# 3D Reconstruction from Cross-Sections

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TGDA 2009

# Motivations

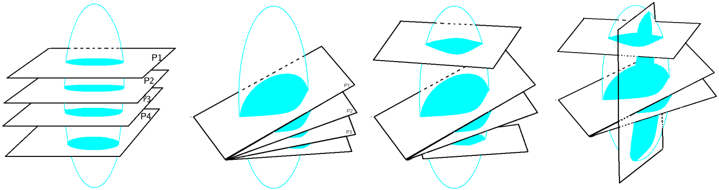

(a) Input: 2D Images + Orientation of the captor

# Motivations

(b) Actual Technology

# Different cross-sections positions






Parallel planes

Non-parallel serial sequence of planes

Arbitrary cutting planes

# Geometric Methods (parallel case)



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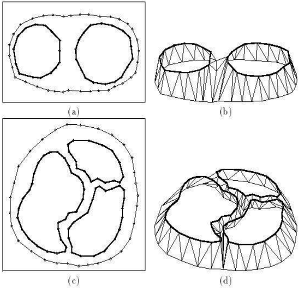
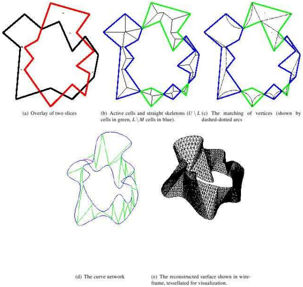
Most of methods use the superposition of the contours.




Bajaj et al. [1996]




Barequet et al. [1996-2004]

# Outline

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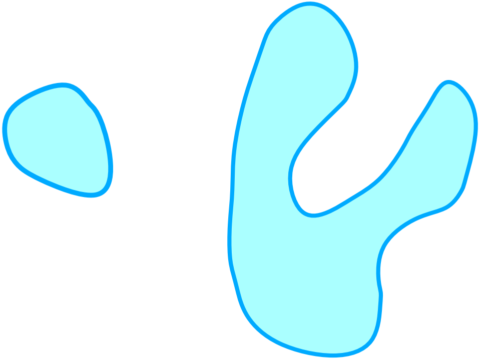
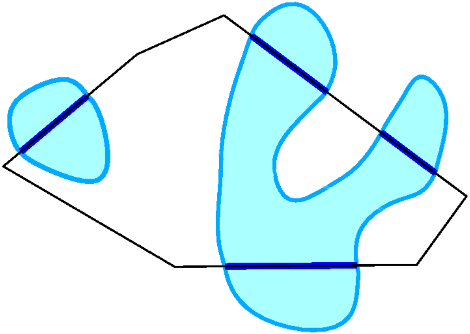
# Planar Cross-Sections of an Object

$\mathcal{O} \subset \mathbb{R}^3$ is a compact 3-manifold with boundary (denoted by $\partial\mathcal{O}$) of class $C^1$.



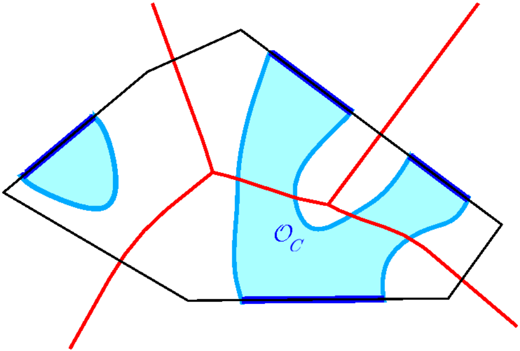
# Planar Cross-Sections of an Object

$\mathcal{O}$ is cut by a set of cutting planes $P$.



# In a Cell of the Arrangement

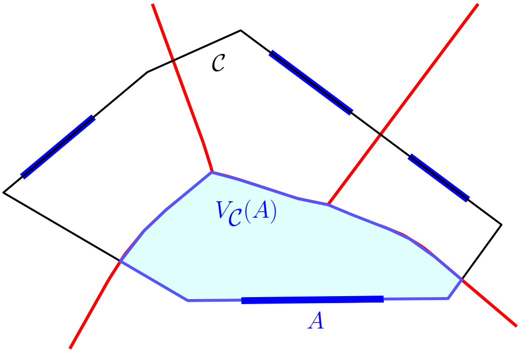
In each cell $\mathcal{C}$ of the arrangement of the cutting planes, $\mathcal{O}_{\mathcal{C}} := \mathcal{O} \cap \mathcal{C}$ will be reconstructed independently.



