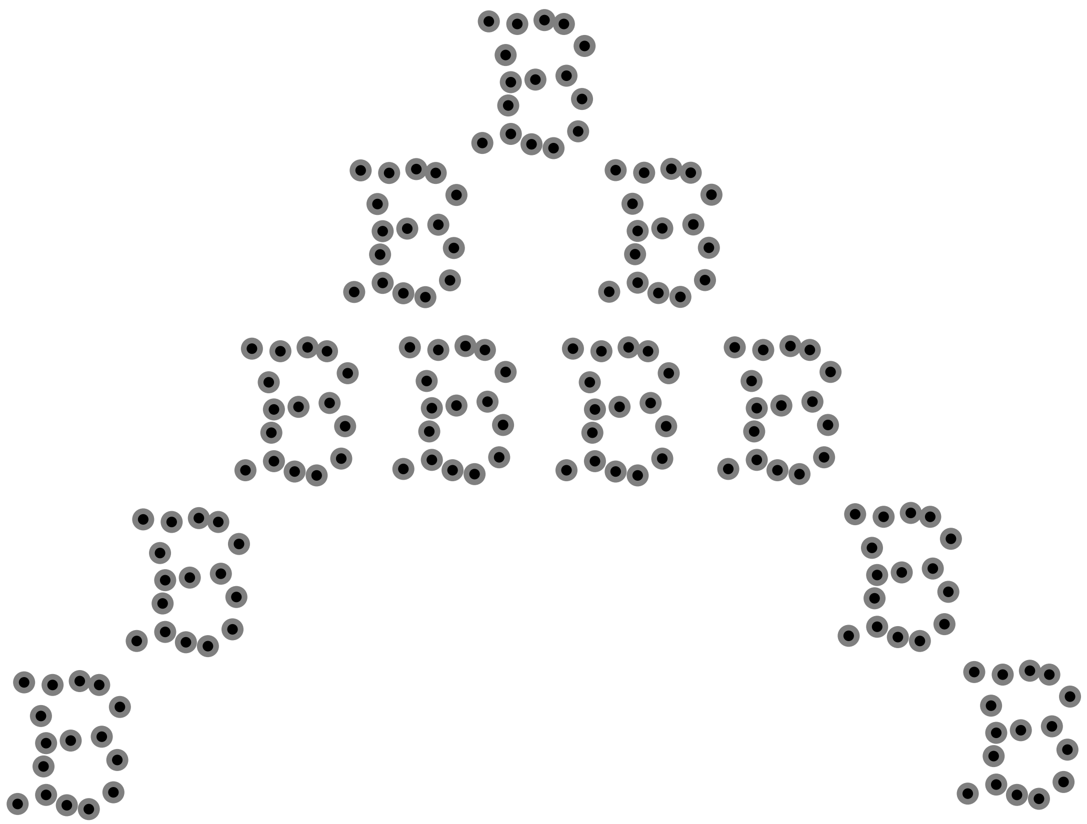

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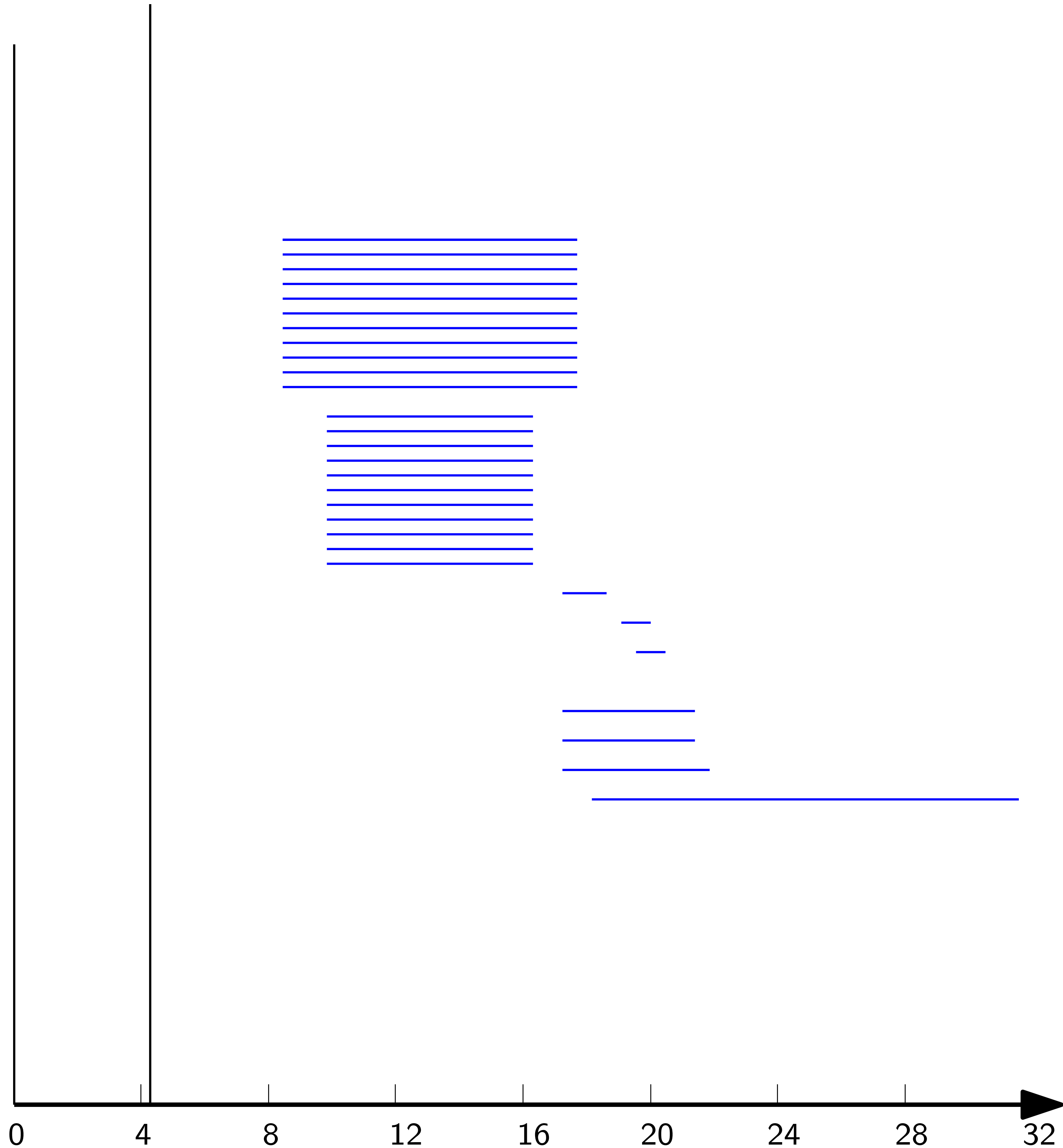

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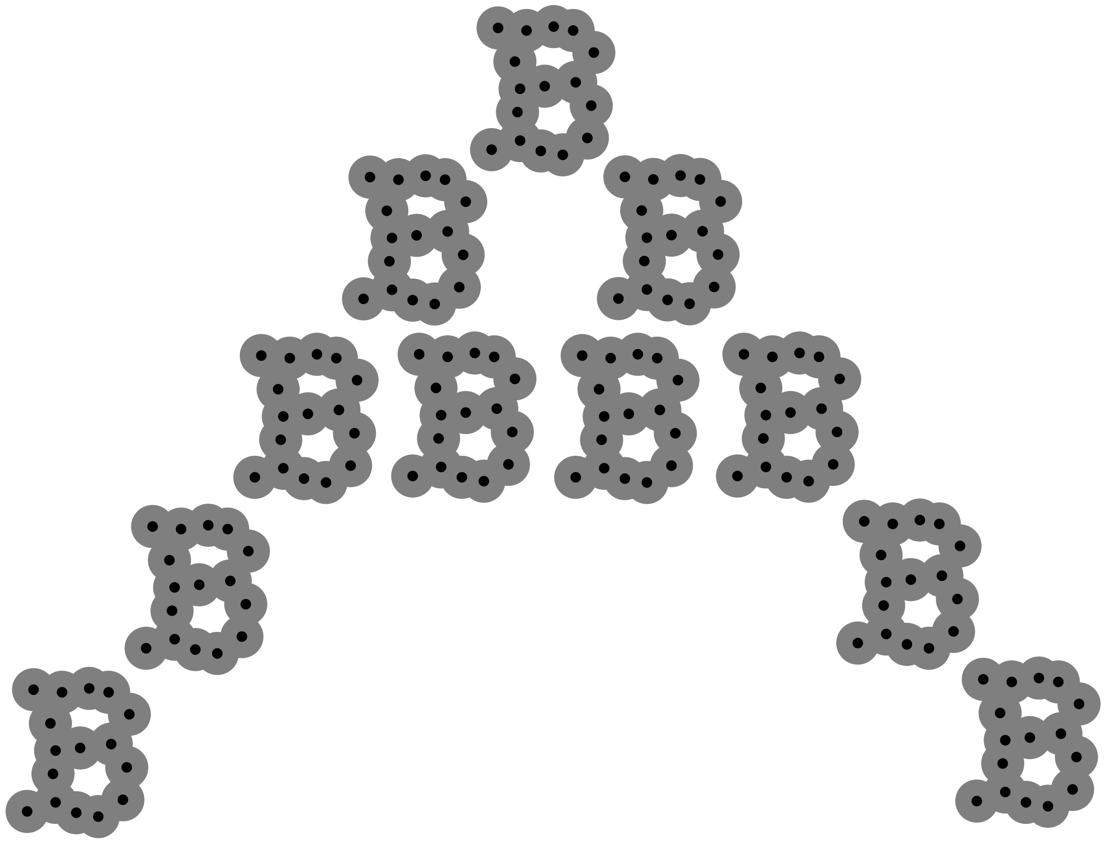

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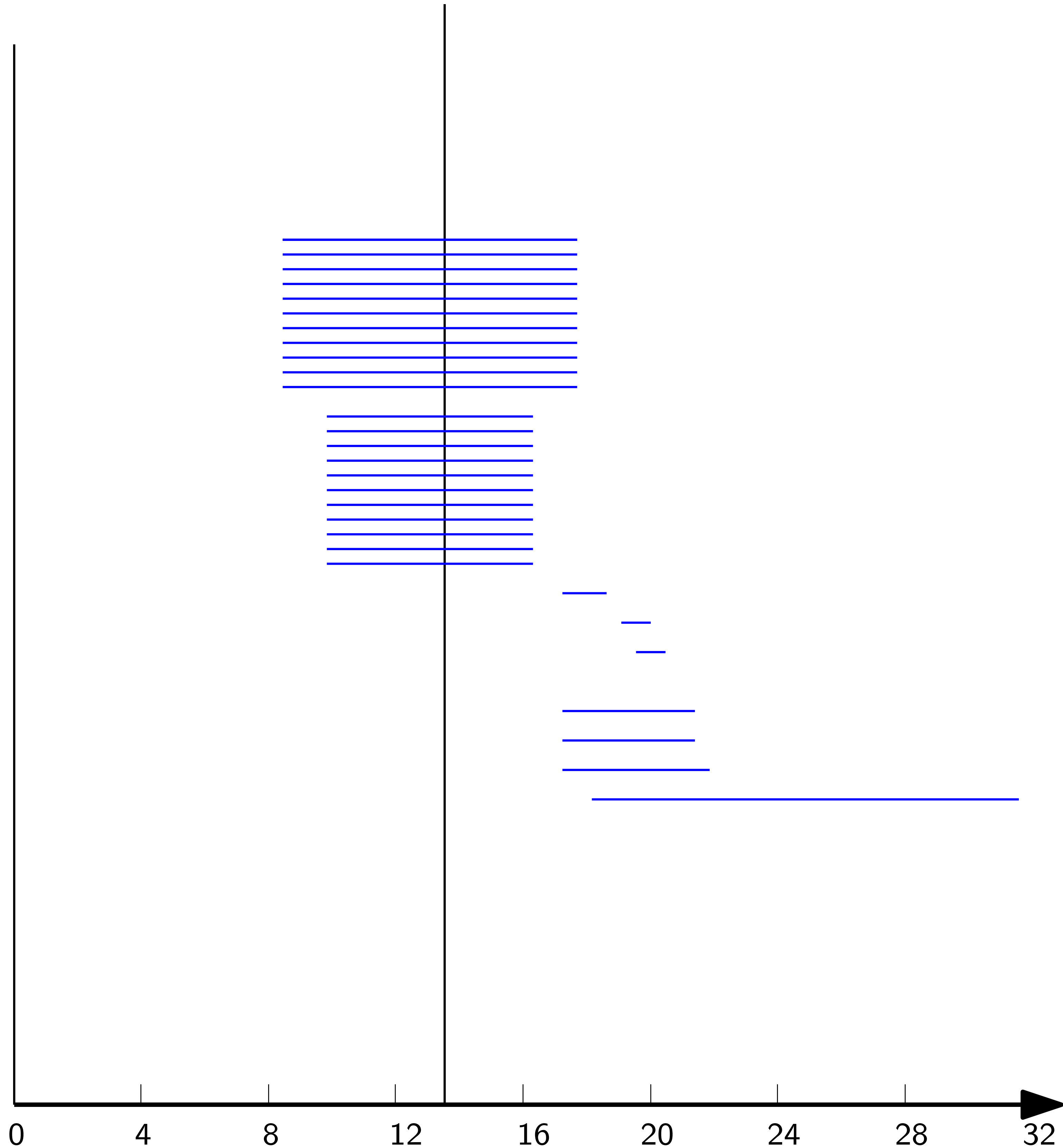

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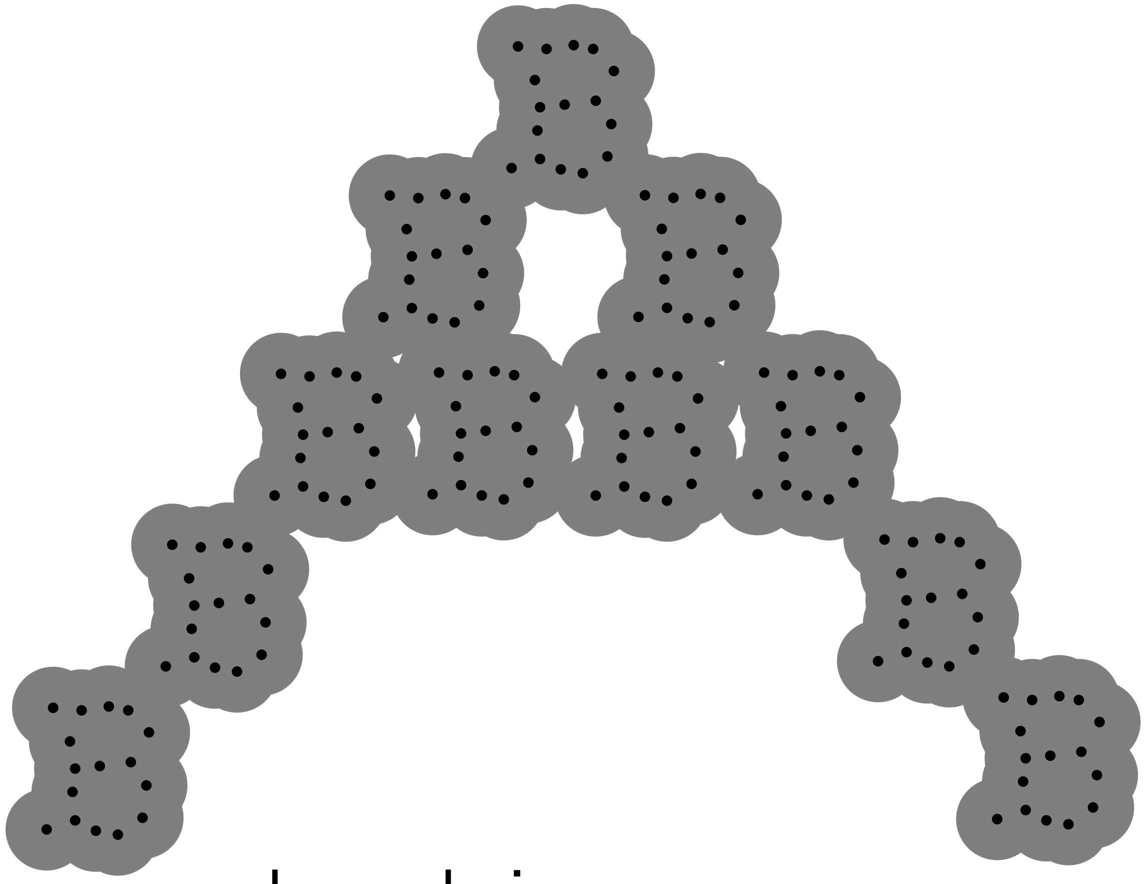

