# Manifold Reconstruction using Tangential Complexes

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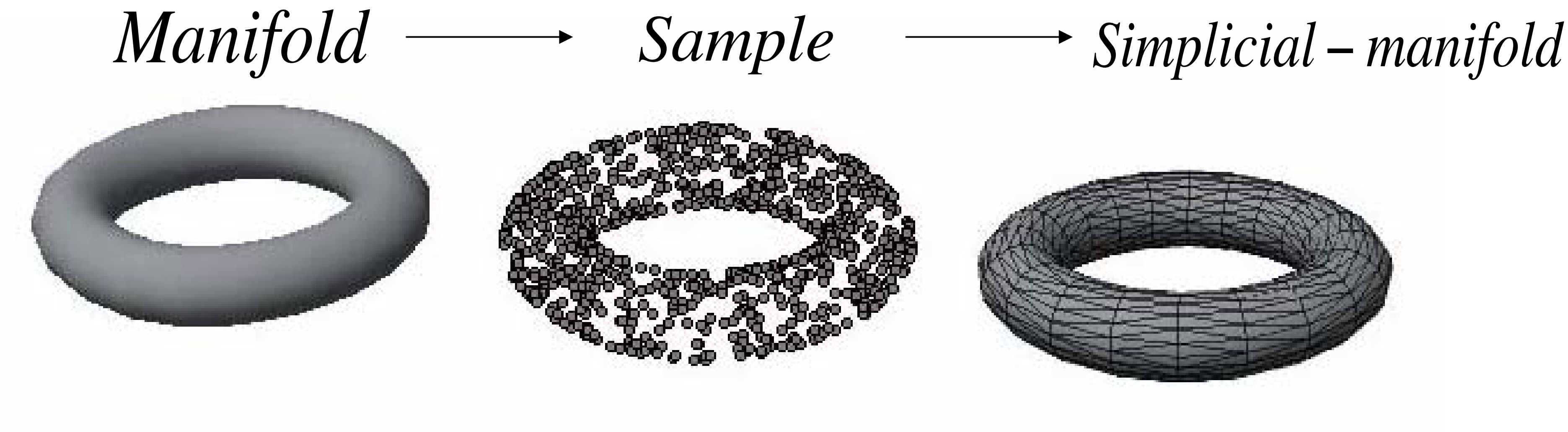
# Content of the talk:

- Problem definition
- Previous works
- Background and preliminaries
- Tangential Complex
- Slivers and inconsistencies
- Algorithm and complexity
- Conclusion

# Problem definition:

- Given a sufficiently dense point sample from a smooth $k-$ dimensional manifold $M$, we want to construct a simplicial manifold that is homeomorphic and geometrically close to $M$.



# Additional assumptions…

- We assume that we know the dimension of the manifold and also tangent space at each sample point, i.e we know $T_p$ for all point p of the point sample.
- In practice neither the dimension nor the tangent space at the sample points are not known but we will deal with this issue later.

# Previous work

When $M \subset R^3$

1. Insert M with a grid or triangulation and deduce a PL approximation of M. [Boissonnat et al.], [Plantiga et al. ]
2. Refine the 3D Delaunay triangulation of an initial small sample and extract the Delaunay triangulation of the sample restricted to M. If we extend this method directly then complexity of the algorithm will be exponential in the dimension of the embedding space.

[Boissonnat, Oudot] [Dey et al.]

# Related previous work…

1. Reconstruction of a k-manifold M of $R^d$

Assuming that the manifold is dense enough, a good PL approximation can be obtained in time

$cn\log n$ (Cocone complex) [Cheng, Dey, Ramos 05]

$c'n^2$ (Witness complex) [B., Guibas, Oudot 07]

$c, c'$ depends exponentially on d

2. Retrieving Homology of manifold M

$c''n^4$ constant depends linerly on d [Chazal, Oudot 08]

# Background

- Weighted points: A weighted point is a pair consisting of a point $p$ of $P$, called the center of the weighted point, and a non-negative real number $\omega(p)$. Relative amplitude of $\omega$, denoted as $ramp(\omega)$ is

$$\max_{p \in P, q \in P \setminus \{p\}} \frac{\omega(p)}{\| p - q \|}$$

From this point onwards we assume $ramp(\omega) \leq \omega_0$ for some $\omega_0 \in [0,1/2)$

$$\omega(p) \in [0, \omega_0^2 L(p)]$$

## …

- Weighted Voronoi diagram: Weighted Voronoi cell of $p$ is

$$V^{\omega}(p) = \{x \in R^d : \| p - x \|^2 - \omega^2(p) \leq \| q - x \|^2 - \omega^2(q), \forall q \in P\}.$$

Dual of the weighted voronoi diagram is the weighted Delaunay diagram, denoted by $Del^{\omega}(P)$.