# Voronoi Diagram of a Section

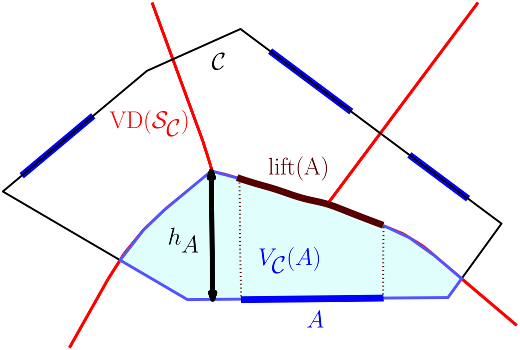


The Voronoi diagram of a section $A$, $\mathrm{V}_{\mathcal{C}}(A)$.



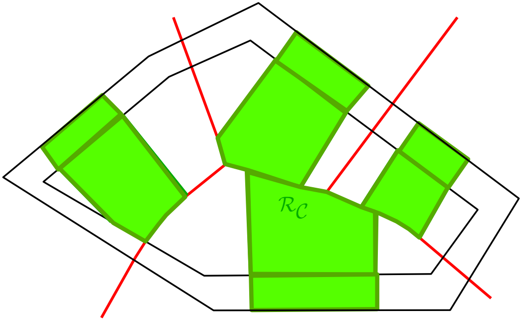
# Lift and Height of a Section





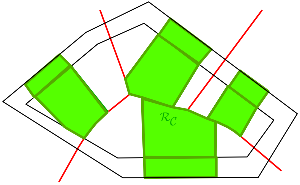

$$h_A := \max_{x \in V_{\mathcal{C}}(A)} d(x, P_A).$$

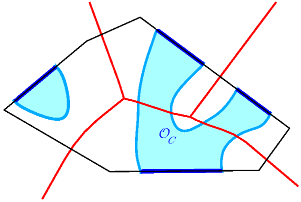
# Reconstructed Object




$$\mathcal{R}_{\mathcal{C}} := \bigcup_{A \in \mathcal{S}_{\mathcal{C}}} \bigcup_{a \in A} [a, \mathrm{lift}(a)].$$

# Approximation Guarantees



# Outline

Voronoi Diagram and Reconstruction

Sampling Conditions

Connectivity between the Sections

Topological Guarantees

# Internal and External Medial Axes of $\partial\mathcal{O}$



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Consider $\partial\mathcal{O}$ as a 2-manifold embedded in $\mathbb{R}^3$. $\mathrm{MA}(\partial\mathcal{O})$, contains two different parts:

- Internal Part: $\mathrm{MA}_i = \mathrm{MA}(\mathbb{R}^3 \backslash \mathcal{O})$, that lies in $\mathcal{O}$.
- External Part: $\mathrm{MA}_e = \mathrm{MA}(\mathcal{O})$, that lies in $\mathbb{R}^3 \backslash \mathcal{O}$.

# Reach



$$\mathrm{reach}(\mathcal{O}) := \min_{m \in \mathrm{MA}(\partial\mathcal{O})} d(m, \partial\mathcal{O}).$$

For a cell $\mathcal{C}$ of the arrangement:

$$\mathrm{reach}_{\mathcal{C}}(\mathcal{O}) = \min_{m \in \mathrm{MA}(\partial\mathcal{O}) \cap \mathcal{C}} d(m, \partial\mathcal{O}).$$

If $\mathrm{MA}(\partial\mathcal{O}) \cap \mathcal{C} = \emptyset$ then $\mathrm{reach}_{\mathcal{C}}(\mathcal{O})$ is infinite.
$reach(\mathcal{O}) = \min_{\mathcal{C}} (reach_{\mathcal{C}}(\mathcal{O}))$ is positive.

# Sampling Conditions



For $\epsilon > 0$, we have an *$\epsilon$-cut sample of $\mathcal{O}$* if :

1. Any connected component of $\mathcal{O}$ is cut by at least one cutting plane.
2. For any cell $\mathcal{C}$ of the arrangement of the cutting planes and for any section $A \in \mathcal{S}_\mathcal{C}$ we have $h_A < \epsilon.\mathrm{reach}_\mathcal{C}(\mathcal{O})$.