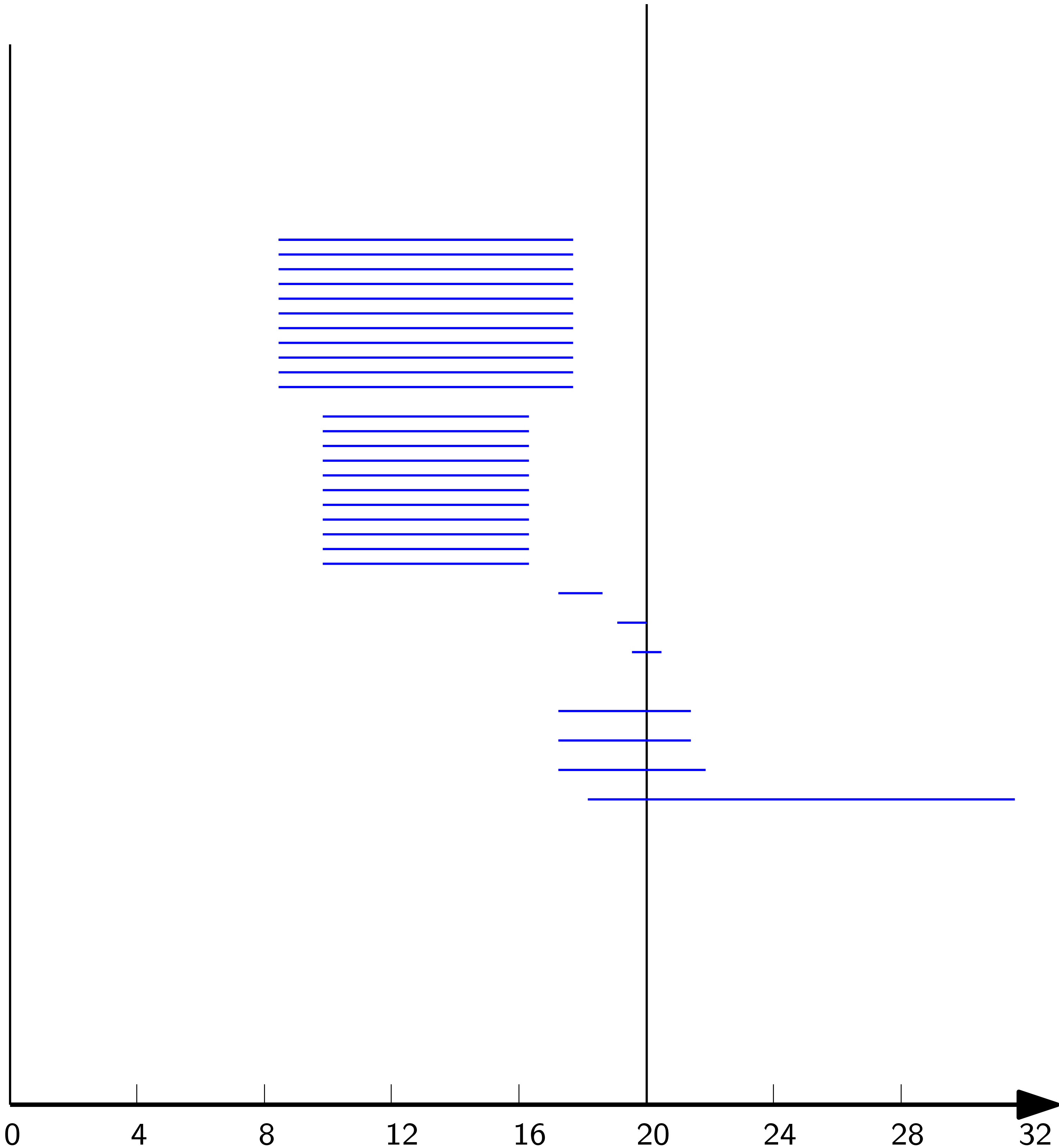

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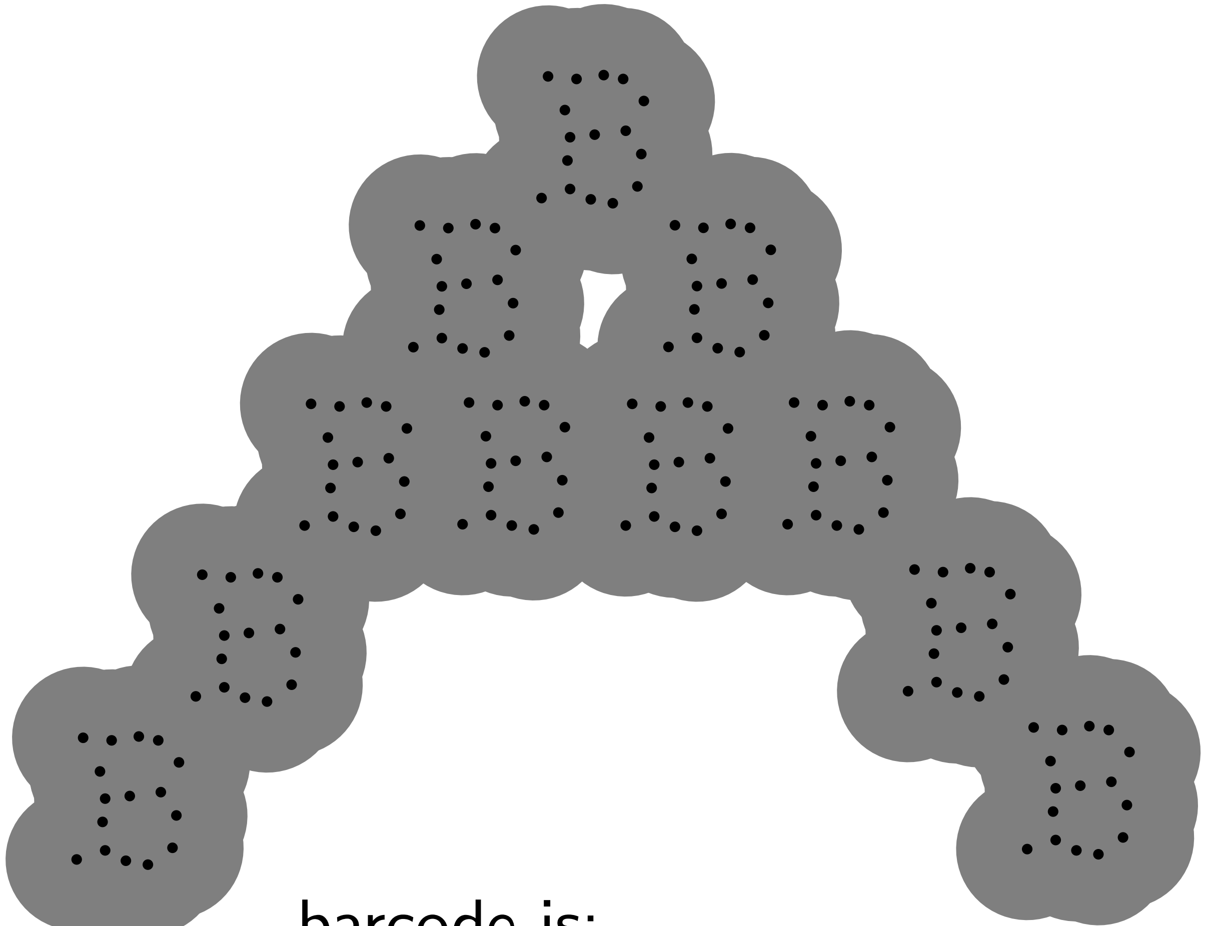

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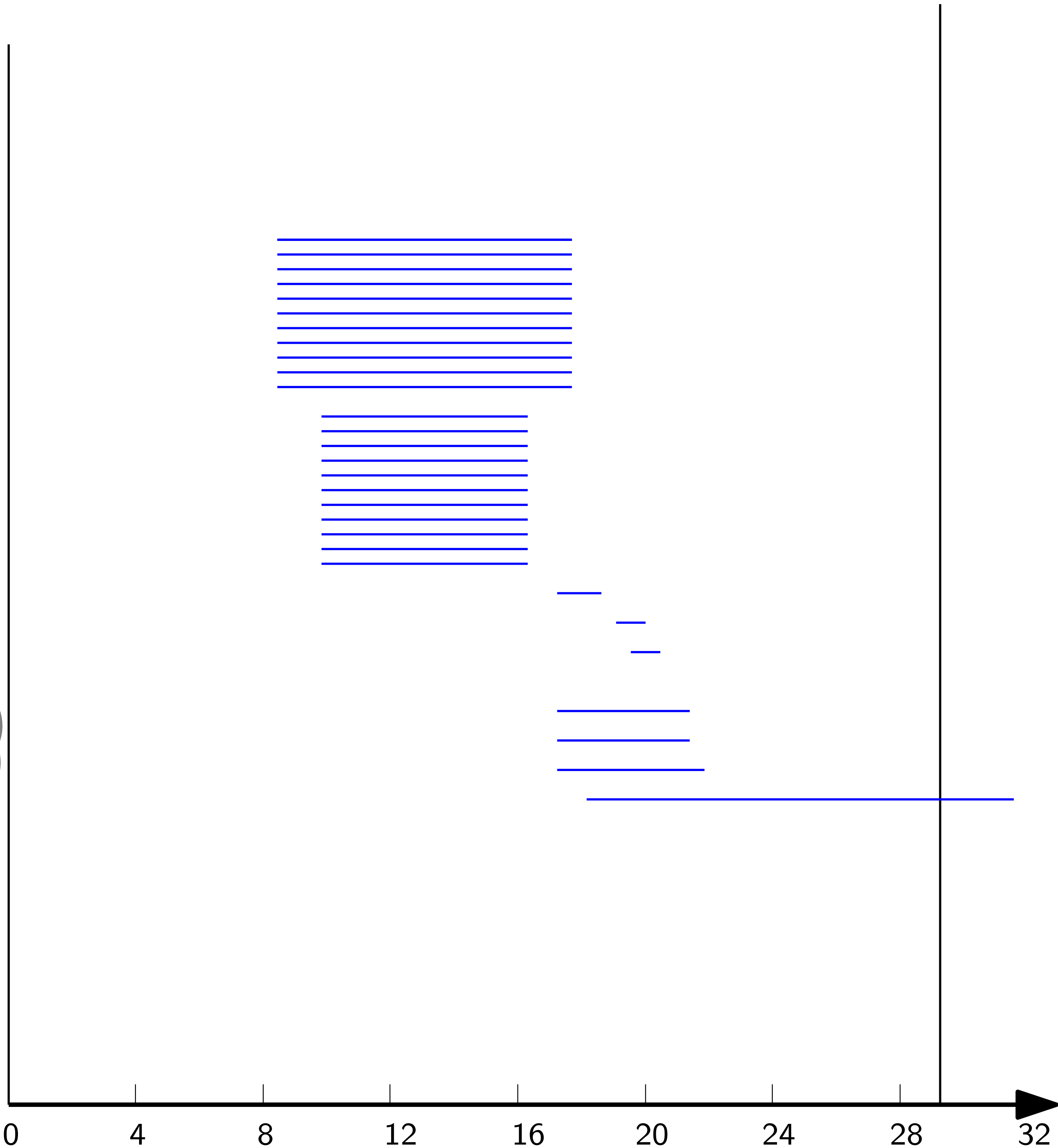

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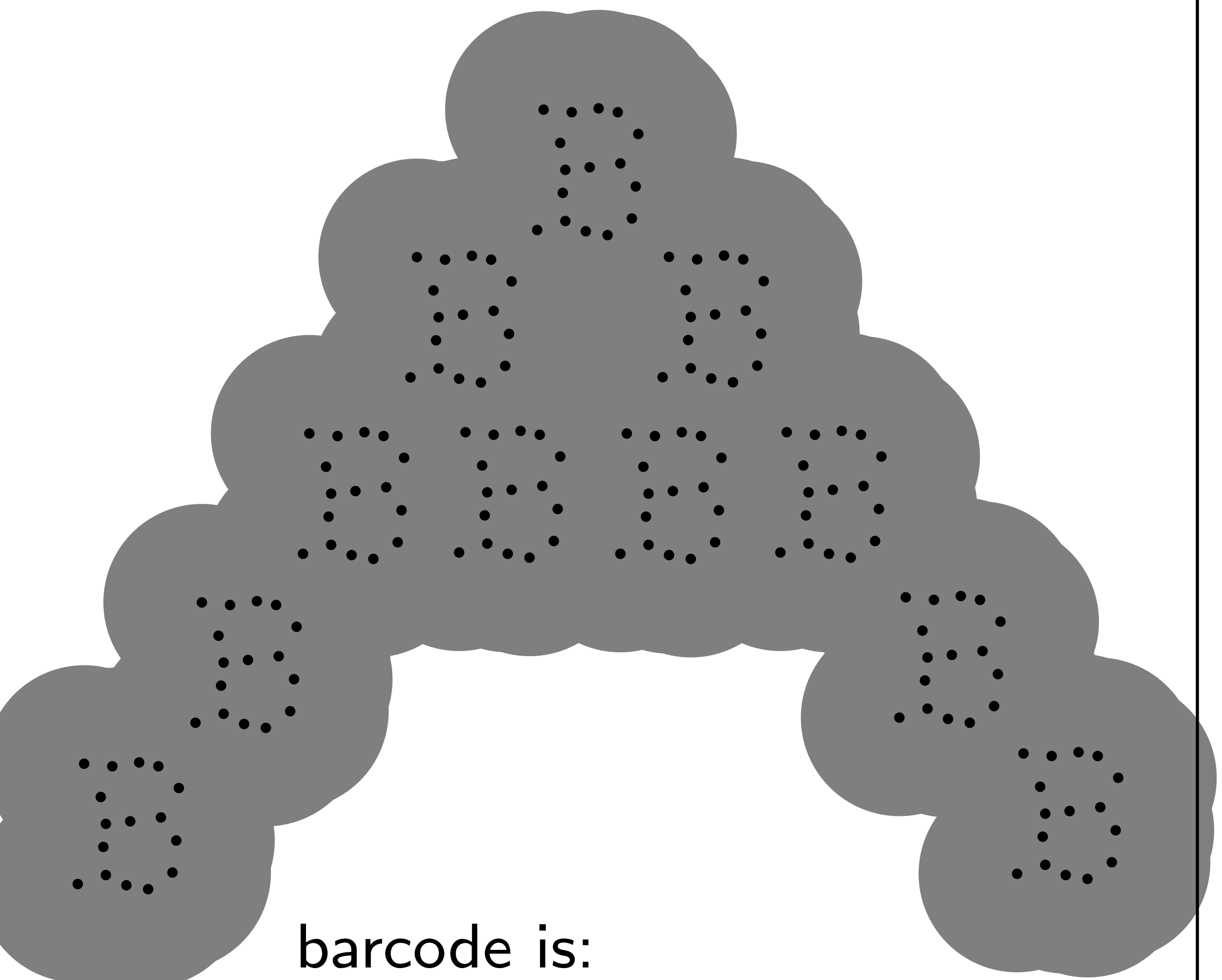

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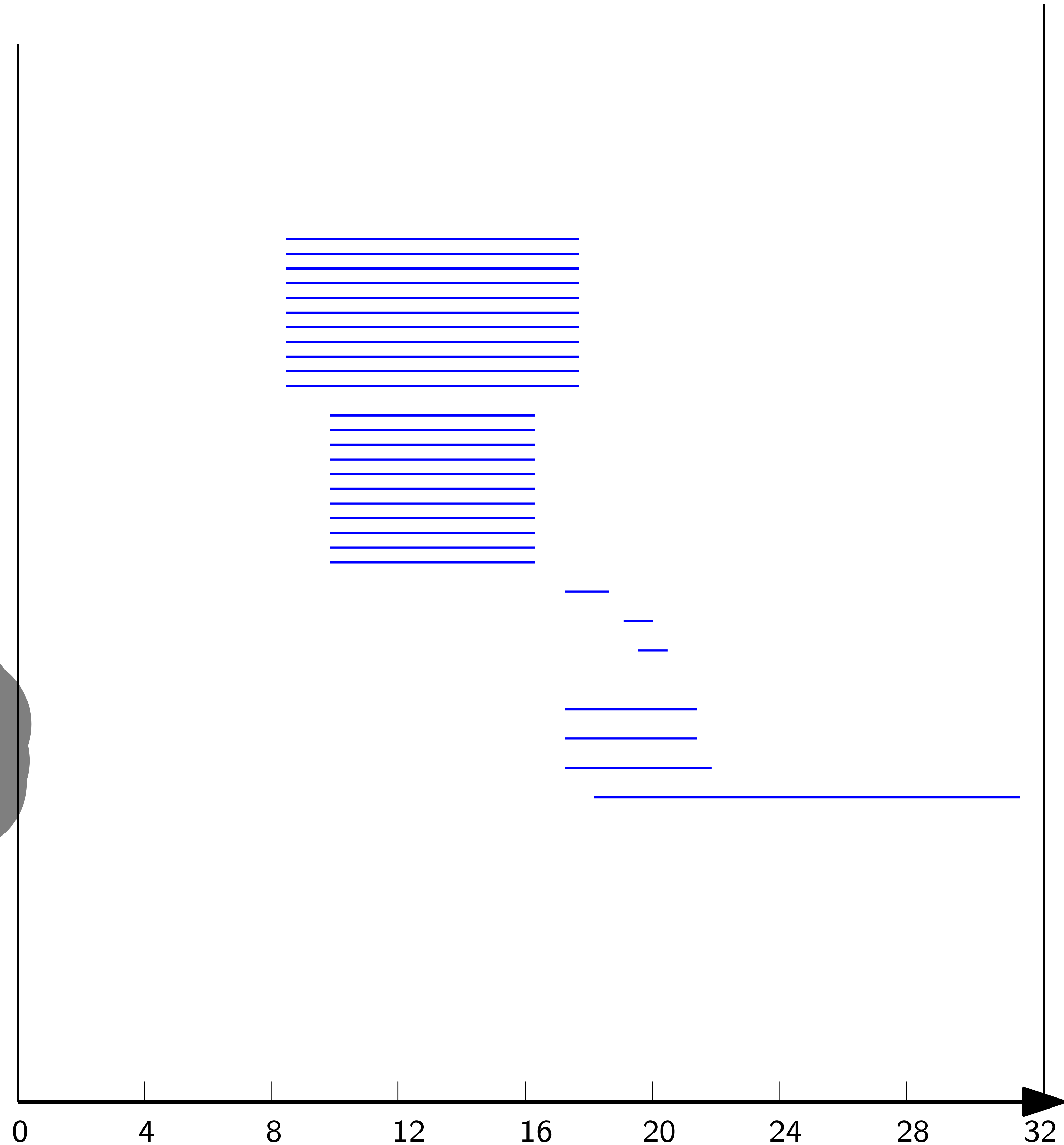

$\exists$ barcodes: decomposition theorems

Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$

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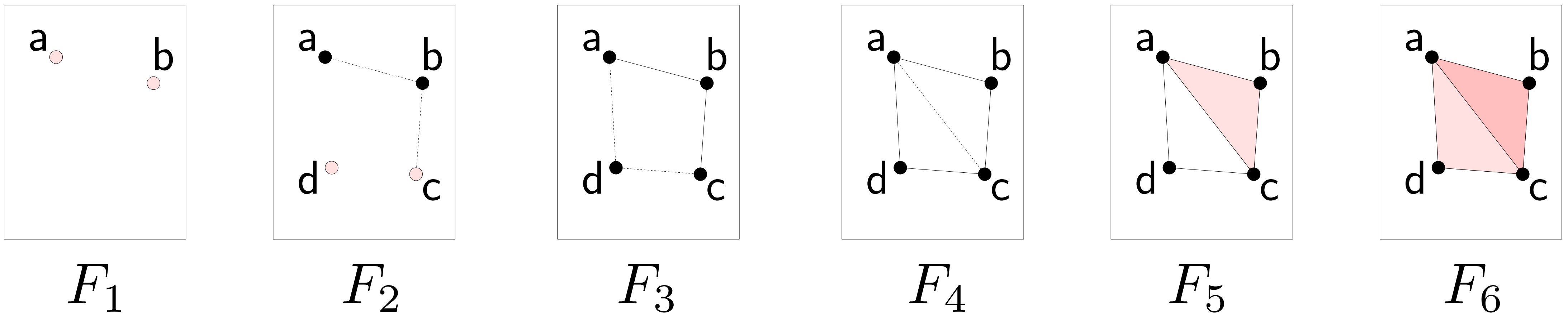