## …

- Adaptive neighborhood: For a constant $\alpha > 1$, the adaptive neigborhood

$$N_\alpha(p) = \{q \in P : \| p - q \| \leq \alpha \min_{x \in P \setminus \{p\}} \| p - x \|\}.$$

# Our approach

Given a $(\varepsilon,\delta)$-sample $P$ of $M$ satisfying $\varepsilon_0 > \varepsilon > \delta \geq \rho_0\varepsilon > 0$

We do the following:

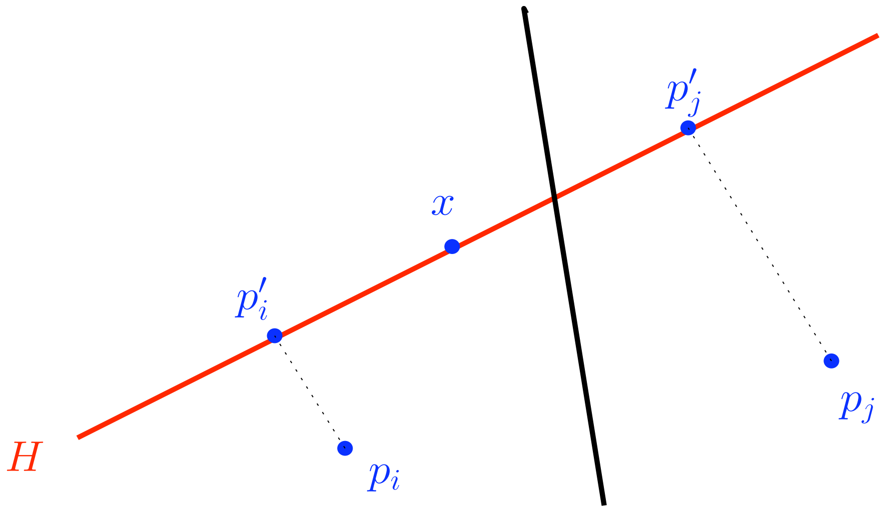
1. Calculate $\omega(p), \forall p \in P$ s.t star of $p$ in w.Delaunay triangulation restricted to $T_p$, denoted by $Del^{\omega}_{T_p}(P)$, does not contain any slivers and $Del^{\omega}_{TM}(P)$ does not contain any inconsistencies.

$Del^{\omega}_{TM}(P)$: the set of stars of $p, p \in P$

Inconsistent simplex: a simplex that does not appear in the star of all its vertex.