Any refinement of an $\epsilon$-cut sample is an $\epsilon$-cut sample as well.

# A Sufficient Condition for Having an $\epsilon$-Cut Sample.



### Definition (Bounding Planes and Height of a Polyhedron)

Let $\mathcal{C}$ be a 3D polyhedron. The supporting planes of the faces of $\mathcal{C}$ are called the *bounding* planes of $\mathcal{C}$. The *height* of $\mathcal{C}$ is defined as the maximum distance between a point in $\mathcal{C}$ and a bounding plane of $\mathcal{C}$.

**Lemma:** For any $\epsilon \leq 2$, if the height of any cell is less than $\epsilon.\text{reach}(\mathcal{O})$, then we have an $\epsilon$-cut sample of $\mathcal{O}$.

# Separation Lemma



### Lemma (Separation Lemma)

If the sample of cutting planes forms an $\epsilon$-cut sample of $\mathcal{O}$, for some $\epsilon \leq 1$, then $\mathrm{MA}_i \subset \mathcal{R}$ and $\mathrm{MA}_e \subset \mathbb{R}^3 \backslash \mathcal{R}$. In other words, $\partial\mathcal{R}$ *separates* the internal and external parts of the medial axis of $\partial\mathcal{O}$.

# Approximation Guarantees.



## Theorem (Hausdorff Distance)

Let $D_1$ be the maximum of $\text{reach}_{\mathcal{C}}(\mathcal{O})$ for all cells $\mathcal{C}$ that intersect $\text{MA}(\partial\mathcal{O})$.
Let $D_2$ be the maximum diameter of the cells that do not intersect $\text{MA}(\partial\mathcal{O})$.

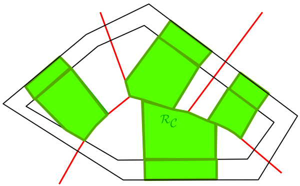
Then in an $\epsilon$-cut sample of $\mathcal{O}$, we have
$d_H(\mathcal{O}, \mathcal{R}) < \max\{2\epsilon D_1, D_2\}$.

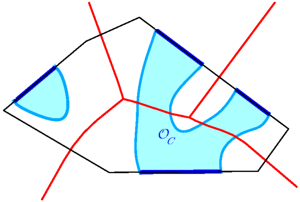
# Outline

# Connectivity Theorem



In an $\epsilon$-cut sample of $\mathcal{O}$, for an $\epsilon \leq 1$, for any cell $\mathcal{C}$ of the arrangement, there is a bijection between the connected components of $\mathcal{O}_\mathcal{C}$ and the connected components of $\mathcal{R}_\mathcal{C}$.



# Connectivity between the Sections



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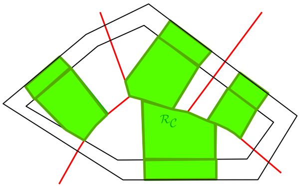
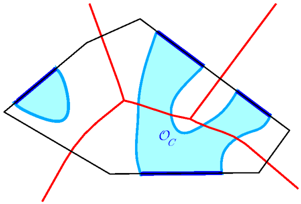
## Lemma

Two sections are connected in $\mathcal{O}_\mathcal{C}$ iff they are connected in $\mathcal{R}_\mathcal{C}$.



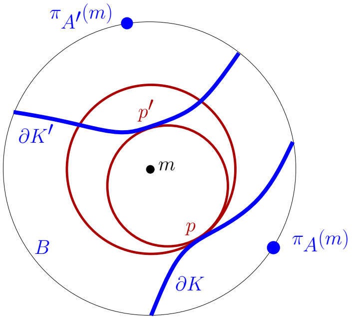



# Proof of $\Longrightarrow$



If two sections are connected in $\mathcal{O}_{\mathcal{C}}$ then they are connected in $\mathcal{R}_{\mathcal{C}}$.

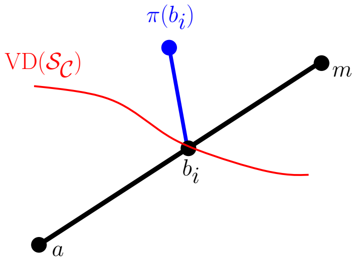
**Proof:**



# Proof of $\Longleftarrow$

If two sections are connected in $\mathcal{R}_\mathcal{C}$ then they are connected in $\mathcal{O}_\mathcal{C}$.