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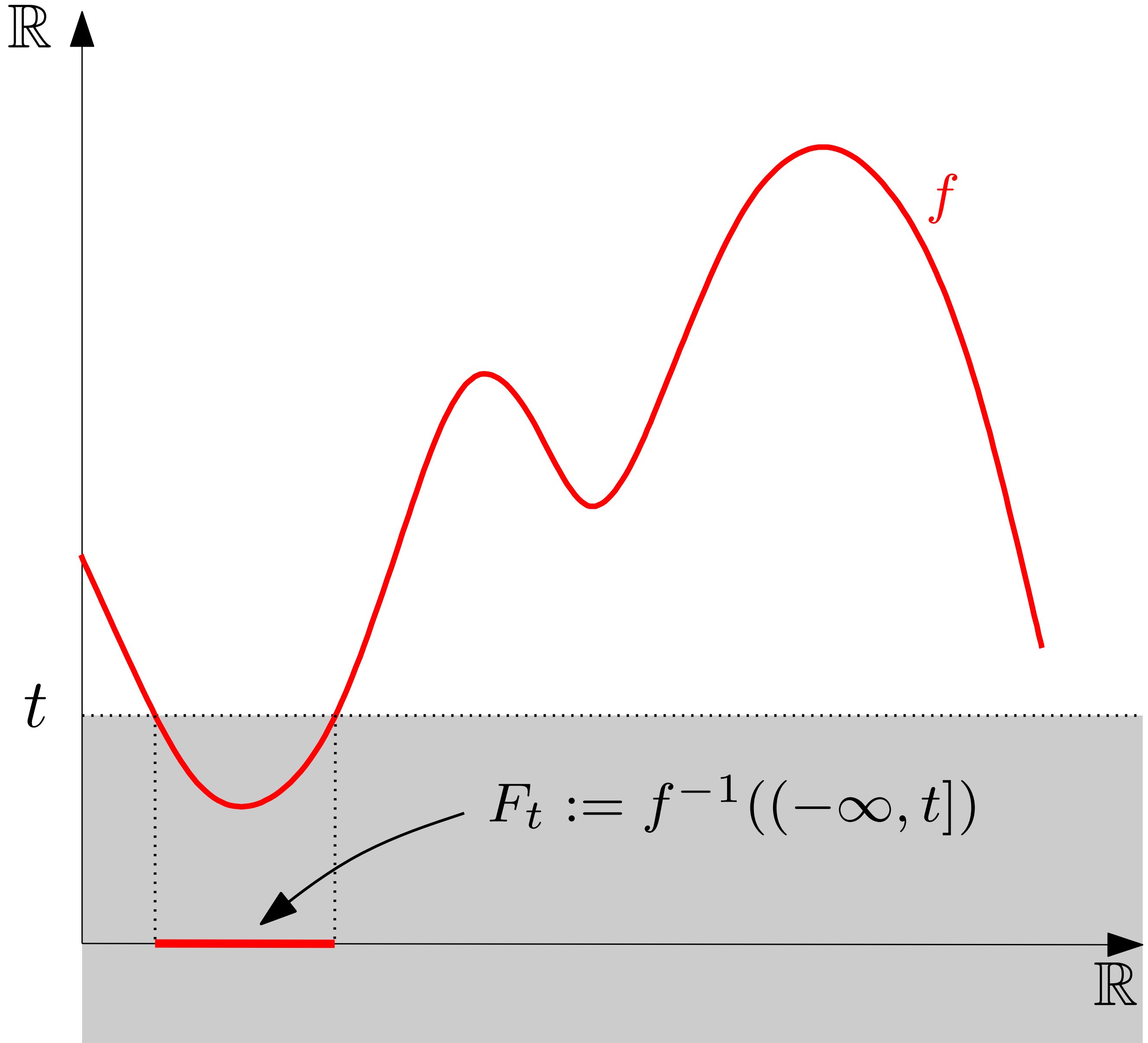

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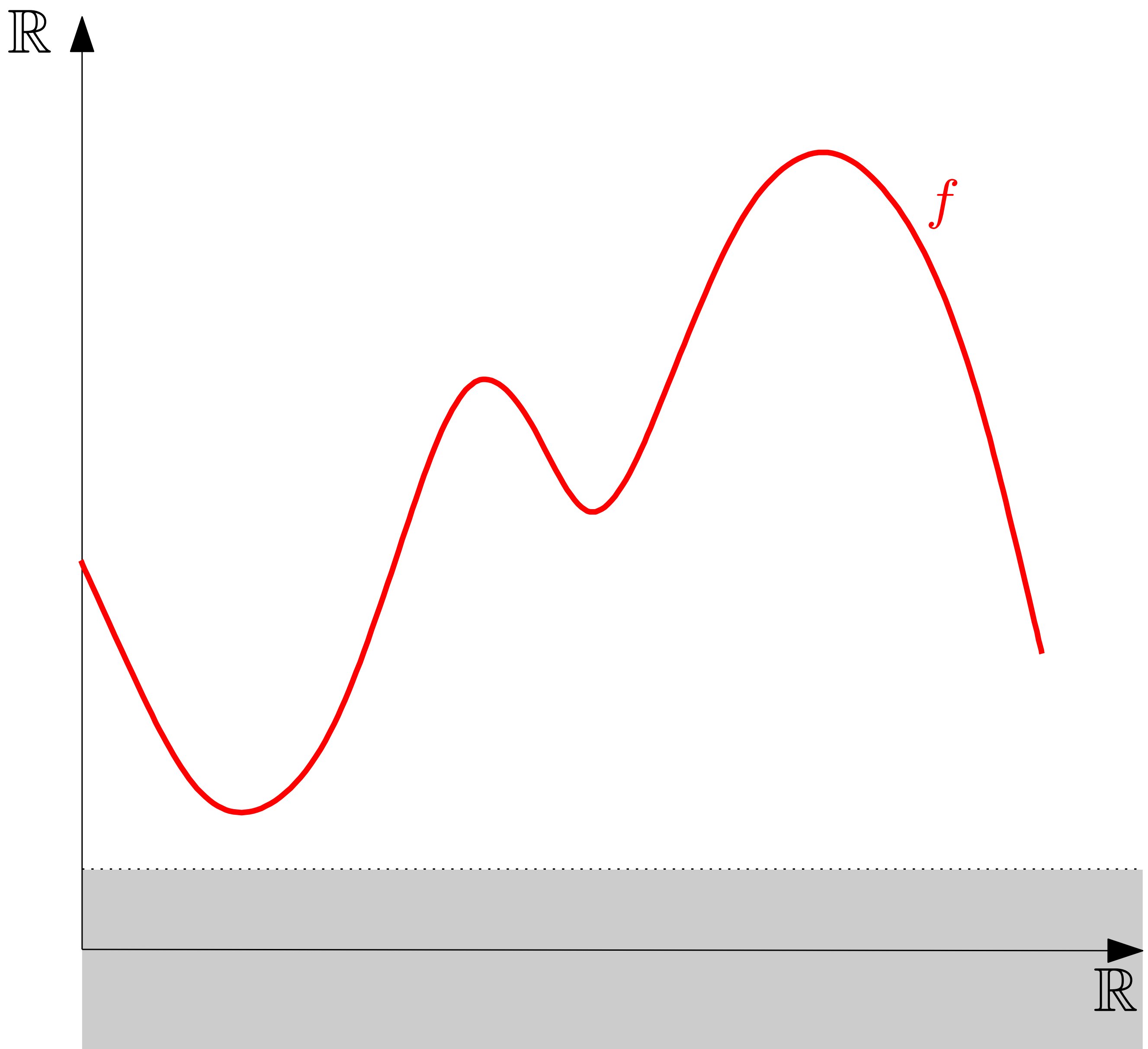

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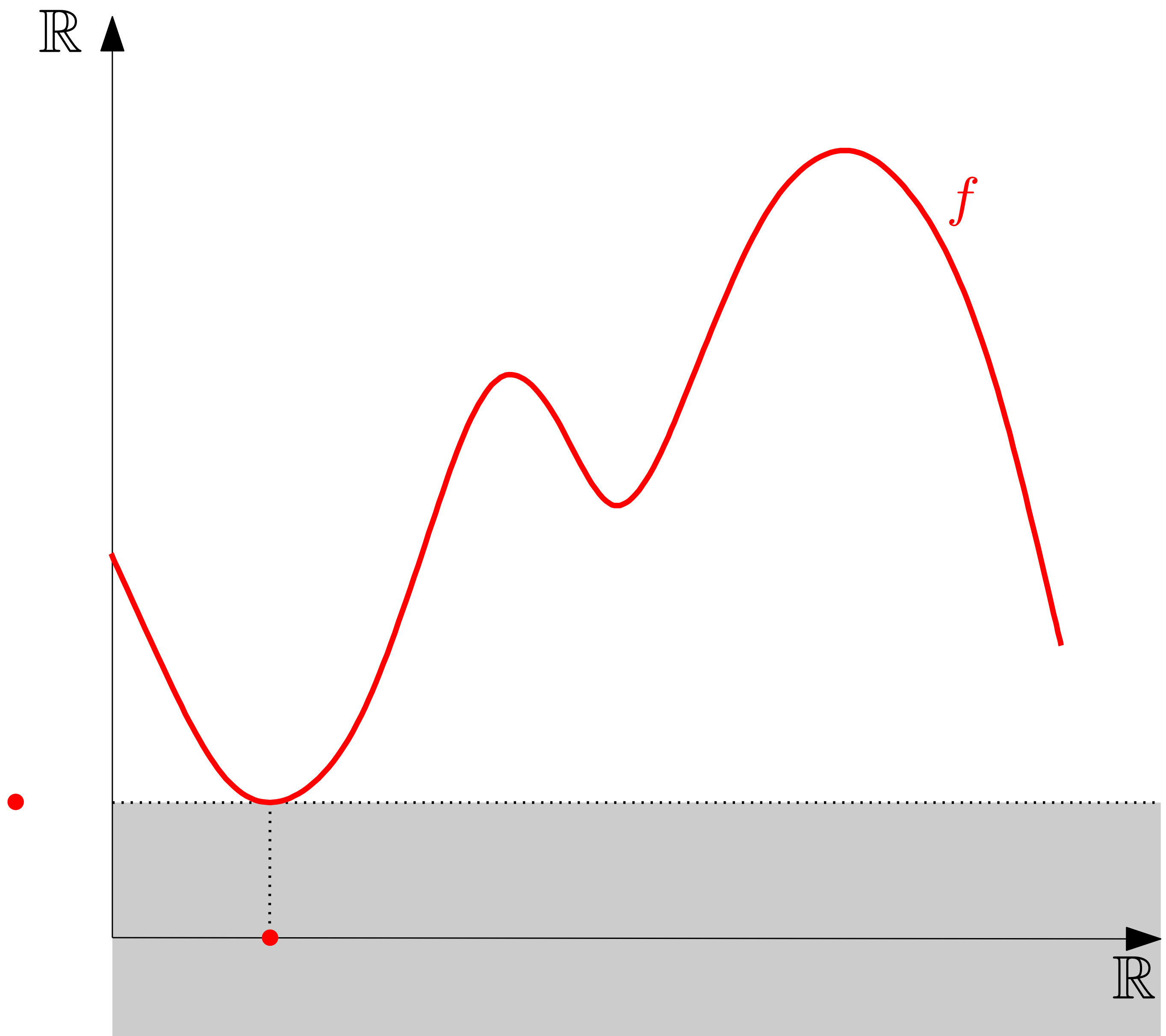

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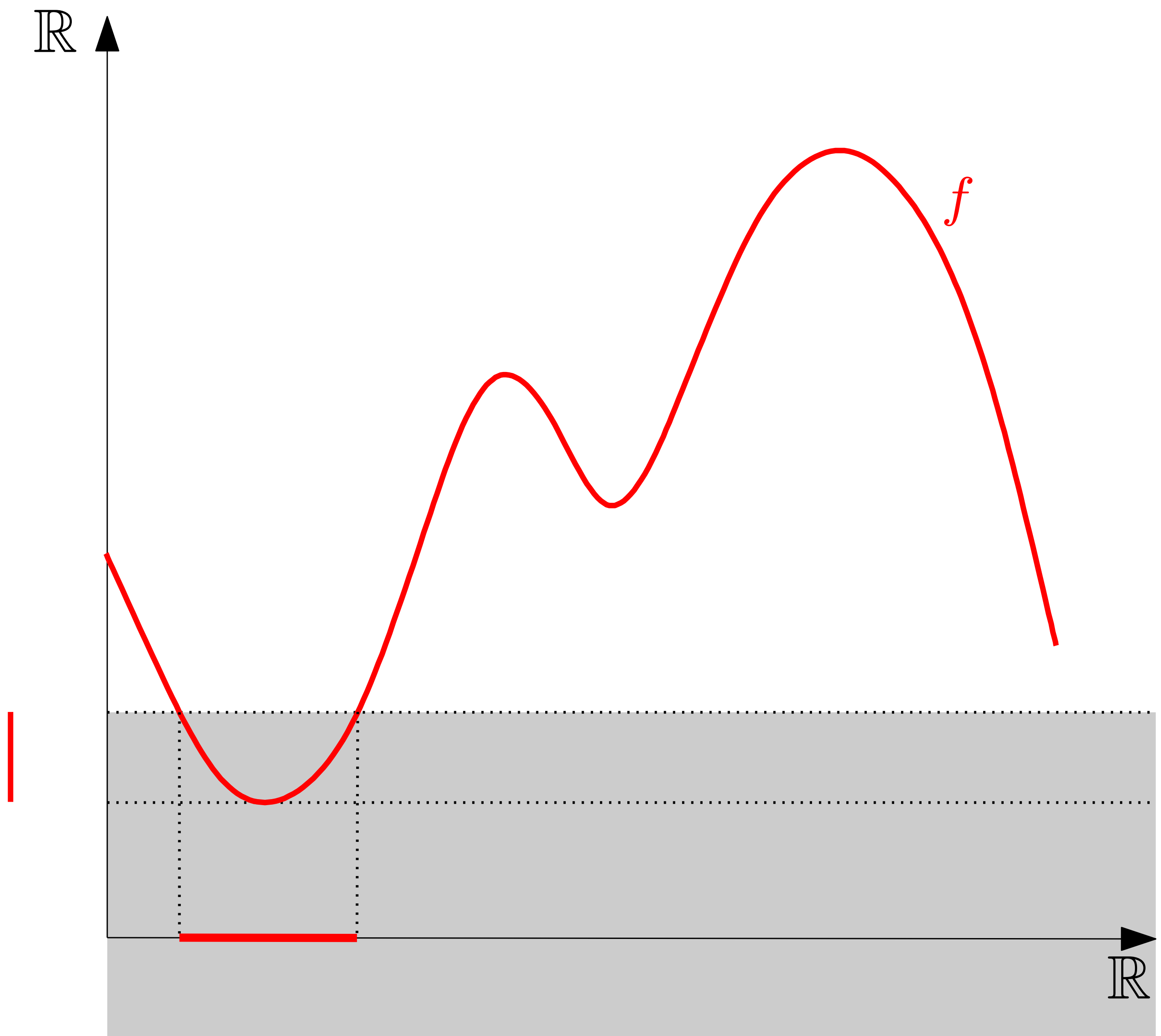

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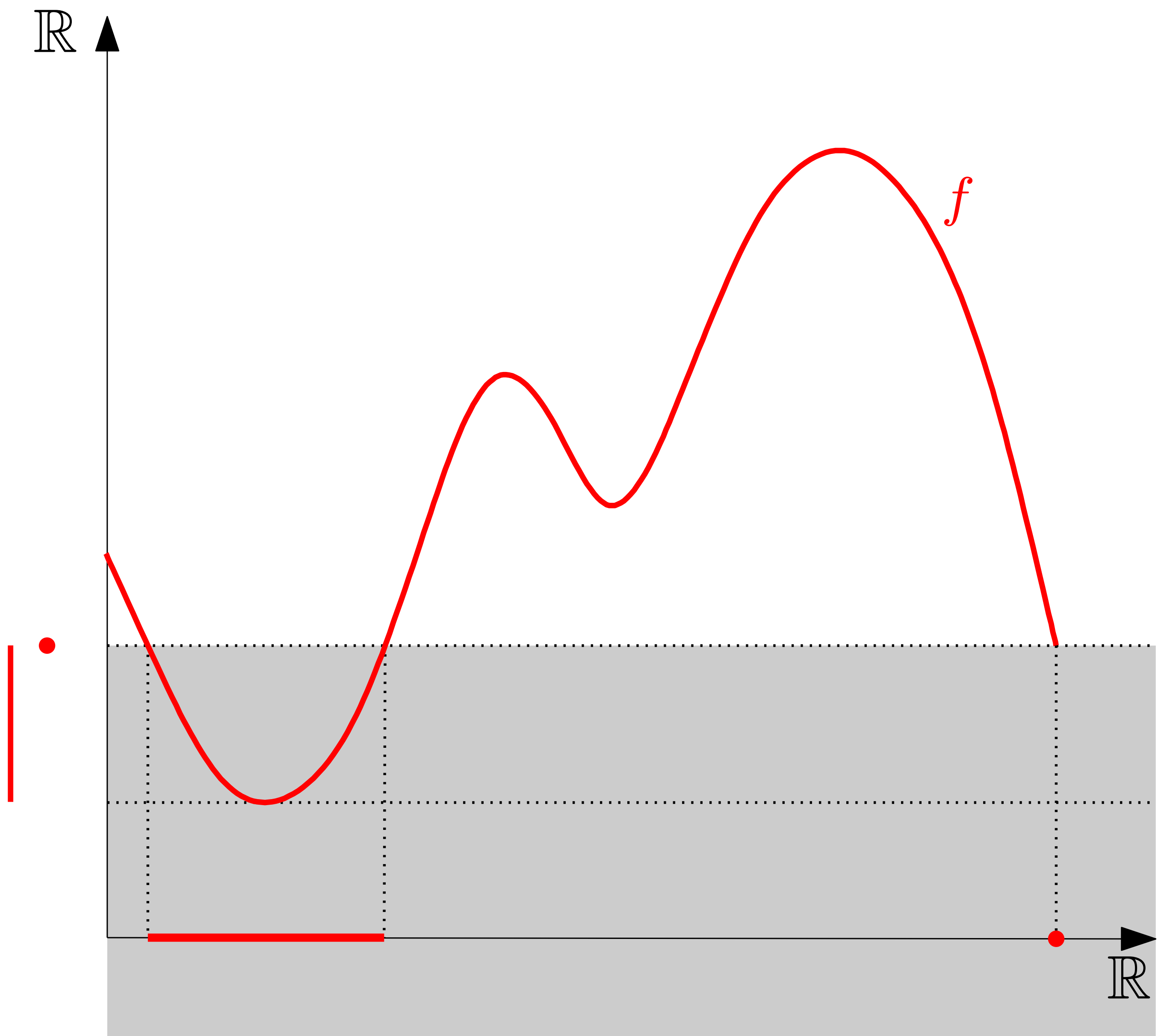