2. Output $Del^{\omega}_{TM}(P)$

# To construct the stars...

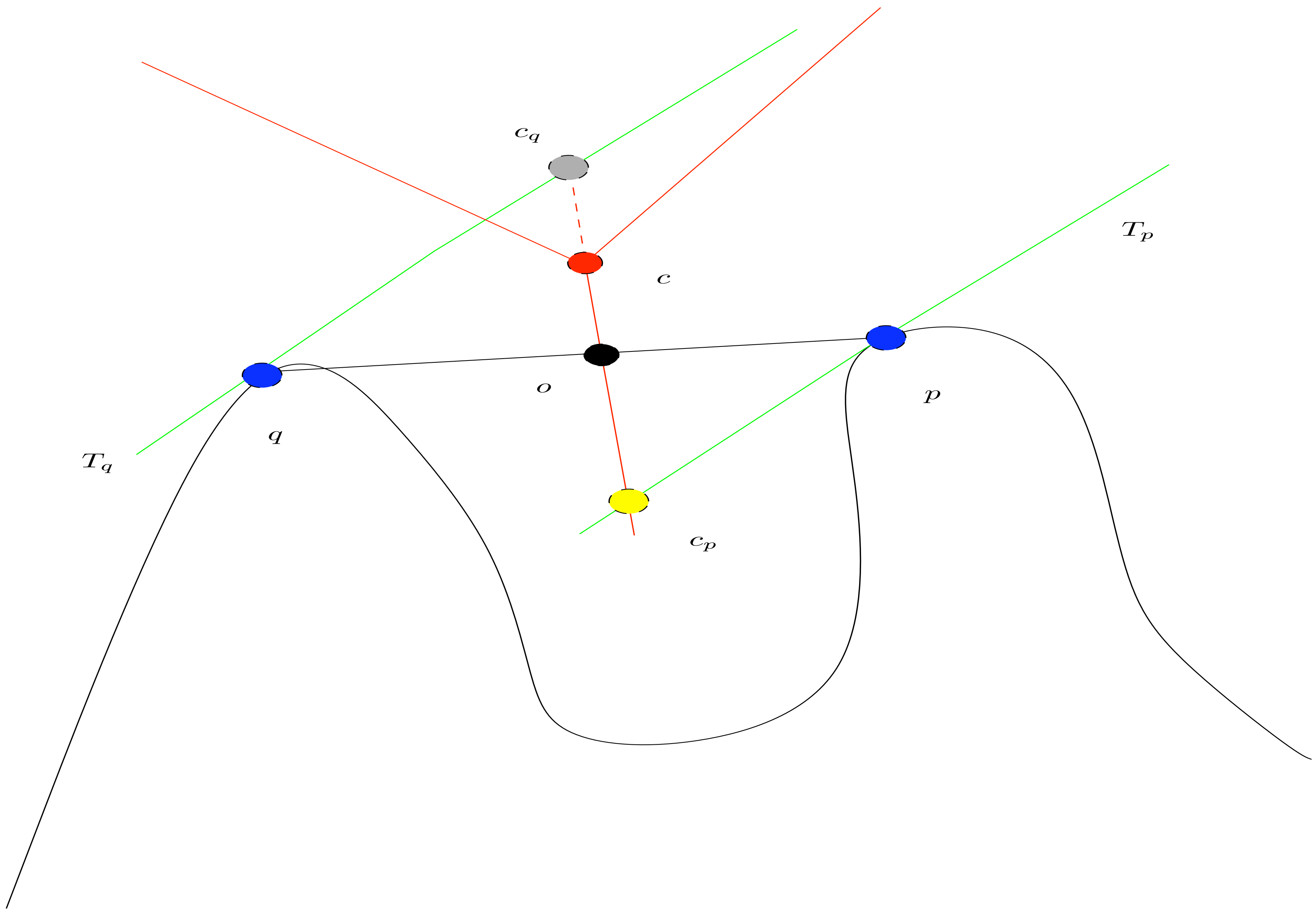



$$\| x - p_i \|^2 - \omega^2(p_i) \le \| x - p_j \|^2 - \omega^2(p_j)$$
$$\Leftrightarrow \| x - p_i' \|^2 - \omega^2(p_i) - \| p_i - p_i' \|^2$$
$$\le \| x - p_j' \|^2 - \omega^2(p_j) - \| p_j - p_j' \|^2$$

$V^\omega(P) \cap T_p$ and therefore $Del_{T_p}(P)$ can be computed by

1. projecting the points on $T_p$ which takes O(dn) time.
2. constructing a Laguerre diagram/regular triangulation in k-space.

# Inconsistencies...



## …

- Note that c is the center of the orthosphere that is orthogonal to vertices of s (is the k-simplex) and also some weighted point r.
- (s,r) is called inconsistent configuration and our goal is to remove all such configurations.

# Adap. Neighborhood

1. If $\alpha \geq 54/\rho_0$ , then adaptive neighborhood $N_\alpha(p)$ contains all the simplices of $Del_{TM}^{\omega}(P)$ that are incident to p and also all the simplices that form inconsistent configuration with p.