Lemma: Let $a$ be in $\partial A$, and $A' \in \mathcal{S}_\mathcal{C}$ be s.t. $[am_i(a)]$ intersects $V_\mathcal{C}(A')$. Then $A$ and $A'$ are connected in $\mathcal{R}_\mathcal{C}$.



# Proof of $\Longleftarrow$

If two sections are connected in $\mathcal{R}_\mathcal{C}$ then they are connected in $\mathcal{O}_\mathcal{C}$.

**Proof:** Take a path $\gamma$ in $\partial\mathcal{O}$ between $a \in \partial A$ and $a' \in \partial A'$, for some section $A' \in \{A_i\}_{i \in I}$. For any point $x \in \partial\mathcal{O}$, we define $J(x) = m_i(x)$ if $m_i(x) \in \mathcal{C}$, and $J(x) = [xm_i(x)] \cap \partial\mathcal{C}$ otherwise. $J$ is a continuous function that maps $\gamma$ to a connected path in $\mathrm{MA}_i \cup \{A_i\}_{i \in I}$, between $J(a')$ and $J(a)$. As $[am_i(a)]$ does not intersect the Voronoi cells of the other sections of $\mathcal{S}_\mathcal{C}$. $J(a)$ is either $m_i(x) \in V_\mathcal{C}(A)$ or is $a$. In both cases, $J(a)$ is in $V_\mathcal{C}(A)$. Two possible cases:

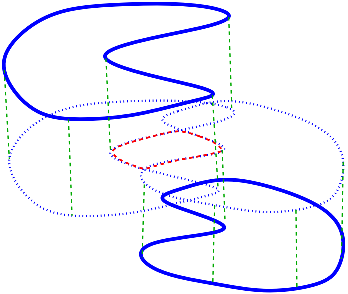
- $\exists A_0 \in \mathcal{S}_\mathcal{C}$, distinct from $A$, s.t. $J(\gamma)$ intersects $V_\mathcal{C}(A_0)$. Thus, $A$ is connected to $A_0$ in $\mathcal{R}_\mathcal{C}$.
- Otherwise, $J(\gamma)$ lies entirely in $V_\mathcal{C}(A)$. In particular, $J(a') \in V_\mathcal{C}(A)$. Hence, $[a'm_i(a')]$ intersects $V_\mathcal{C}(A)$, and $A'$ is connected to $A$ in $\mathcal{R}_\mathcal{C}$.

# Outline

# Topology?



Good Connections $\Longrightarrow$ Good Topology ?

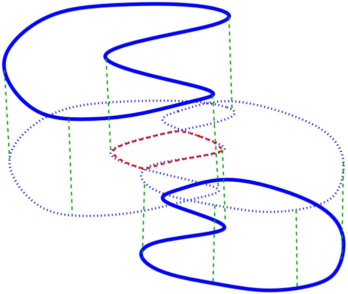


Yes, with an $\epsilon$-cut sample of $\mathcal{O}$, for $\epsilon \leq \frac{1}{2}$.

# Topology?



Good Connections $\Longrightarrow$ Good Topology ?

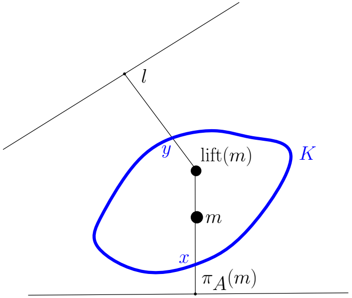


Yes, with an $\epsilon$-cut sample of $\mathcal{O}$, for $\epsilon \leq \frac{1}{2}$.

# $\frac{1}{2}$-Cut Sample's Property



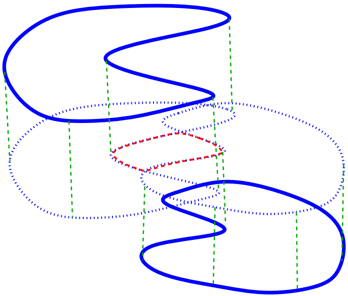
Let $m \in V_{\mathcal{C}}(A)$ s.t. $\text{lift}(m) \in \partial V_{\mathcal{C}}(A')$. In an $\frac{1}{2}$-cut sample of $\mathcal{O}$: If $m \in \text{MA}_i$, then $\text{lift}(m) \in \text{lift}(A) \cap \text{lift}(A')$.
If $m \in \text{MA}_e$, then $\text{lift}(m) \notin \text{lift}(A)$ and $\text{lift}(m) \notin \text{lift}(A')$.