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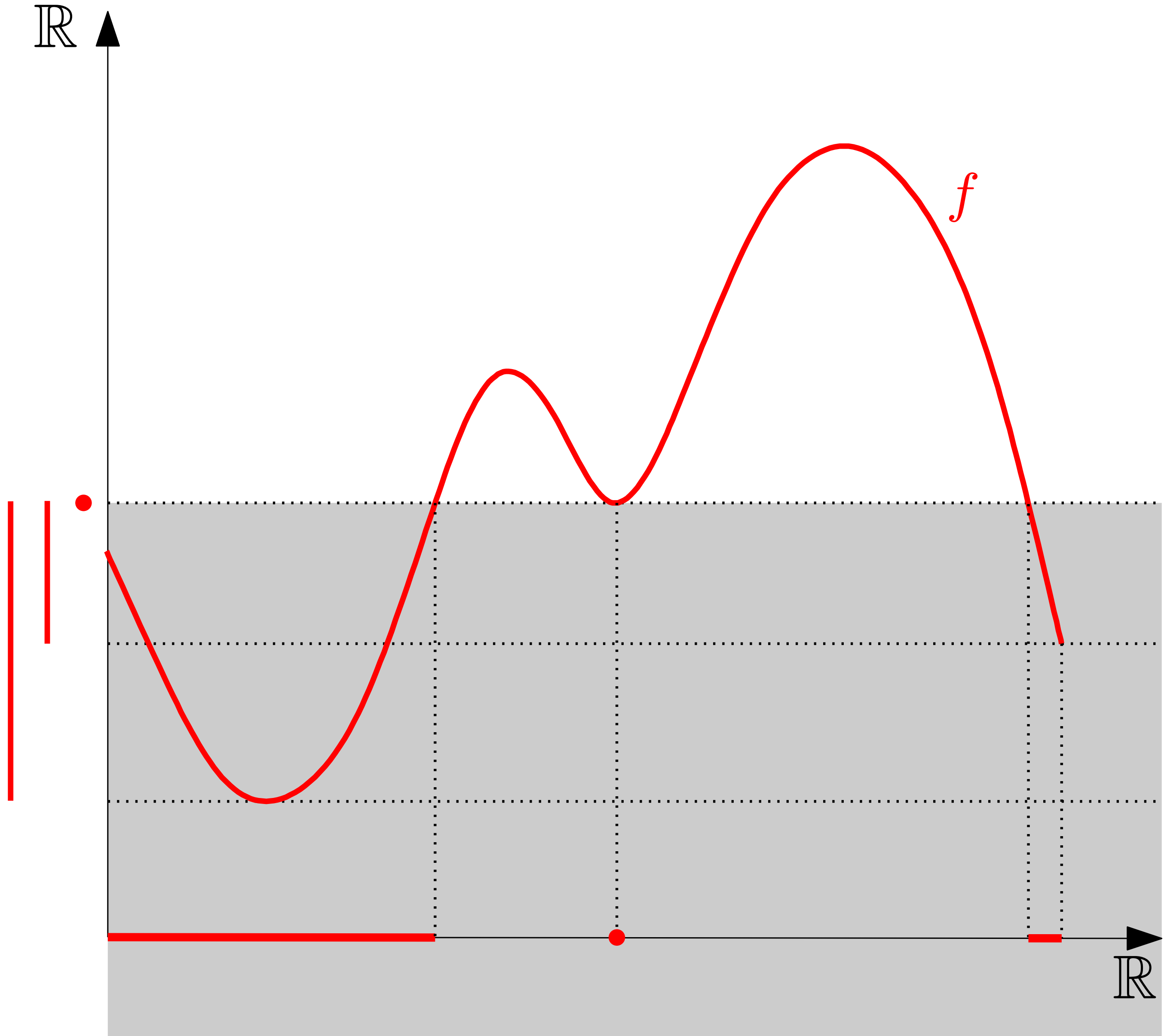

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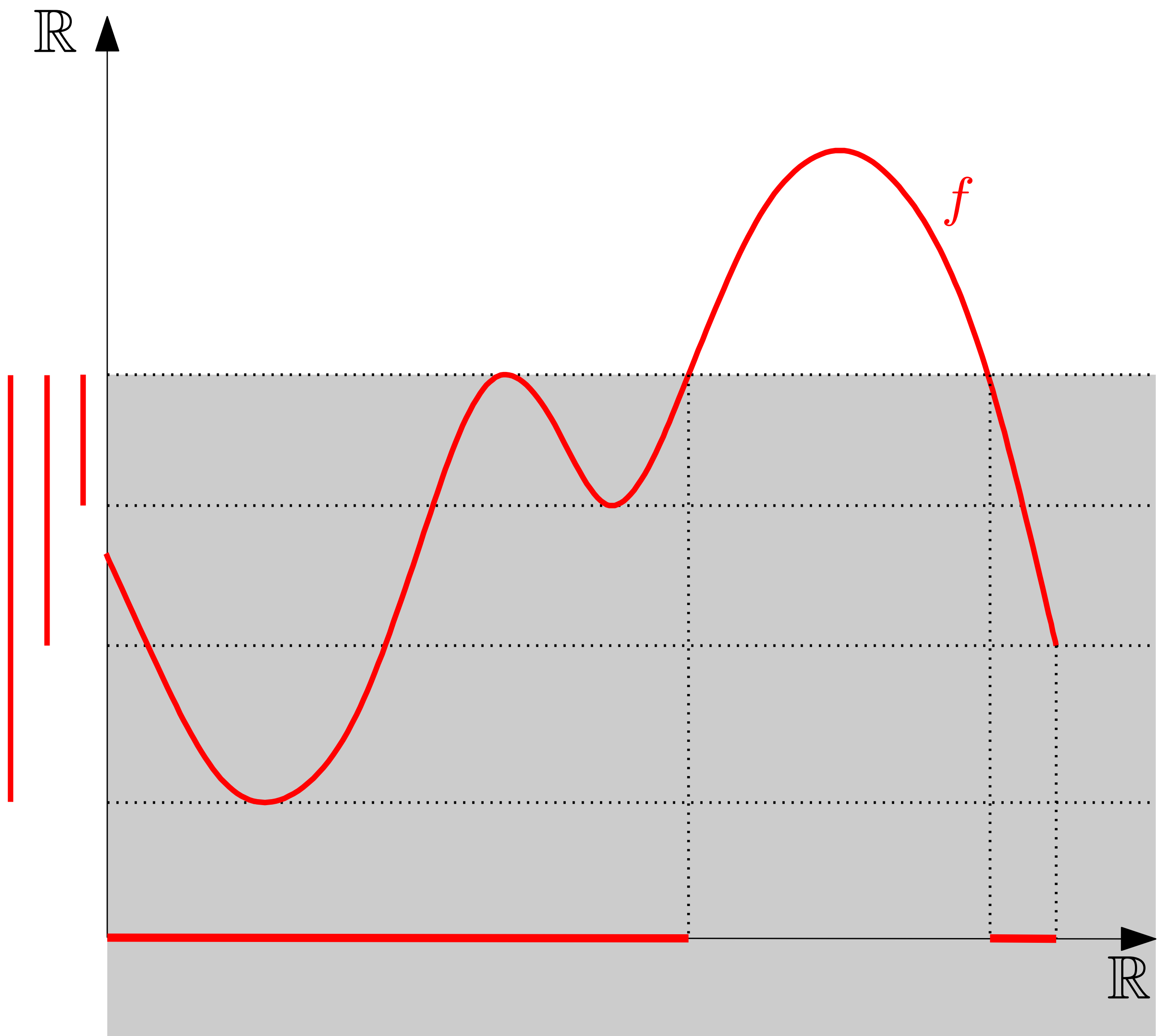

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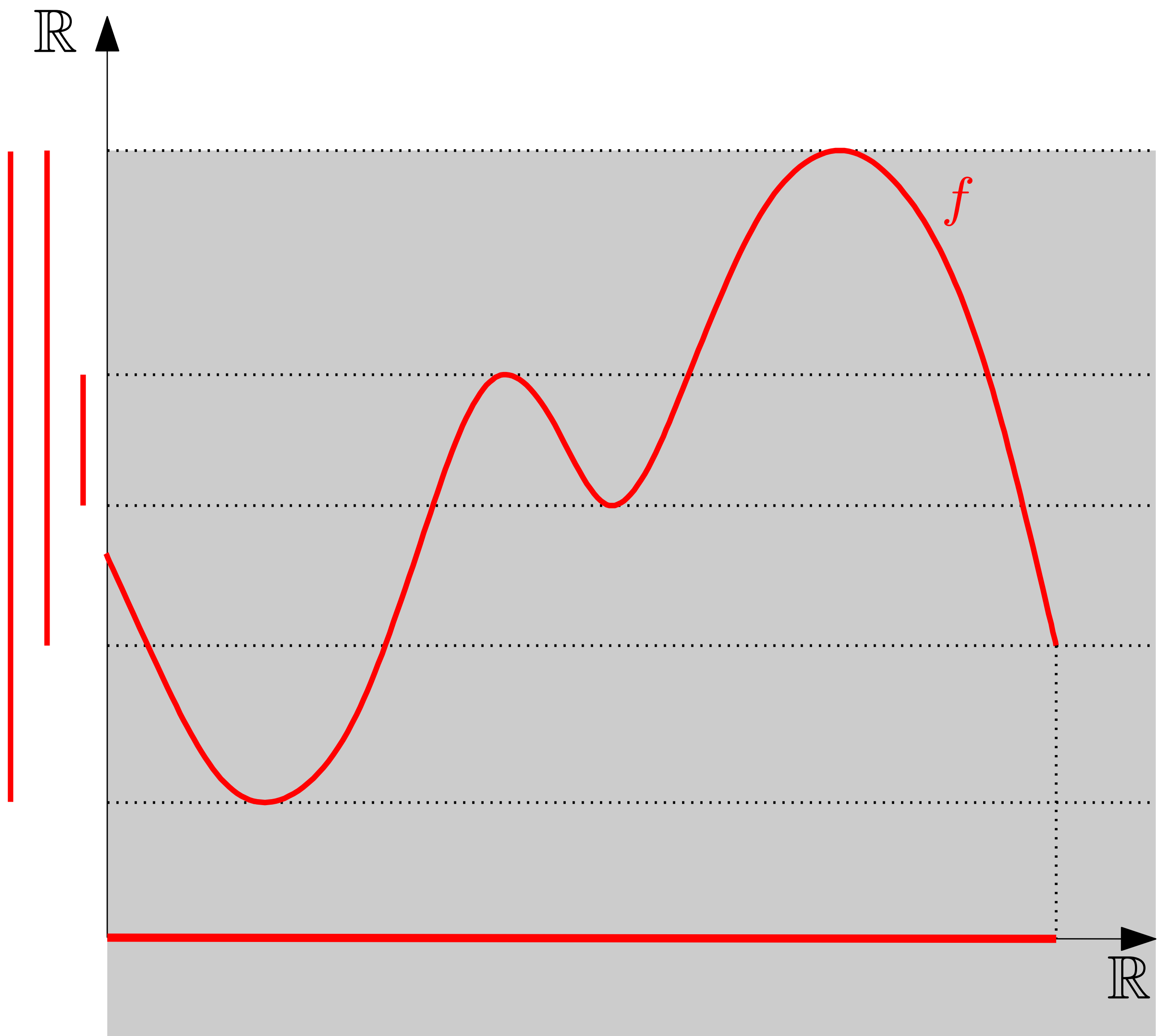

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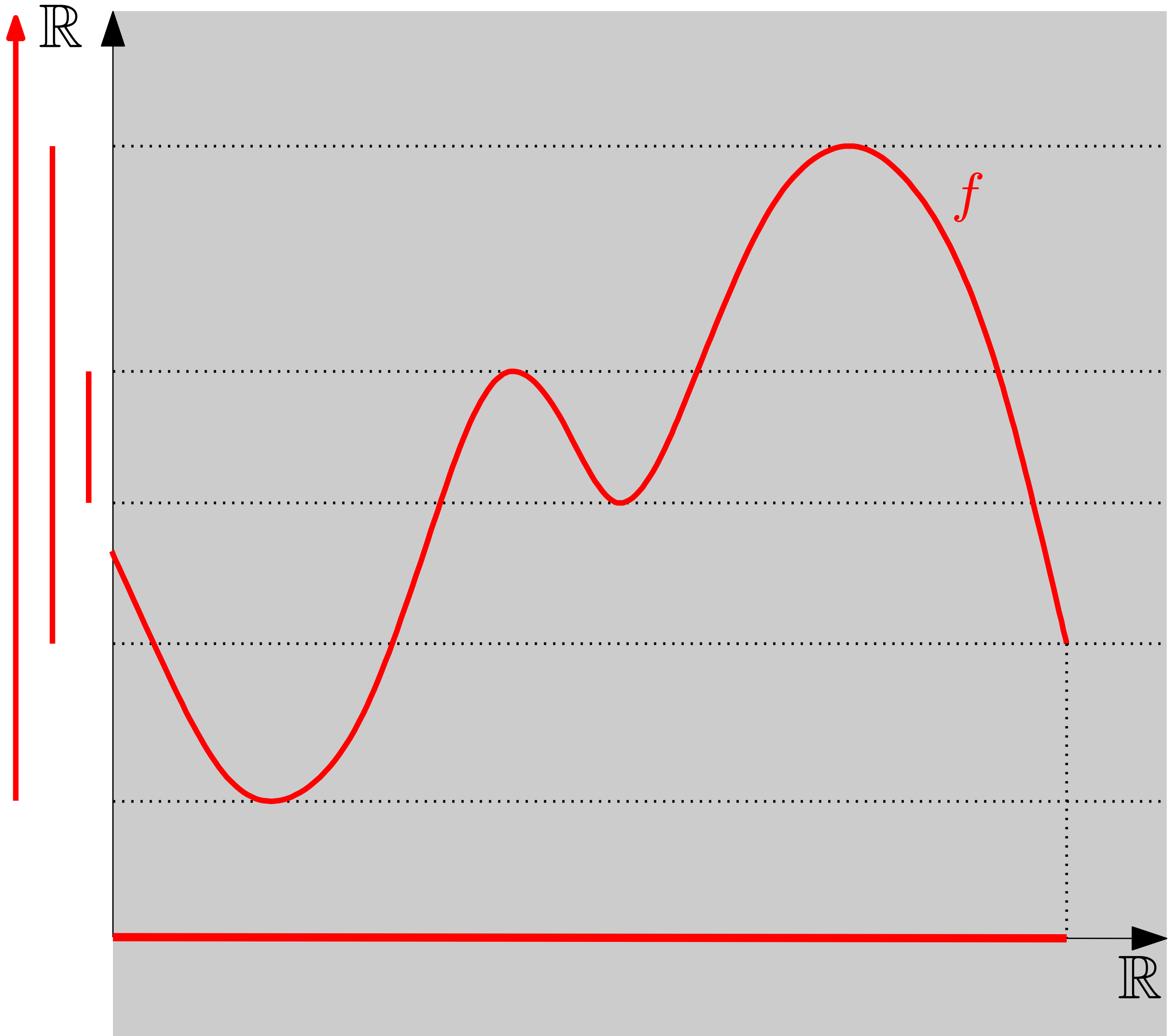

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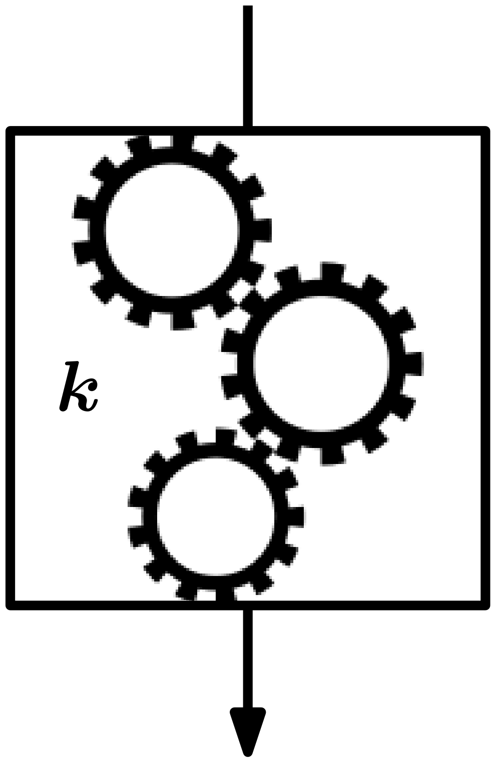

(homology functor)

topological level

algebraic level

Persistence module: $H_*(F_1) \to H_*(F_2) \to H_*(F_3) \to H_*(F_4) \to H_*(F_5) \cdots$

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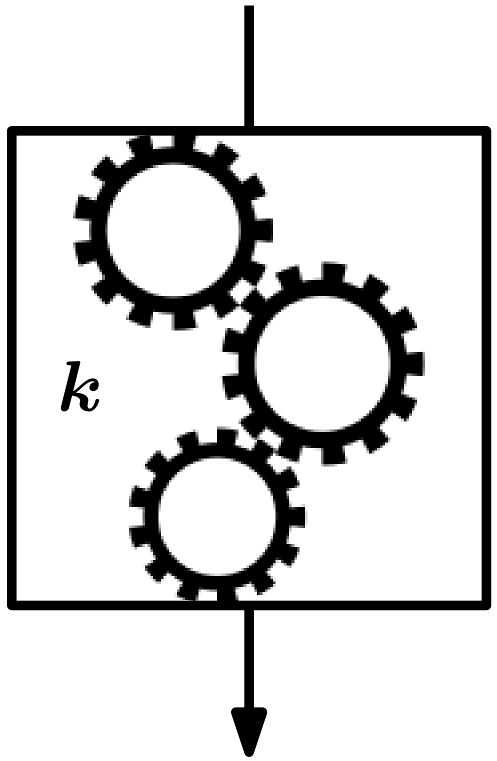