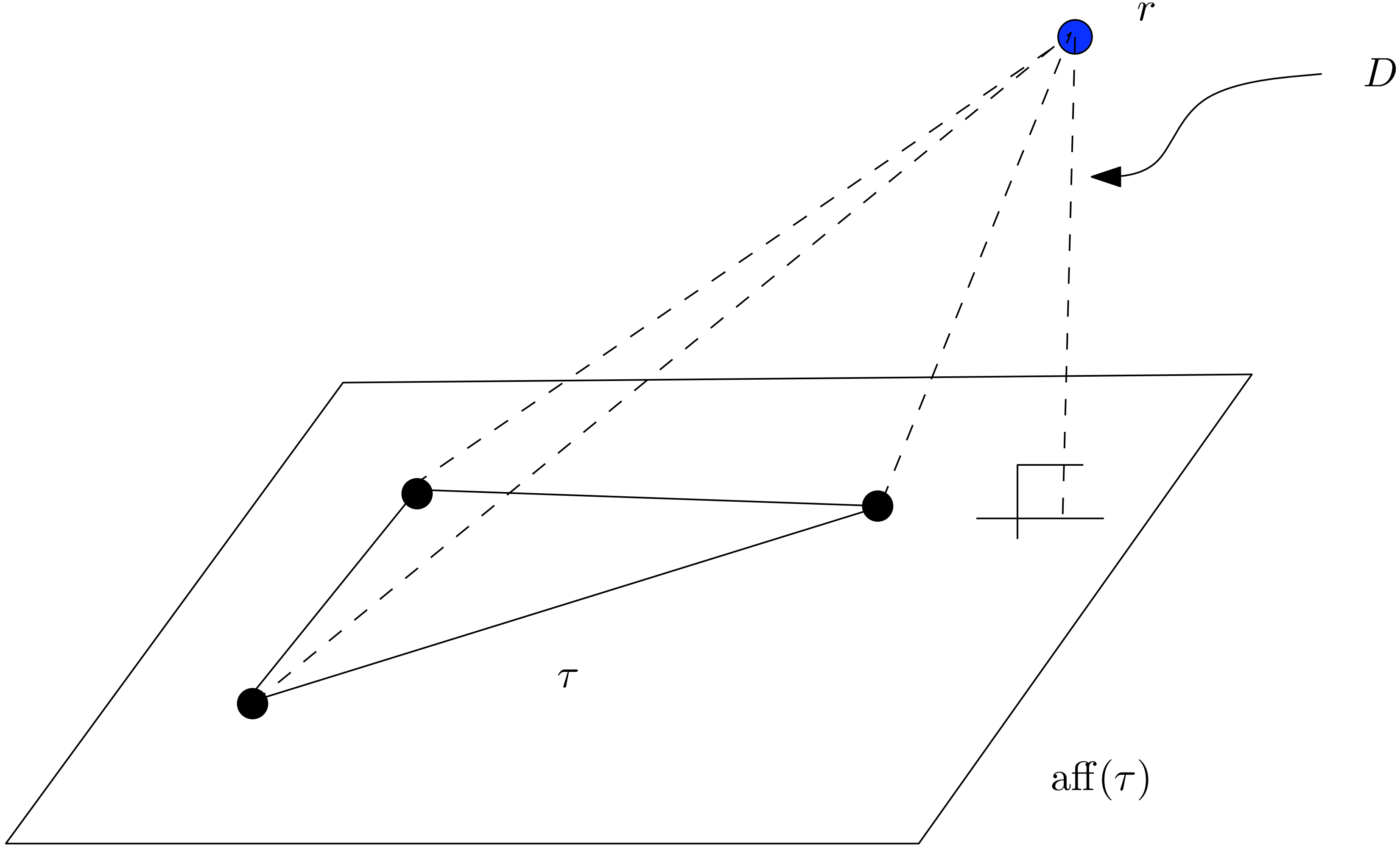
2. $\# N_\alpha(p) = 2^{O(k)}$

3. Observation: $Del_{T_p}^{\omega}(P) = Del_{T_p}^{\omega}(N_\alpha(p))$

# Inconsistencies...

1. Candidate j-slivers of p is a j-dimensional simplex made up of p and points from the adaptive neighborhood of p and it is a sliver.
2. 2. If sliverity ratio $\sigma_0$ is small enough then we can remove the candidate slivers by pumping at p. [Cheng et al.]
3. In a similar way we define candidate inconsistent configuration and show that there exists weight assignment that will remove the candidate inconsistent configuration.

# Cand. Inconsistency …



Theorem: Algorithm terminates without any candidate inconsistencies or slivers if

1. $\omega_0 \in [0,1/2)$
2. $\varepsilon_0 > \varepsilon > \delta \geq \rho_0 \varepsilon > 0$
3. $\sigma < \sigma_0 = f(\rho_0, \omega_0)$

Thm: Removal of candidate slivers and inconsistencies implies $Del_{TM}^{\omega}(P)$ has no inconsistencies or slivers.

# Basic operations

1. Computing weight of sample points in the adaptive neighborhood.
2. Projecting a point on a k-flat.
3. Calculating star at each point.
4. Merging of the stars at the end to obtain the final output, $Del_{TM}^{\omega}(P)$.

# Complexity

- Time to build the adaptive neighborhood is $dn^2$.
- Weight calculation takes $d2^{O(k^2)}$ for each point.
- Calculation of star takes $d2^{O(k^{3/2})}$ for each point.
- In practice dimension of the manifold or the tangent spaces at each sample point is not know. But algorithms by Cheng-Wang-Wu and Giesen-Wagner can be used to estimate the dimension and approx. tangent spaces.
- Complexity of their algorithm is linear in size of the sample and d. And our algorithm is robust for approx tangent space.

# Properties of the output

Upon termination, the are coherent (i.e no inconsistent configuration), the simplices are small and there is no sliver.

Hence $Del^{\omega}_{TM}(P)$

- is a PL k-manifold
- homeomorphic to $M$
- geometrically close to $M$, i.e $d(M, Del^{\omega}_{TM}(P)) = O(\varepsilon^2)$ and $\angle(N_\tau, N_p) = O(\varepsilon)$

# Conclusion

Differents variants ….

1. Locally uniform anisotropic mesh generation in any dimension [B., Wormser, Yvinec 08]
2. Manifold Reconstruction from Point Sample [Cheng et al.]
3. Sampling and meshing of manifold in higher dimensional space

Work in progress:

1. Implementation of the current algorithm
2. Extension of the algorithm to non-sparse samples

Thank you…