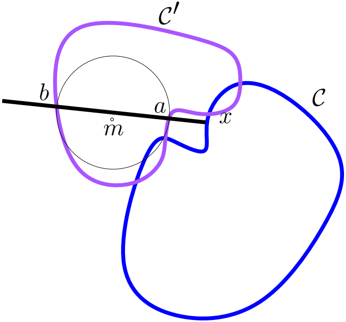



**Proof:** $d(m, \partial\mathcal{O}) < \|ml\| \leq \|m\text{lift}(m)\| + \|\text{lift}(m)l\| \leq 2.\epsilon reach_{\mathcal{C}}(\mathcal{O}) \leq reach_{\mathcal{C}}(\mathcal{O}) \leq d(m, \partial\mathcal{O})$, a contradiction. A similar proof shows that if $m \in \text{MA}_e$ then $\pi_{A'}(m) \notin A'$.

# Lemma

In an $\frac{1}{2}$-cut sample of $\mathcal{O}$, any hole in $\mathrm{lift}(\mathcal{R}_{\mathcal{C}})$ is in the intersection of the lift of some holes in the sections. In particular, this situation does not happen:



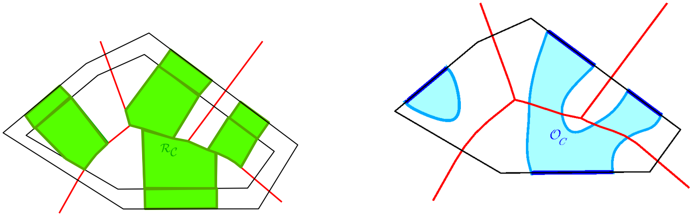


# Main Theorem



In an $\epsilon$-cut sample of $\mathcal{O}$, for $\epsilon \leq \frac{1}{2}$, for any cell $\mathcal{C}$ of the arrangement:

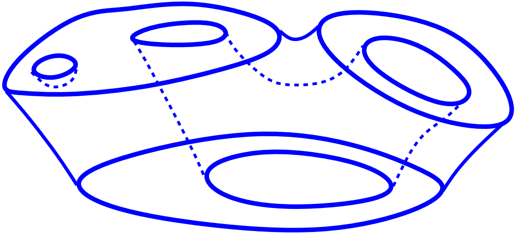
$\mathcal{R}_{\mathcal{C}}$ is homotopy equivalent to $\mathcal{O}_{\mathcal{C}}$.



# Pant Definition



**Pant Definition** A topological sphere from which two dimensional disks have been removed.
For a set of disjoint closed curves $\{D_i\}_{i\in I}$, a $\{D_i\}_{i\in I}$-pant is a topological sphere passing through $\{D_i\}_{i\in I}$ from which the disks bounded by $\{D_i\}_{i\in I}$ have been removed.



# $\partial\mathcal{R}_\mathcal{C}$ and $\partial\mathcal{O}_\mathcal{C}$-Patches



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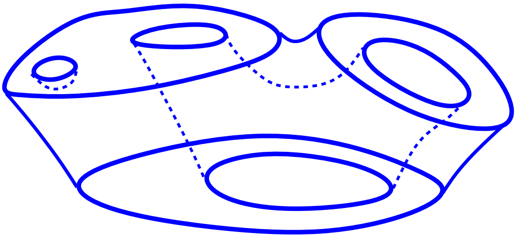
A $\partial\mathcal{R}_\mathcal{C}$-patch is a connected component of $\partial\mathcal{R} \cap \mathcal{C}$.
A $\partial\mathcal{O}_\mathcal{C}$-patch is a connected component of $\partial\mathcal{O} \cap \mathcal{C}$.



# $\partial\mathcal{R}_\mathcal{C}$ and $\partial\mathcal{O}_\mathcal{C}$-Patches are topological Pants

## Lemma (1)

In an $\frac{1}{2}$-cut sample of $\mathcal{O}$, let $\bar{\mathcal{F}}$ be a connectivity class of the holes of sections. The $\partial\mathcal{R}_\mathcal{C}$-patch that passes through the holes of $\bar{\mathcal{F}}$ is a $\bar{\mathcal{F}}$-pant.