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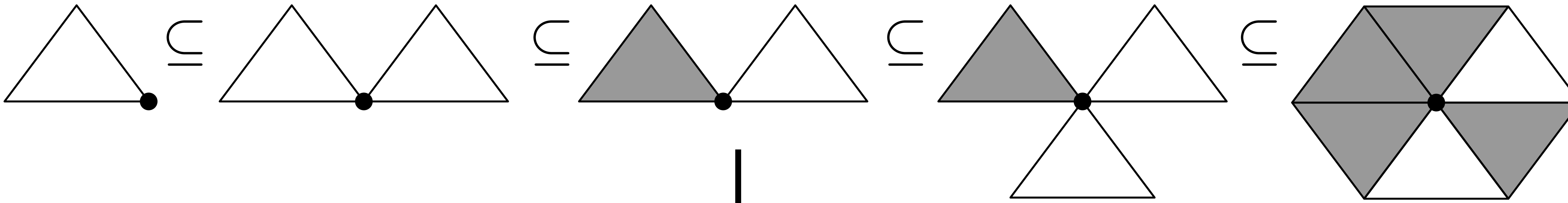
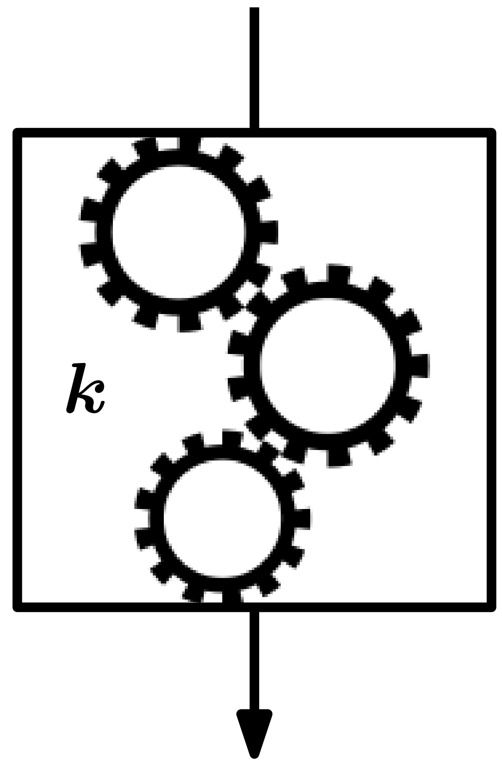
$\rightarrow$ algebraic structure of module is described by a barcode

∃ barcodes: decomposition theorems

Example:





(1-homology functor)

$$\boldsymbol{k} \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} \boldsymbol{k} \xrightarrow{\begin{pmatrix}0\\1\end{pmatrix}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}} \boldsymbol{k}^2 \quad \cdots$$

∃ barcodes: decomposition theorems

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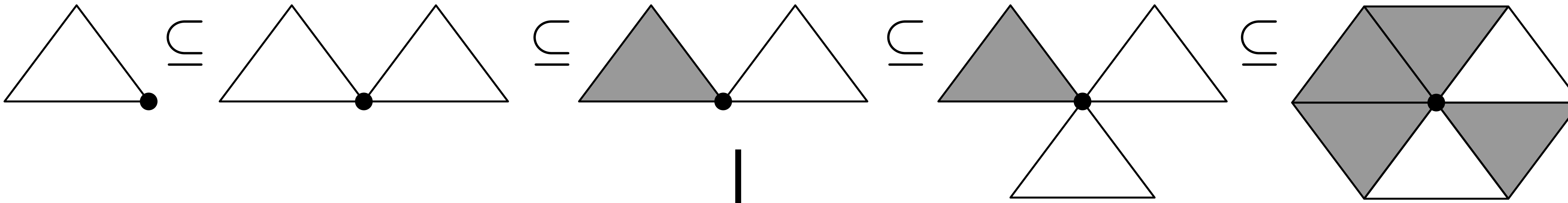

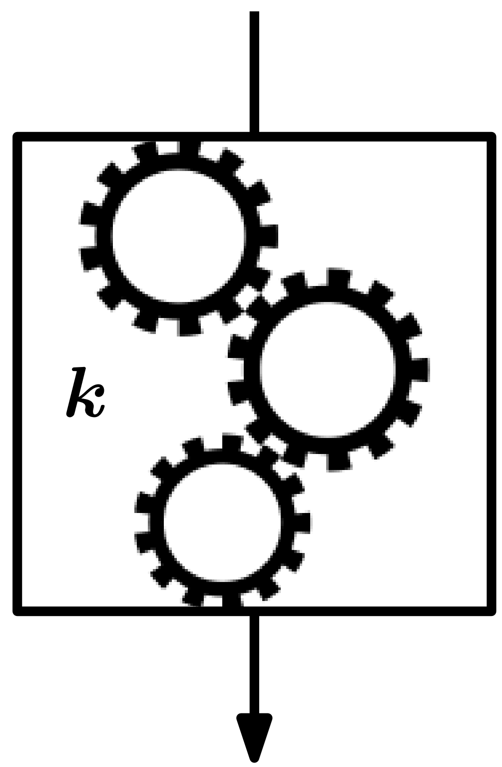

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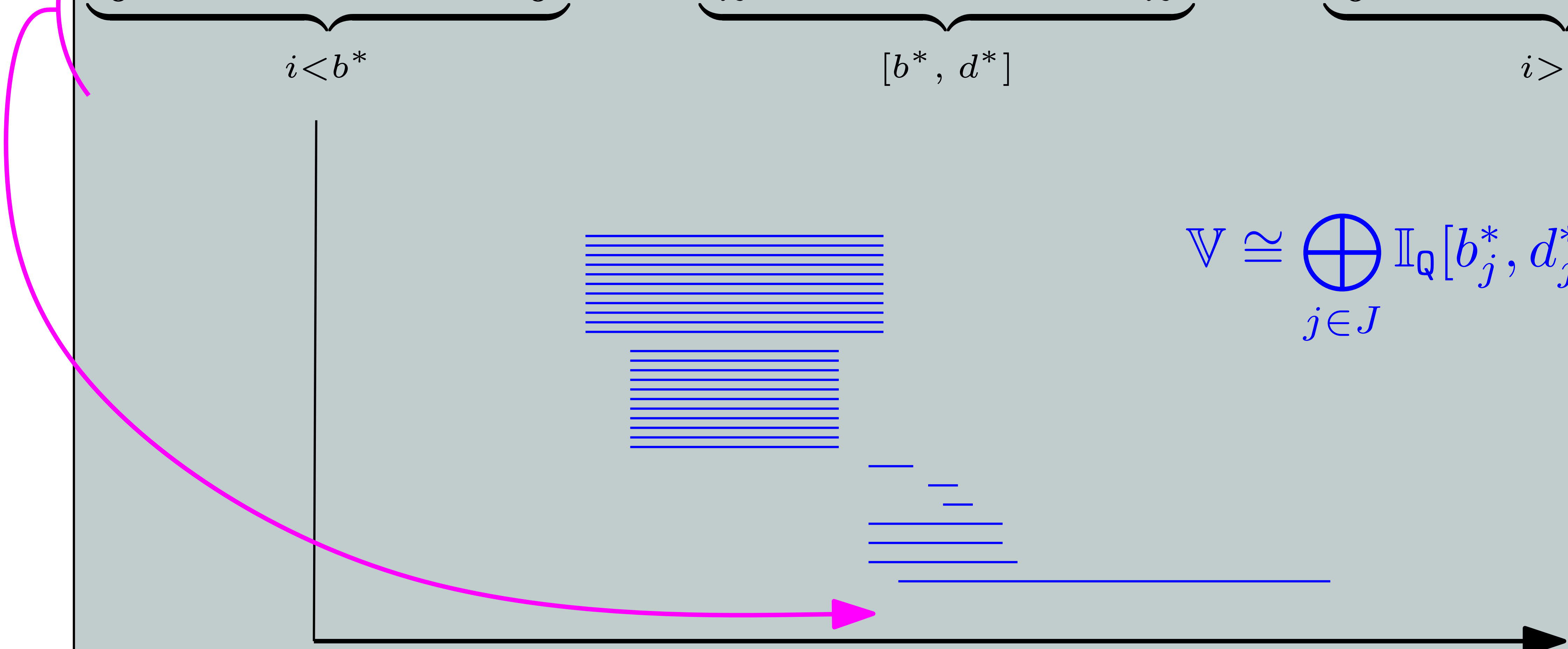
$\exists$ barcodes: decomposition theorems

Theorem. Let $\mathbb{V}$ be a persistence module over some index set $T \subseteq \mathbb{R}$. Then, $\mathbb{V}$ decomposes as a direct sum of **interval modules** $\mathbb{I}_{\mathtt{Q}}[b^*, d^*]$:

$$\underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0} 0}_{i<b^*} \xrightarrow{0} \underbrace{\boldsymbol{k} \xrightarrow{\mathbb{1}} \cdots \xrightarrow{\mathbb{1}} \boldsymbol{k}}_{[b^*,\, d^*]} \xrightarrow{0} \underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0} 0}_{i>d^*}$$



$$\mathbb{V} \simeq \bigoplus_{j \in J} \mathbb{I}_{\mathtt{Q}}[b_j^*, d_j^*]$$

(the barcode is a complete descriptor of the algebraic structure of $\mathbb{V}$)

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in the following cases:

- T is finite [Gabriel 1972] [Auslander 1974],
- $\mathbb{V}$ is *pointwise finite-dimensional* (every space V_t has finite dimension) [Webb 1985] [Crawley-Boevey 2012].

Moreover, when it exists, the decomposition is **unique** up to isomorphism and permutation of the terms [Azumaya 1950].

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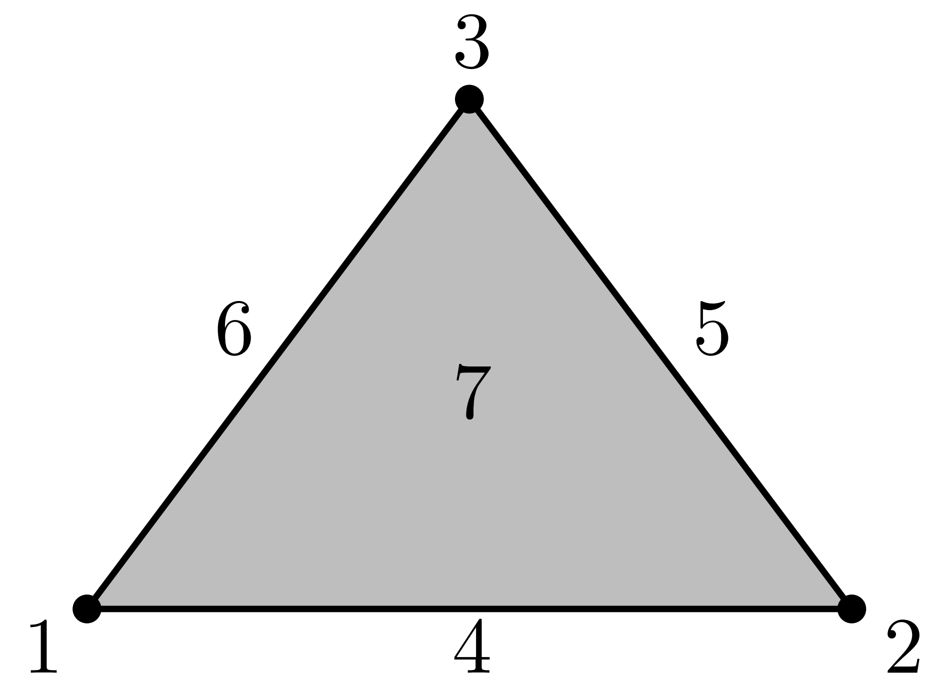
simple case: $\mathbb{V}$ is finitely generated (finite T and finite-dim. V_t)

(the barcode is a complete descriptor of the algebraic structure of $\mathbb{V}$)

Computation of barcodes: matrix reduction

[Edelsbrunner, Letscher, Zomorodian 2002] [Carlsson, Zomorodian 2005] . . .

Input: simplicial filtration



Computation of barcodes: matrix reduction

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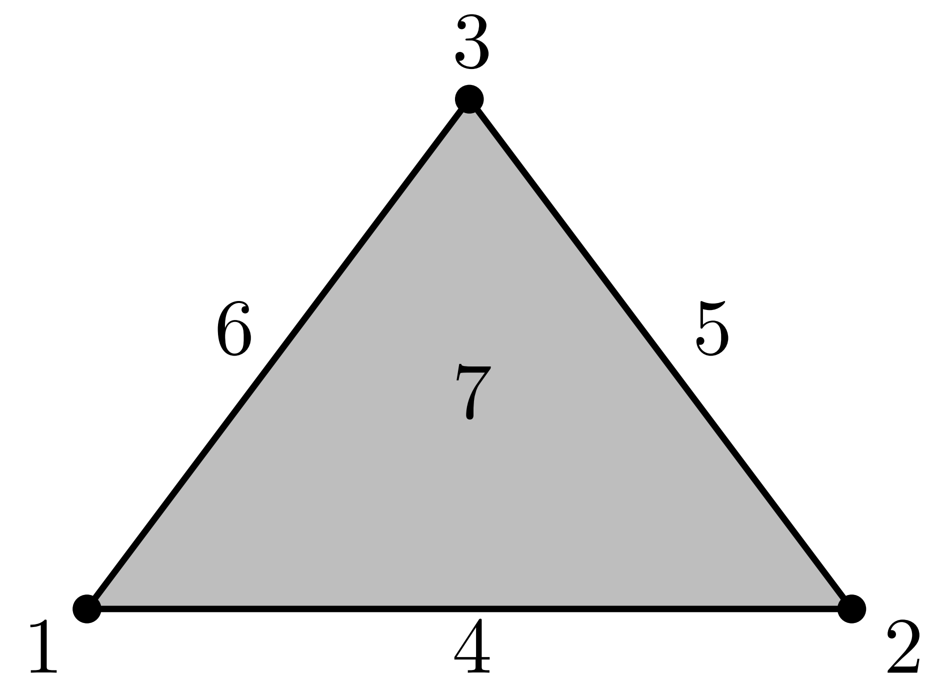
Input: simplicial filtration

Output: boundary matrix



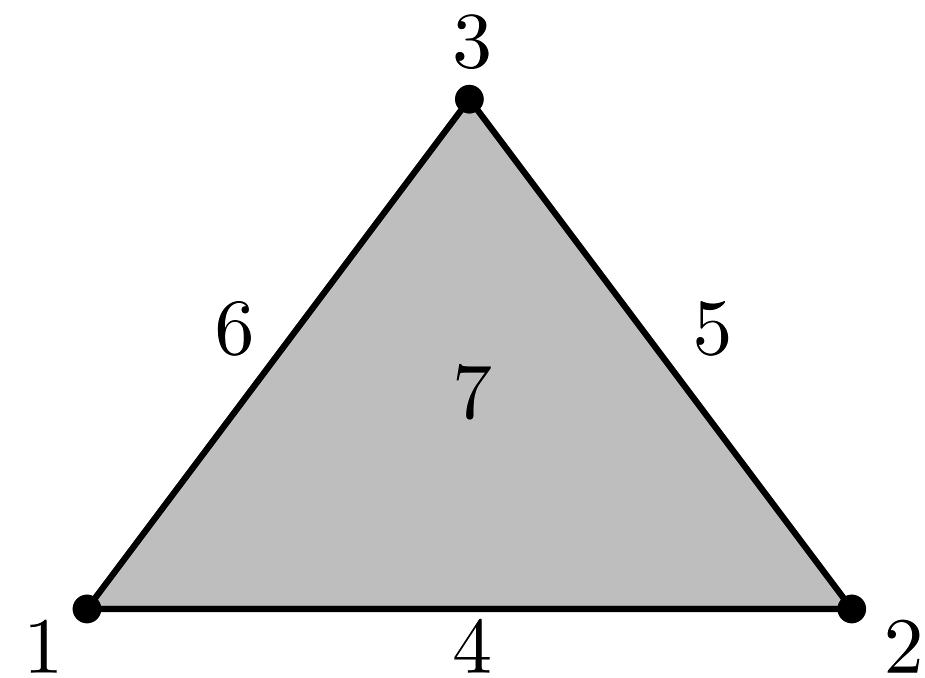
	1	2	3	4	5	6	7
1				*		*	
2				*	*		
3					*	*	
4							*
5							*
6							*
7							

Computation of barcodes: matrix reduction

[Edelsbrunner, Letscher, Zomorodian 2002] [Carlsson, Zomorodian 2005] . . .

Input: simplicial filtration

Output: boundary matrix
reduced to column-echelon form



	1	2	3	4	5	6	7
1				*		*	
2				*	*		
3					*	*	
4							*
5							*
6							*
7							

	1	2	3	4	5	6	7
1				*			
2				1	*		
3					1		
4							*
5							*
6							1
7							

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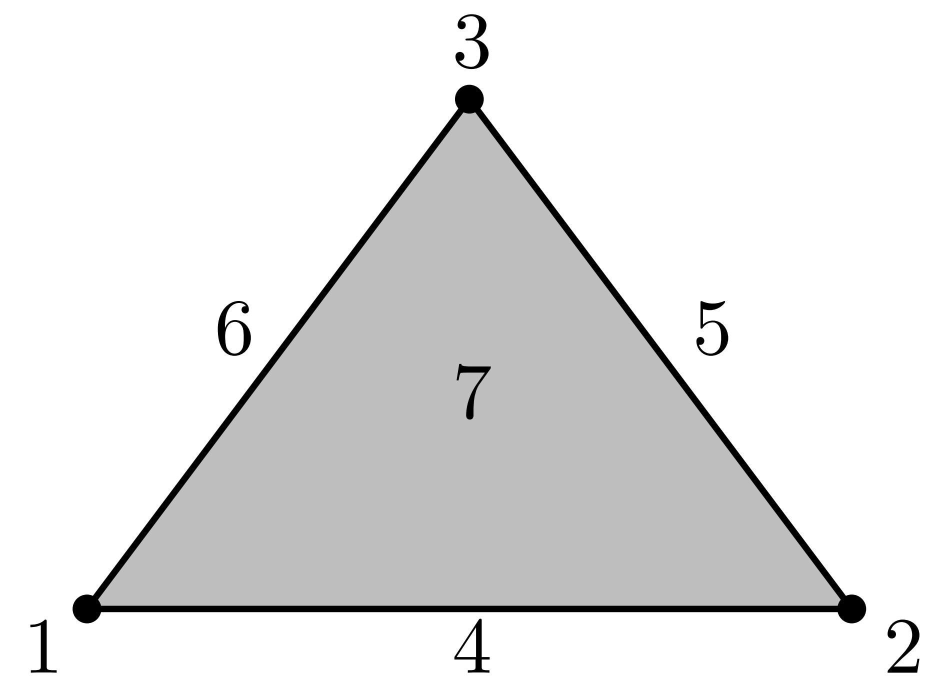

Input: simplicial filtration

Output: boundary matrix
reduced to column-echelon form

simplex pairs give finite intervals:

$[2,4)$, $[3,5)$, $[6,7)$

unpaired simplices give infinite intervals: $[1,+\infty)$

	1	2	3	4	5	6	7
1				$*$		$*$	
2				$*$	$*$		
3					$*$	$*$	
4							$*$
5							$*$
6							$*$
7							

	1	2	3	4	5	6	7
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4							$*$
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[Edelsbrunner, Letscher, Zomorodian 2002] [Carlsson, Zomorodian 2005] . . .

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PLU factorization:

- Gaussian elimination
- fast matrix multiplication (divide-and-conquer) [Bunch, Hopcroft 1974]
- random projections?

Computation of barcodes: matrix reduction

[Edelsbrunner, Letscher, Zomorodian 2002] [Carlsson, Zomorodian 2005] . . .

Input: simplicial filtration

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PLU factorization:

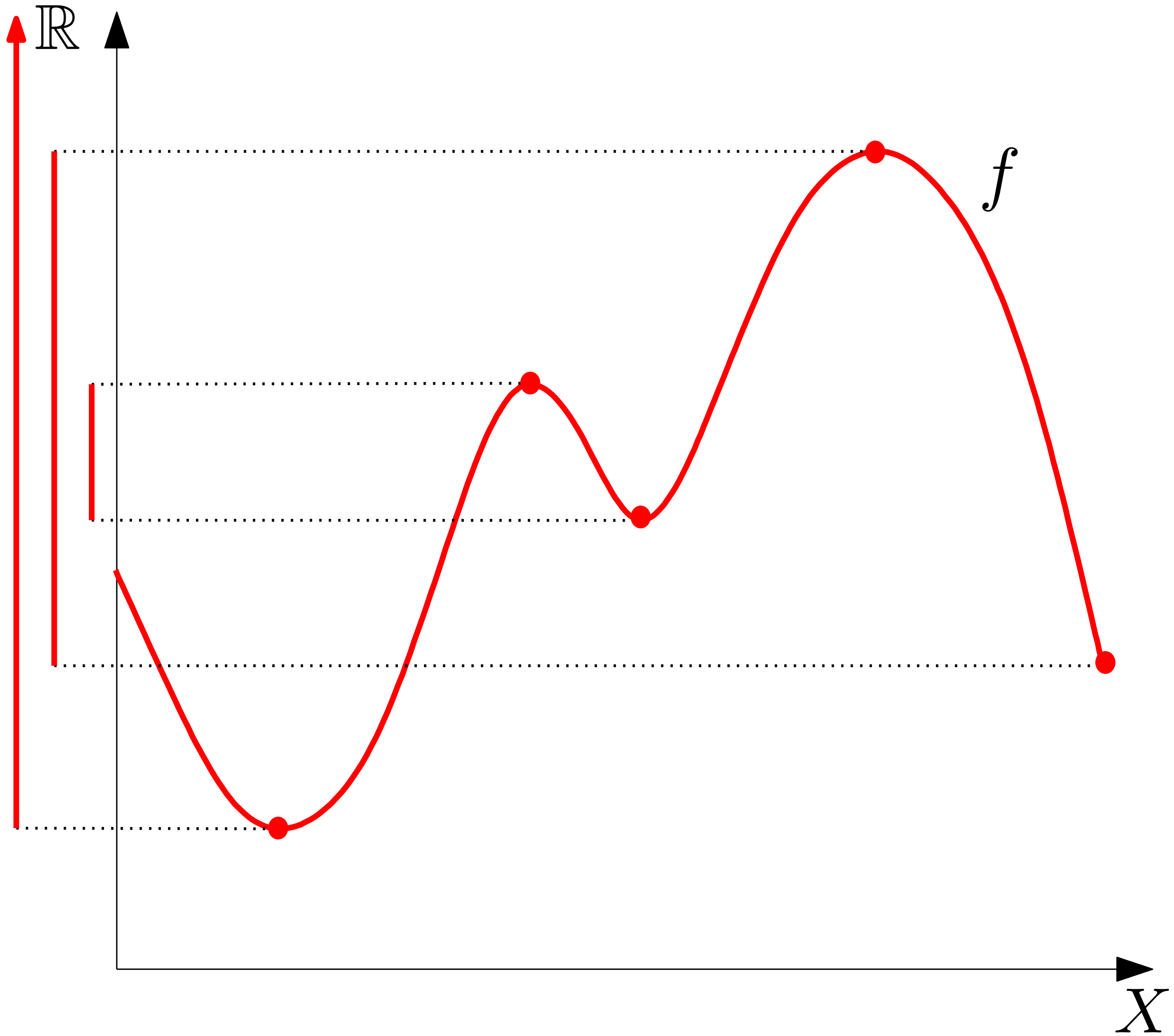
- Gaussian elimination
  - PLEX / JavaPLEX (http://appliedtopology.github.io/javaplex/)
  - Dionysus (http://www.mrzv.org/software/dionysus/)
  - Perseus (http://www.sas.upenn.edu/~vnanda/perseus/)
  - Gudhi (http://gudhi.gforge.inria.fr/)
  - PHAT (https://bitbucket.org/phat-code/phat)
  - DIPHA (https://github.com/DIPHA/dipha/)
  - CTL (https://github.com/appliedtopology/ctl)

Stability of persistence barcodes

X topological space, $f : X \to \mathbb{R}$ function

sublevel-sets filtration $\to$ barcode

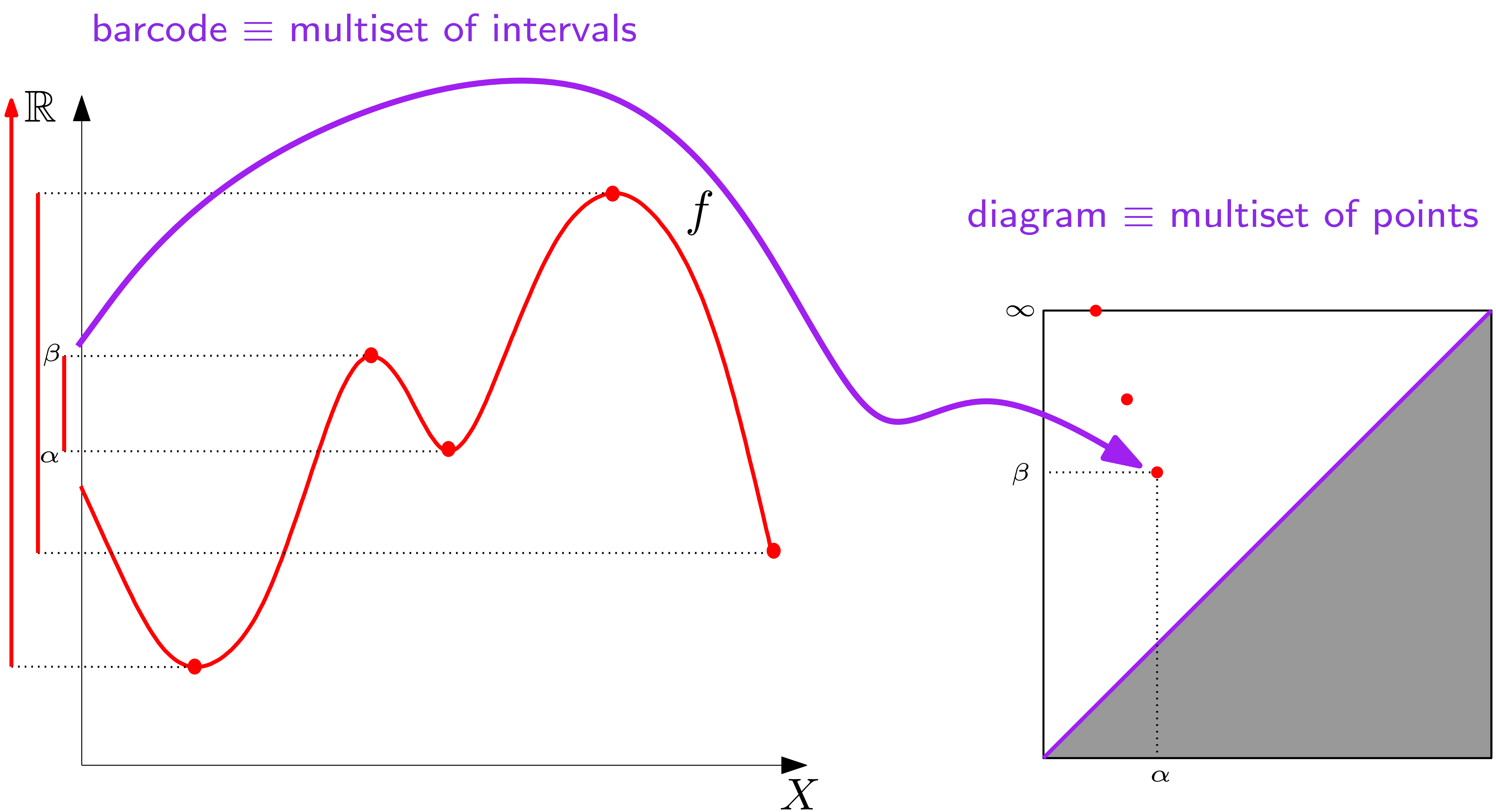
barcode $\equiv$ multiset of intervals



Stability of persistence barcodes

X topological space, $f : X \to \mathbb{R}$ function

sublevel-sets filtration $\to$ barcode / diagram



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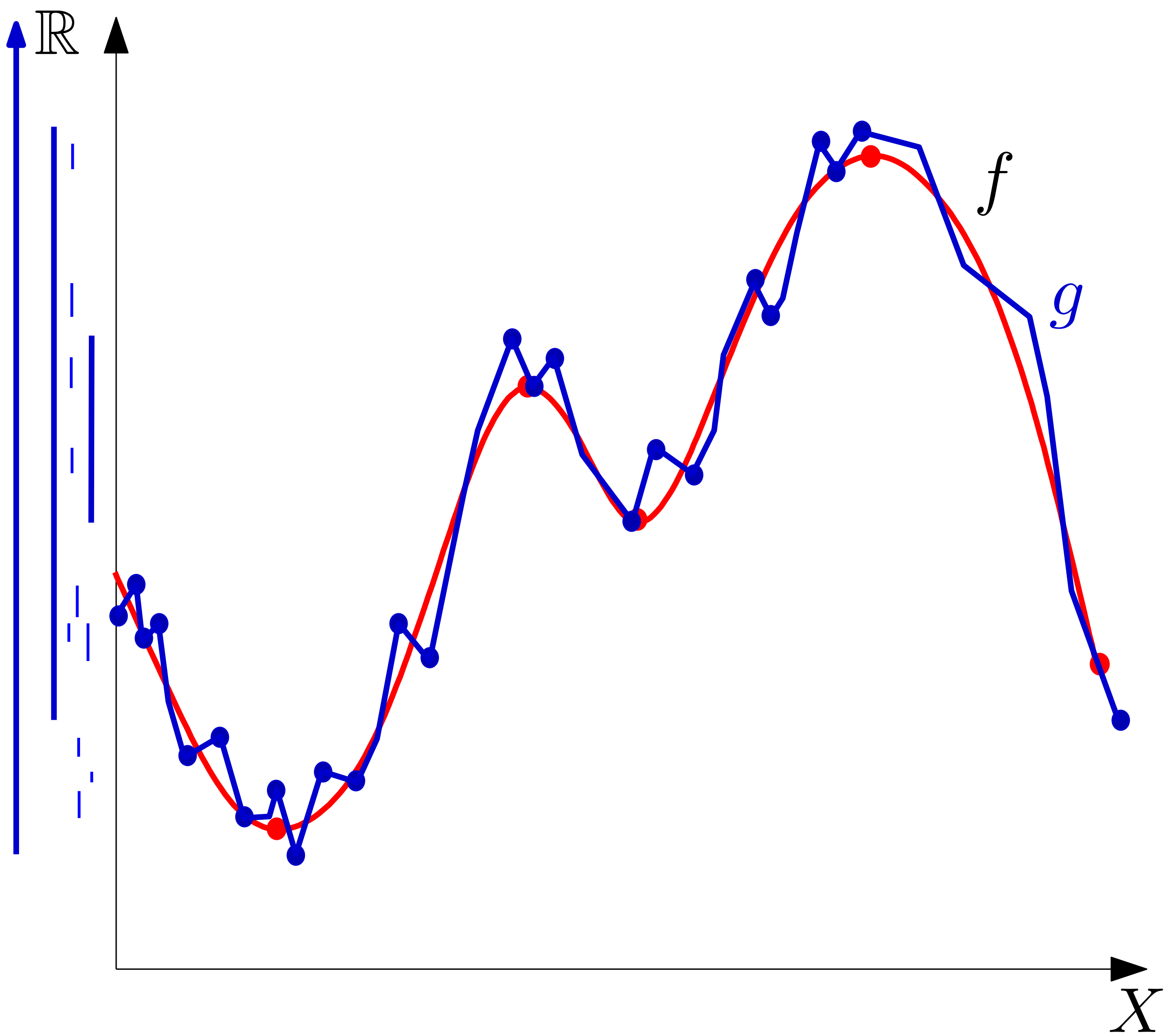
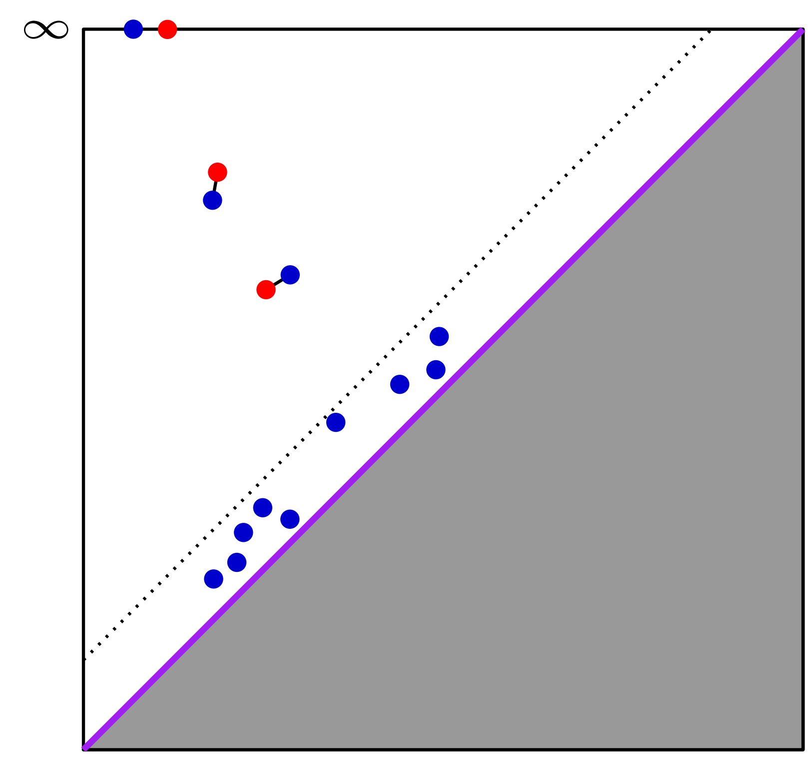
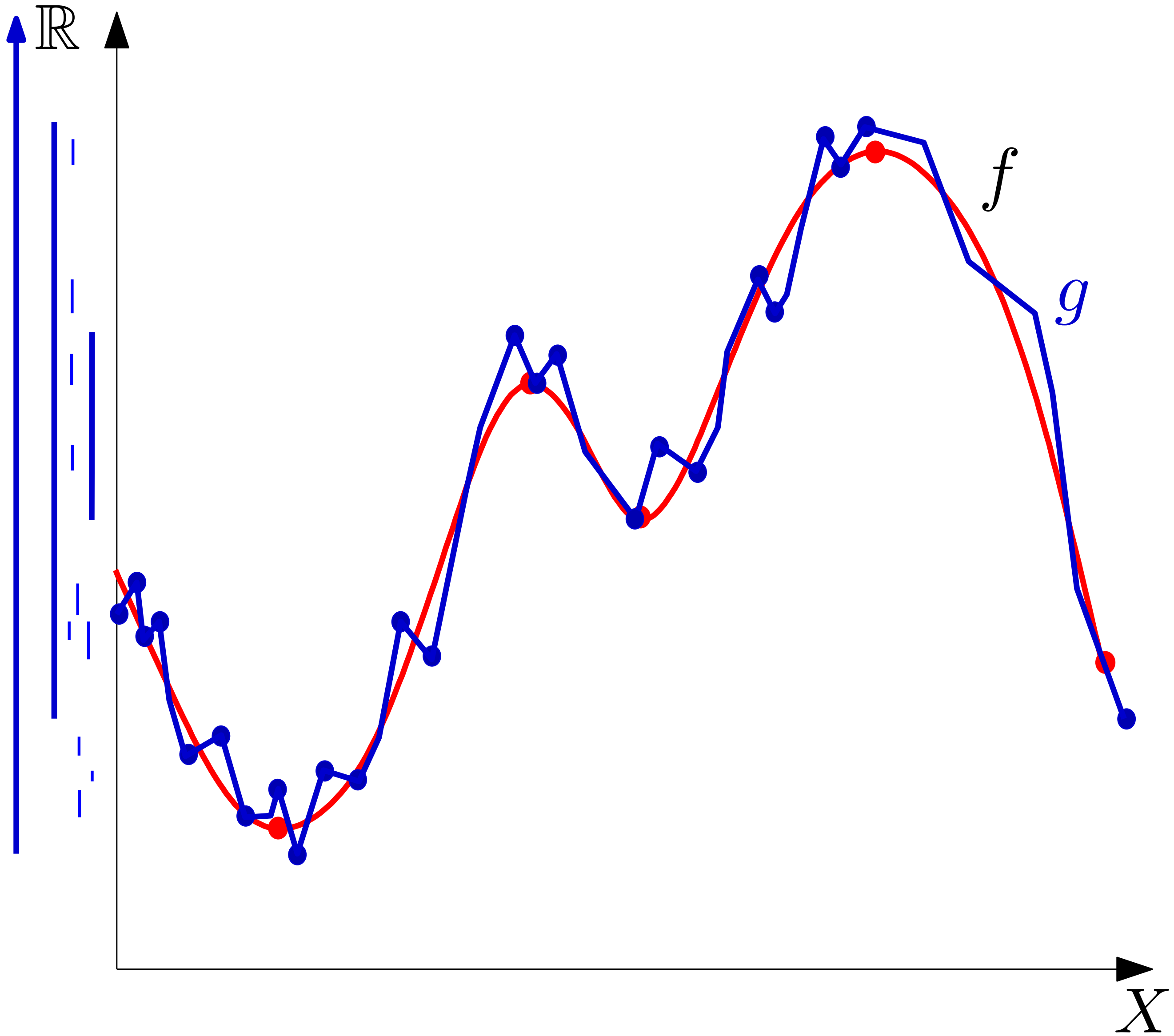