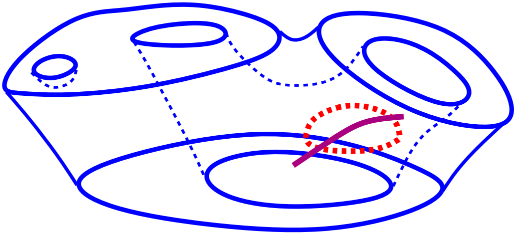
## Lemma (2)

In an $\frac{1}{2}$-cut sample of $\mathcal{O}$, let $\bar{\mathcal{F}}$ be a connectivity class of the holes of sections. The $\partial\mathcal{O}_\mathcal{C}$-patch that passes through the holes of $\bar{\mathcal{F}}$ is a $\bar{\mathcal{F}}$-pant.

# Proof of Lemma (2)

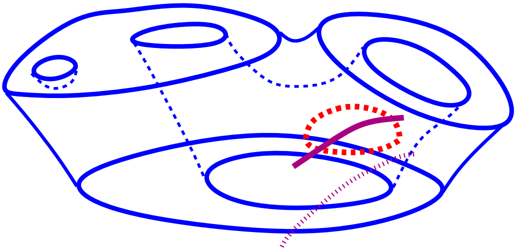


$K'$ is the $\partial\mathcal{O}_{\mathcal{C}}$-patch that passes through the holes of $\bar{\mathcal{F}}$. Let $H'$ be a handle in $K'$. Take a path $\lambda' \subset \mathrm{MA}_i$ that passes through this handle.



# Proof of Lemma (2)

$K'$ is the $\partial\mathcal{O}_\mathcal{C}$-patch that passes through the holes of $\bar{\mathcal{F}}$. Let $H'$ be a handle in $K'$. Take a path $\lambda' \subset \mathrm{MA}_i$ that passes through this handle. $\mathrm{lift}(\lambda')$ in $\mathrm{VD}(\mathcal{S}_\mathcal{C})$ intersects the lift of the holes of $\bar{\mathcal{F}}$. Contradiction with the $\frac{1}{2}$-cut sample's property.



# Homotopy



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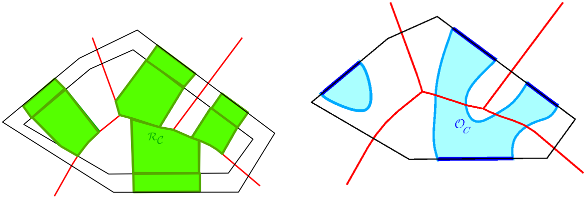
Any $\partial\mathcal{R}_\mathcal{C}$-patch passes through a contour-section. Any $\partial\mathcal{O}_\mathcal{C}$-patch passes through a contour-section as well. Thus, there is a bijection $Q$ from the $\partial\mathcal{R}_\mathcal{C}$-patches to the $\partial\mathcal{O}_\mathcal{C}$-patches, such that for any $\partial\mathcal{R}_\mathcal{C}$-patch $K$, $K$ is homotopy equivalent to $Q(K)$.

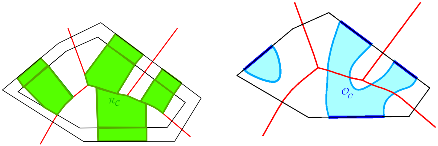



$\mathcal{R}_\mathcal{C}$ is homotopy equivalent to $\mathcal{O}_\mathcal{C}$.

# Conclusion

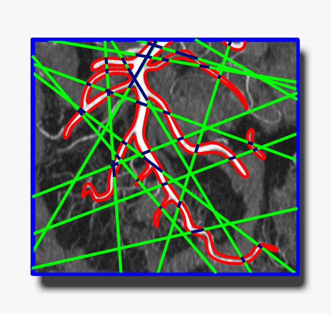


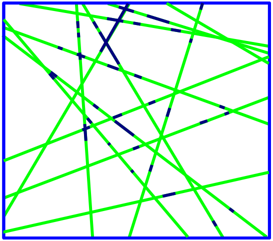
- Generalization of the classical overlapping criterion
- Topologically correct solution for the correspondence problem
- Justification of most of existing methods in parallel case




**Thank you**

# 2D Problem





# 2D Method