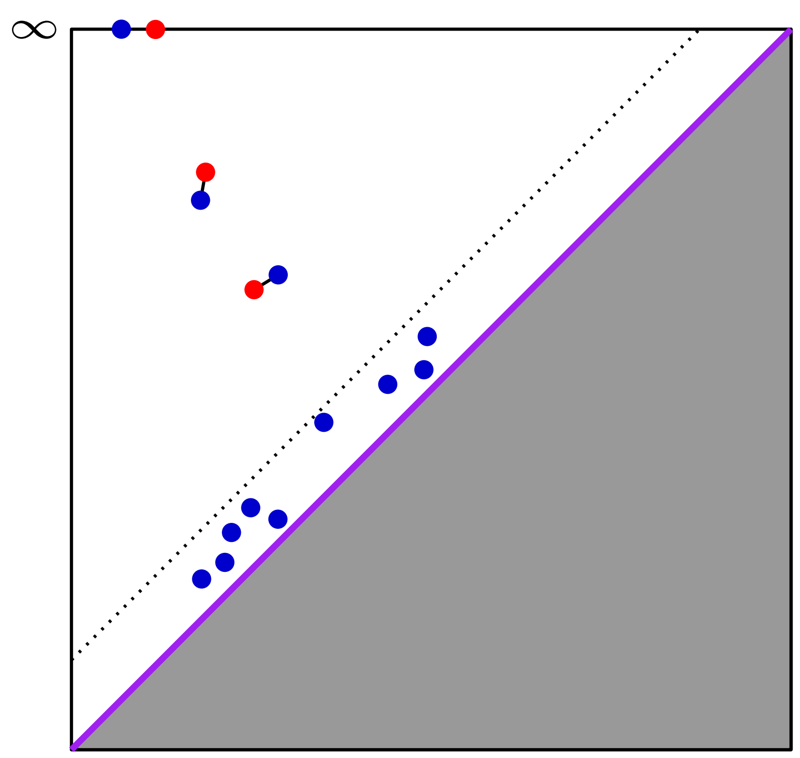


Stability of persistence barcodes

Theorem: For any pfd functions $f, g : X \to \mathbb{R}$,

$$d_b^\infty(\operatorname{dgm} f, \operatorname{dgm} g) \leq \|f - g\|_\infty$$





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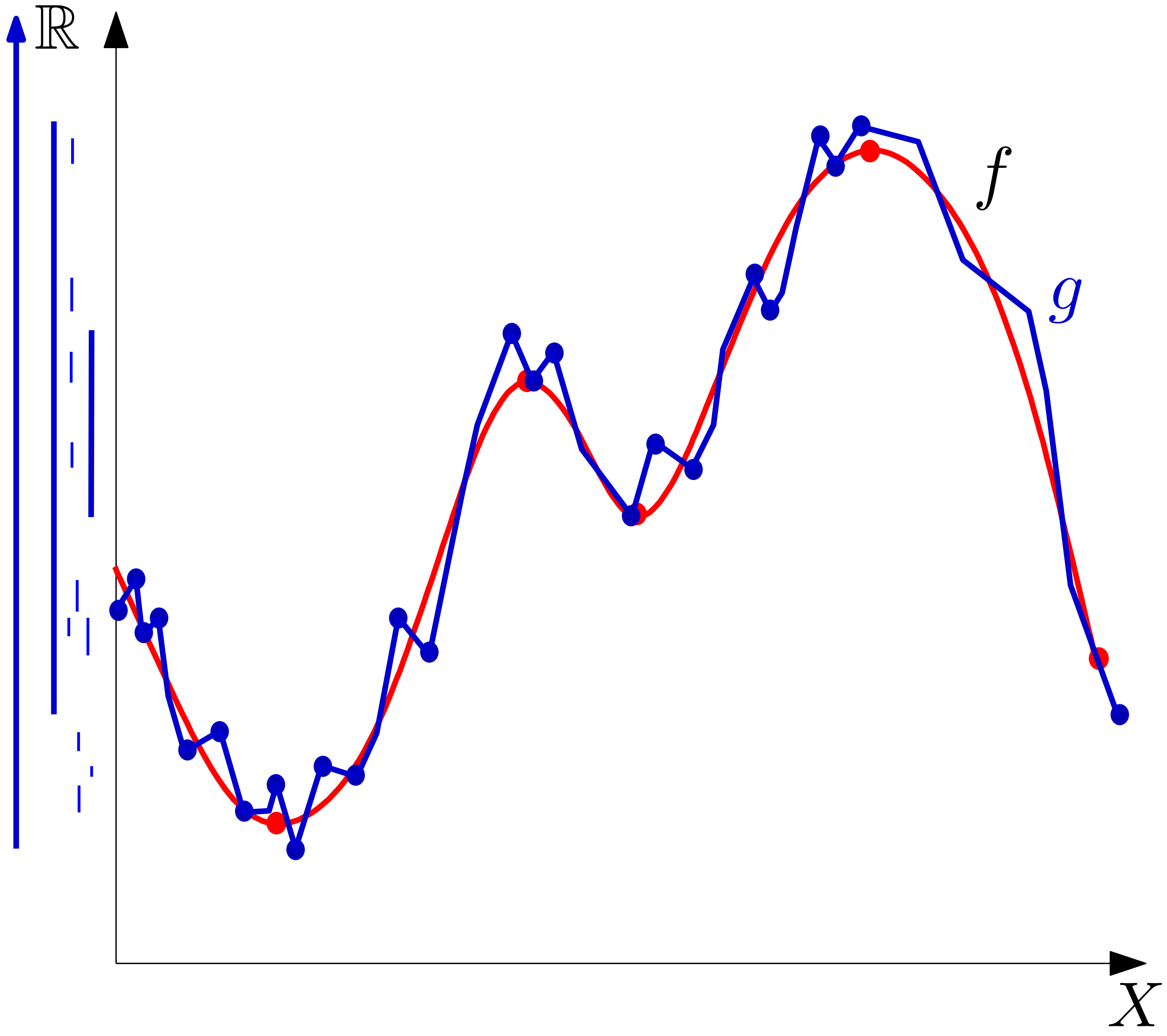
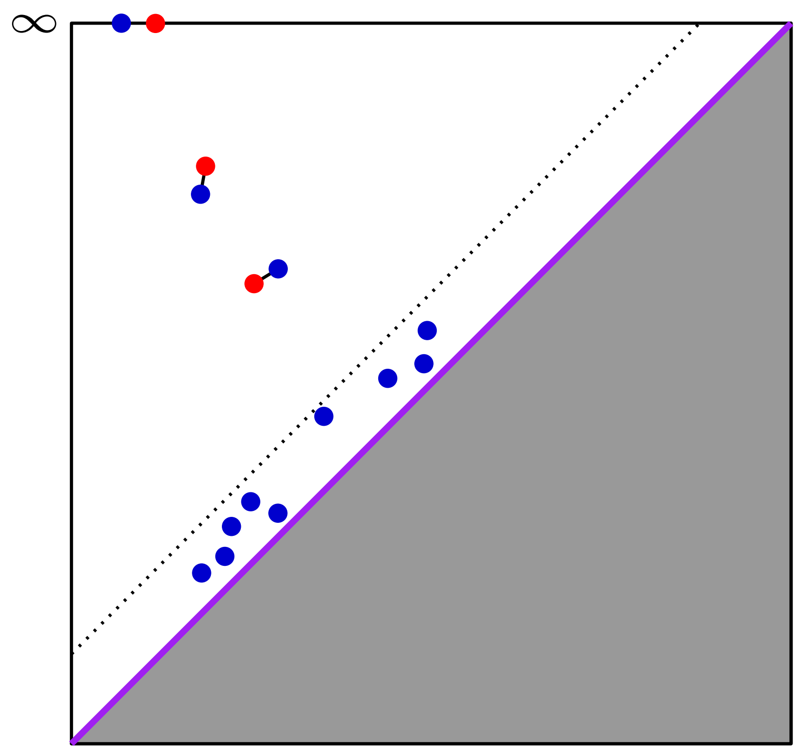
Note: f, g **do not have to have to be defined over the same domain** X





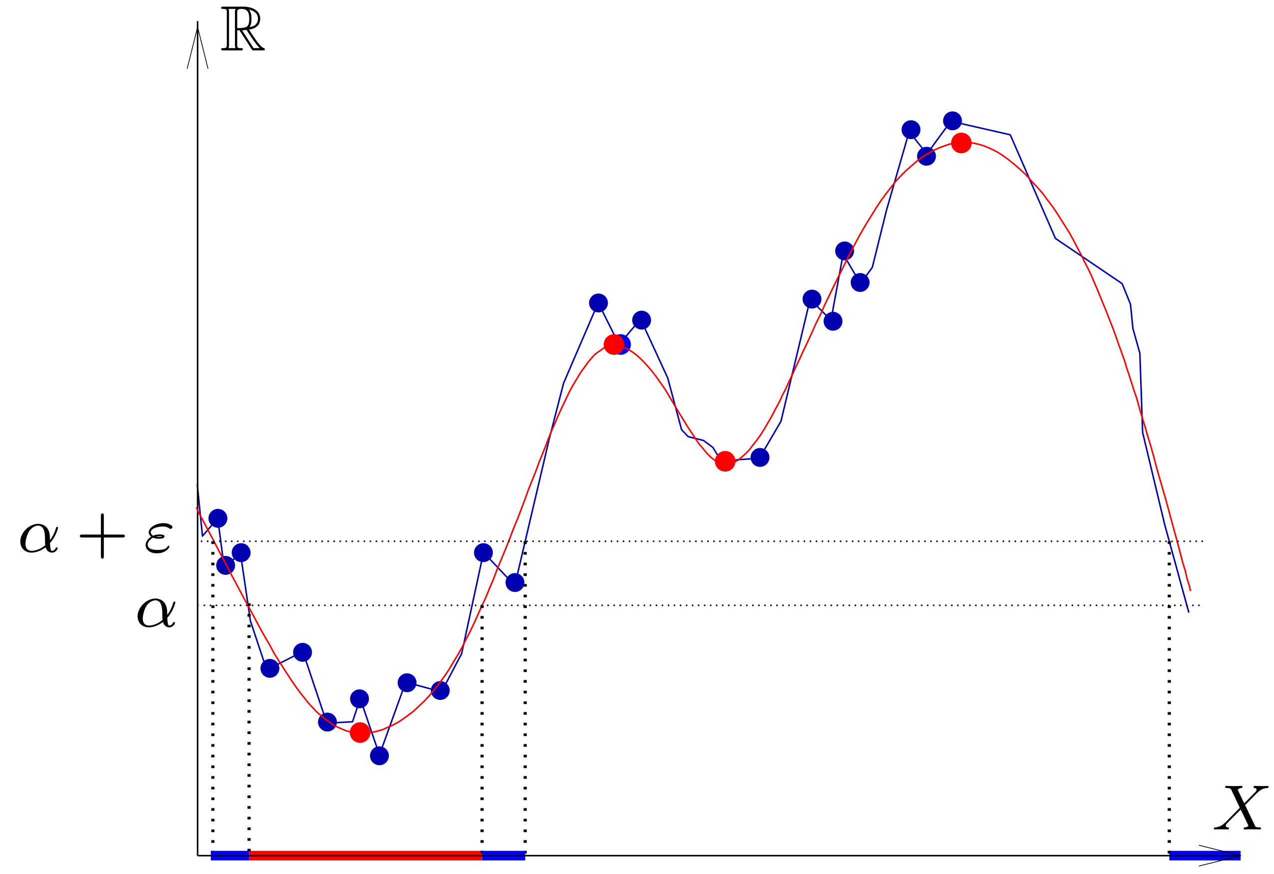
Intuition behind the proof

Let $f, g : X \to \mathbb{R}$ be pfd, and let $\varepsilon = \|f - g\|_\infty$.

$$F_\alpha := f^{-1}((-\infty, \alpha])$$
$$G_\alpha := g^{-1}((-\infty, \alpha])$$

- Key observation: $\{F_\alpha\}_\alpha$ and $\{G_\alpha\}_\alpha$ are ε-**interleaved** w.r.t. inclusion:

$$\forall \alpha \in \mathbb{R},\ G_{\alpha-\varepsilon} \subseteq F_\alpha \subseteq G_{\alpha+\varepsilon}$$



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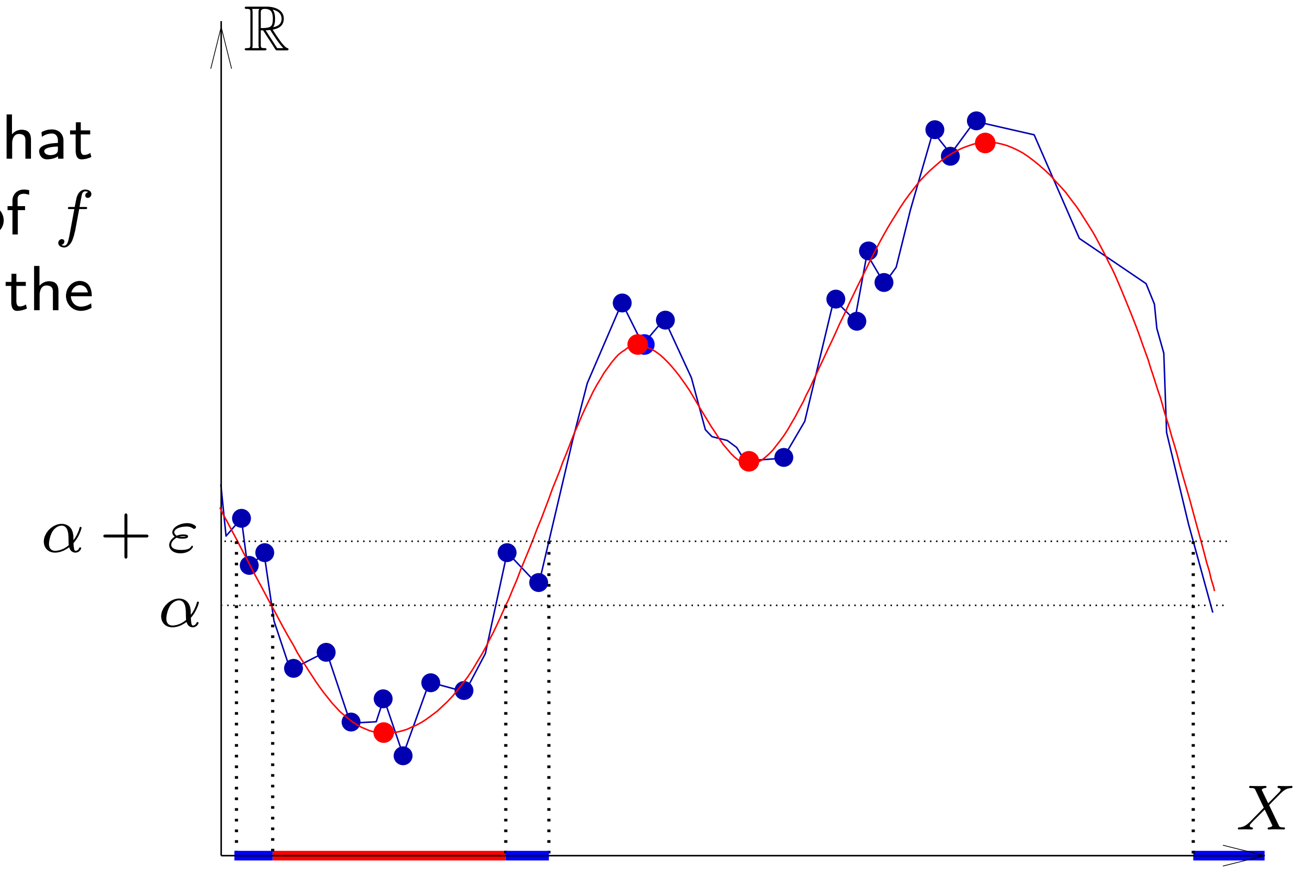
$\to$ Intuition: every homological feature that appears/dies at time α in the filtration of f appears/dies at time $\alpha + \varepsilon$ at the latest in the filtration of g, and vice versa.



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- both filtrations are 2ε-discretizations of $\{H_{n\varepsilon}\}_{n\in\mathbb{Z}}$,where $H_{n\varepsilon} = \begin{cases} F_{n\varepsilon} & \text{if } n \text{ is even} \\ G_{n\varepsilon} & \text{if } n \text{ is odd} \end{cases}$

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$\to$ **goal**: bound distances between diagrams of filtrations and discretizations

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Let $f, g : X \to \mathbb{R}$ be pfd, and let $\varepsilon = \|f - g\|_\infty$.

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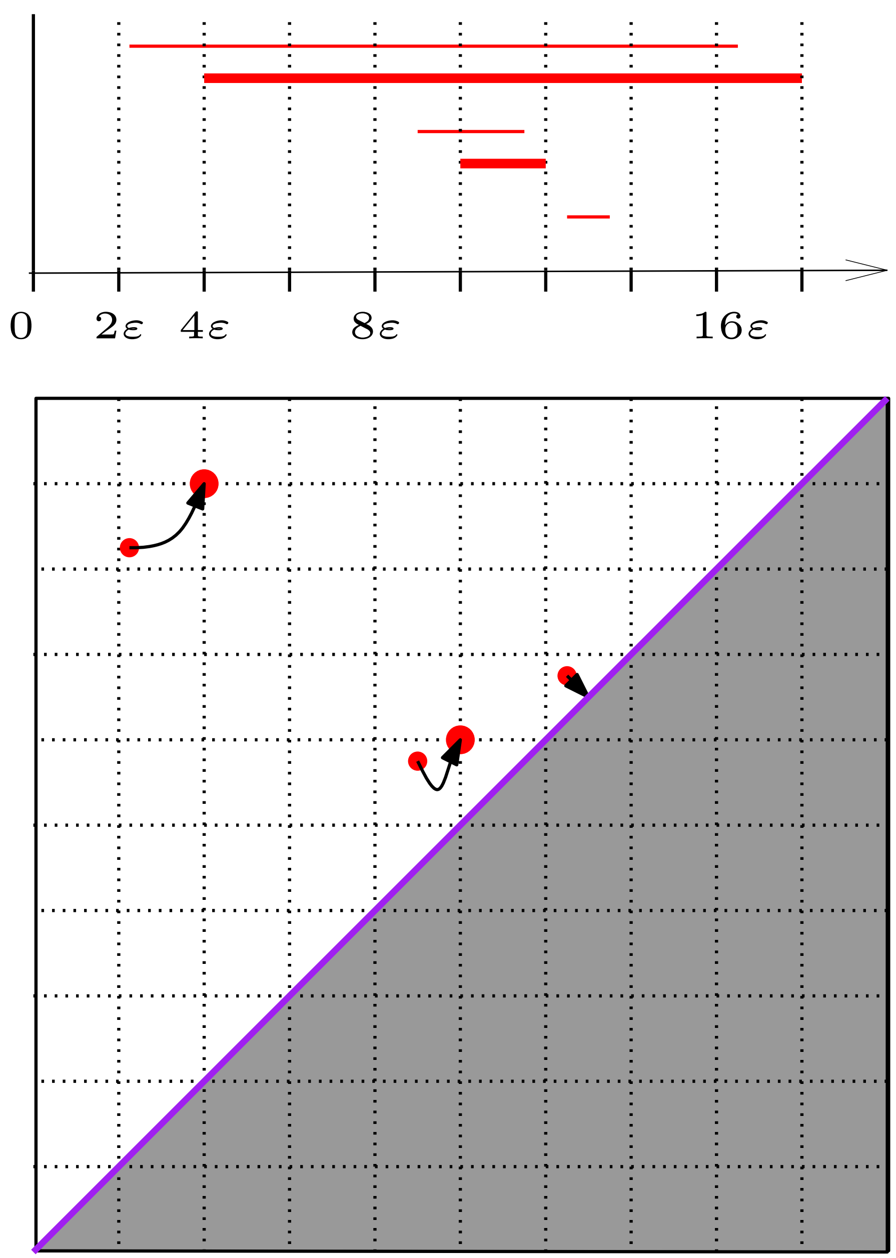
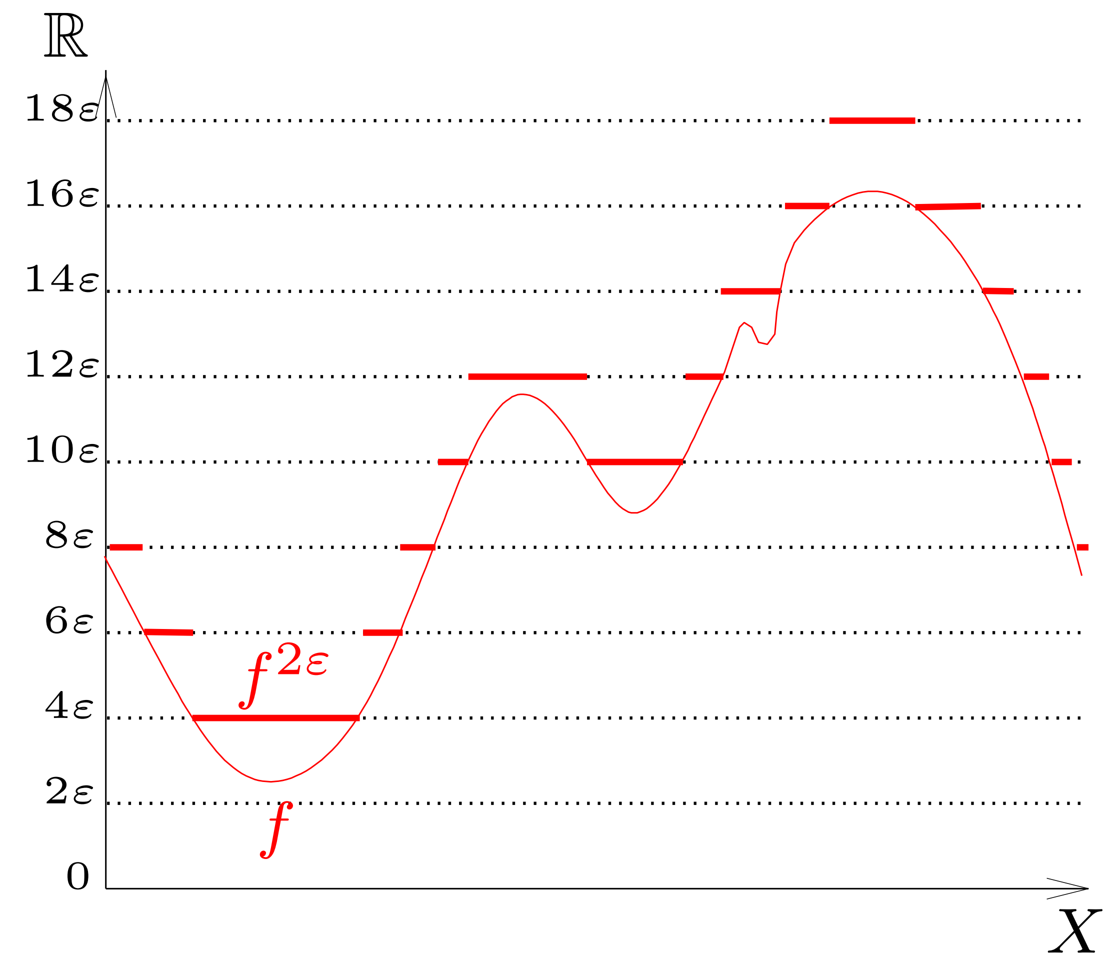
- Discretization $\Rightarrow$ pixelization effect on the barcodes / diagrams:





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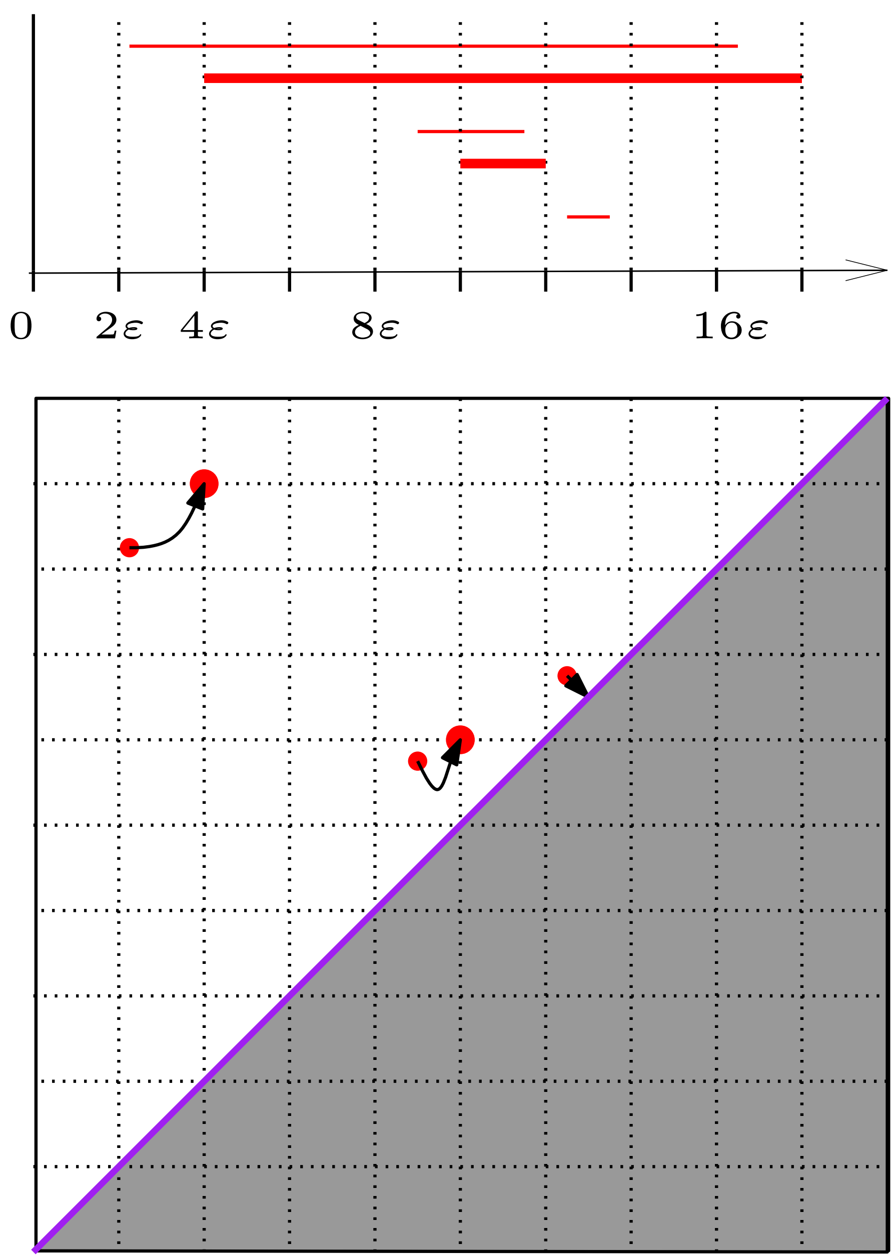

Pixelization map: $\forall \alpha \leq \beta$,

$$\pi_{2\varepsilon}(\alpha, \beta) = \begin{cases} (\lceil \frac{\alpha}{2\varepsilon} \rceil 2\varepsilon, \lceil \frac{\beta}{2\varepsilon} \rceil 2\varepsilon) \text{ if } \lceil \frac{\beta}{2\varepsilon} \rceil > \lceil \frac{\alpha}{2\varepsilon} \rceil \\ (\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}) \text{ if } \lceil \frac{\beta}{2\varepsilon} \rceil = \lceil \frac{\alpha}{2\varepsilon} \rceil \end{cases}$$

Theorem: If $f : X \to \mathbb{R}$ is q-tame, then $\pi_{2\varepsilon}$ induces a bijection $\mathrm{dgm}\, f \to \mathrm{dgm}\, f^{2\varepsilon}$.

$$\Rightarrow \mathrm{d}_b^\infty(\mathrm{dgm}\, f, \mathrm{dgm}\, f^{2\varepsilon}) \leq 2\varepsilon$$

Intuition behind the proof

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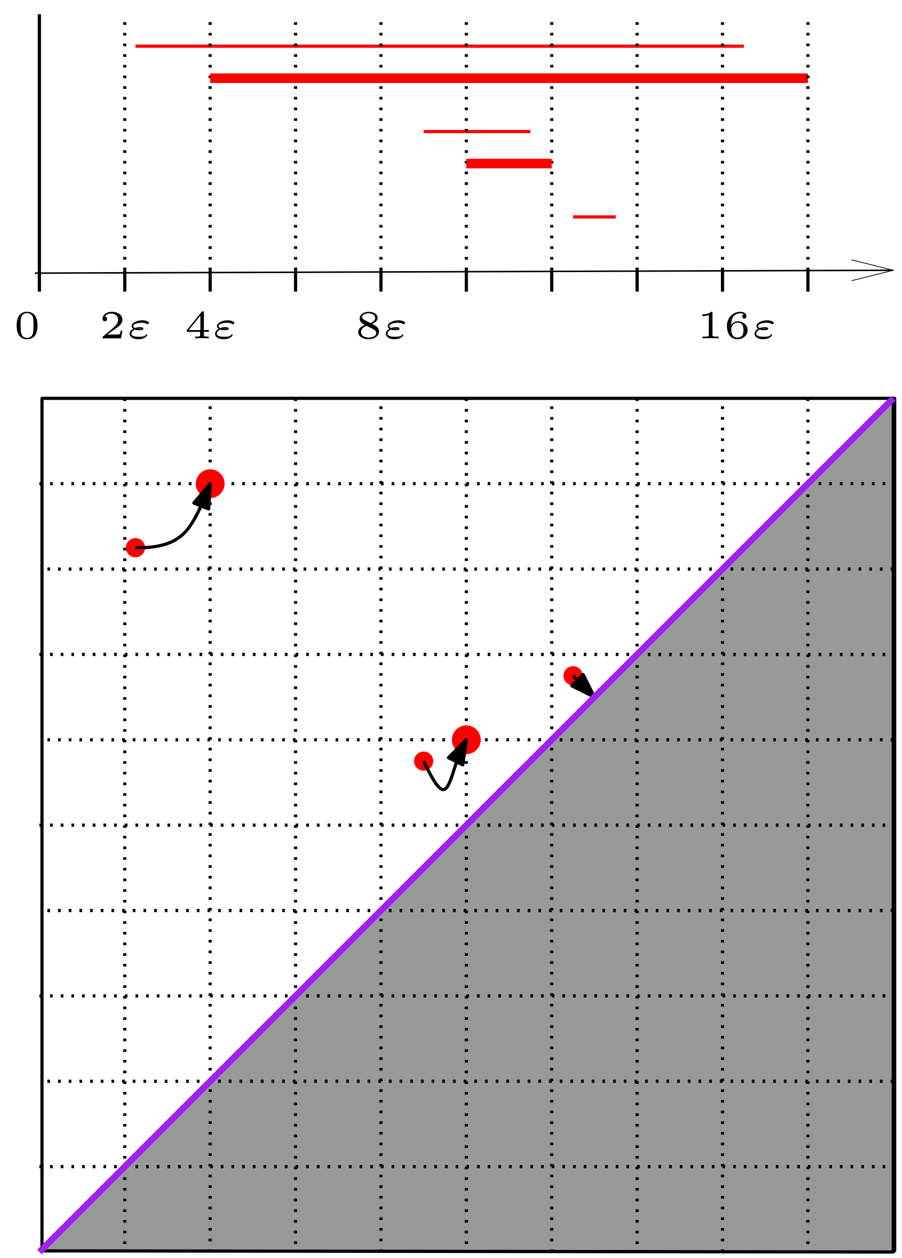

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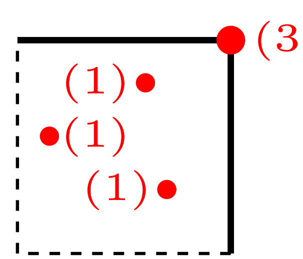
Theorem: If $f : X \to \mathbb{R}$ is q-tame, then $\pi_{2\varepsilon}$ induces a bijection $\mathrm{dgm}\, f \to \mathrm{dgm}\, f^{2\varepsilon}$.

$\to$ proof: show that the multiplicities of $\mathrm{dgm}\, f$ and $\mathrm{dgm}\, f^{2\varepsilon}$ are the same inside each grid cell that does not intersect the diagonal.

The case of diagonal cells is trivial.



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- Back to interleaved filtrations:

$$\cdots \subseteq F_0 \subseteq G_\varepsilon \subseteq F_{2\varepsilon} \subseteq \cdots \subseteq G_{(2n-1)\varepsilon} \subseteq F_{2n\varepsilon} \subseteq G_{(2n+1)\varepsilon} \subseteq \cdots$$

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Previous theorem + triangle inequality $\Rightarrow \mathrm{d}_b^\infty(\mathrm{dgm}\, f, \mathrm{dgm}\, g) \leq 8\varepsilon$

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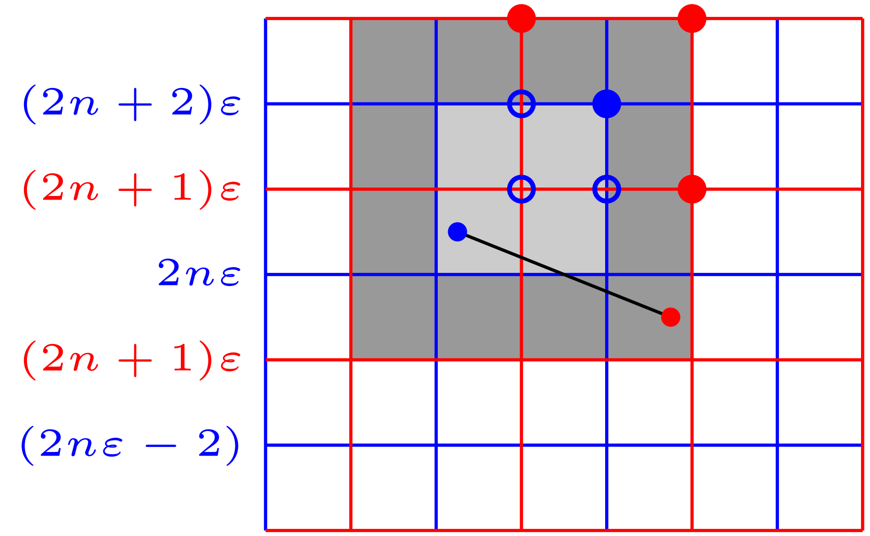
Improvement: $\boxed{\mathrm{d}_b^\infty(\mathrm{dgm}\, f, \mathrm{dgm}\, g) \leq 3\varepsilon}$



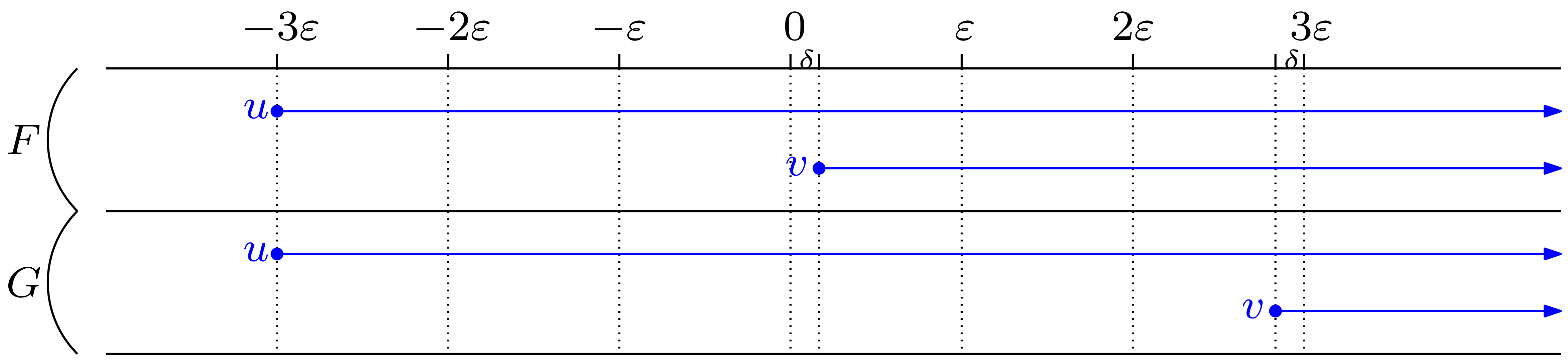
Intuition behind the proof

- Comments:

  - sketch of proof based on [Chazal, Cohen-Steiner, Glisse, Guibas, O. 2009].

  - uses only the fact that F, G are interleaved over the scale $\varepsilon\mathbb{Z}$.

  - bound 3ε is tight under this assumption.



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  - bound 3ε is tight under this assumption.

$$\forall \alpha \in \mathbb{R},\ G_{\alpha-\varepsilon} \subseteq F_\alpha \subseteq G_{\alpha+\varepsilon}$$

  - full interleaving hypothesis gives bound ε via an **interpolation argument**:

    # at the functional level (requires extra conditions)

    [Cohen-Steiner, Edelsbrunner, Harer 2005]

    # at the algebraic level directly ($\rightarrow$ **isometry theorem**)

    [Chazal, Cohen-Steiner, Glisse, Guibas, O. 2009] [Chazal, de Silva, Glisse, O. 2016]

    [Bauer, Lesnick 2015]

    [Botnan, Lesnick 2016]

    ...

Recap'

3 pillars to the theory (topological persistence):

- decomposition theorems ($\exists$ barcodes)
- algorithms (computation of barcodes)
- stability theorems (barcodes as stable descriptors)