an introduction to

zigzag persistence

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Topology

What is topology?

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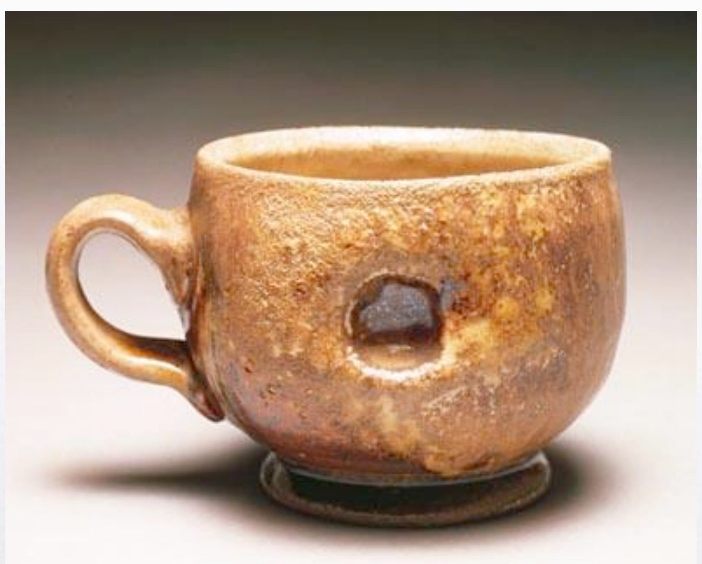
It is the branch of mathematics which does not distinguish between a teacup and a bagel

one popular answer



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Digression: what is topology?

Topology gives answers to qualitative questions

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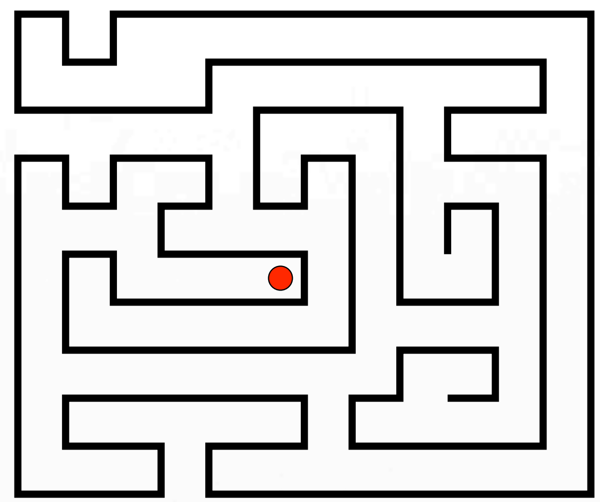
Topology gives answers to qualitative questions



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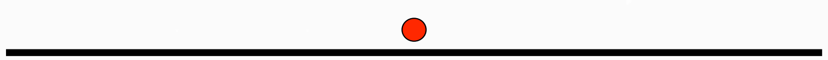
Topology gives answers to qualitative questions



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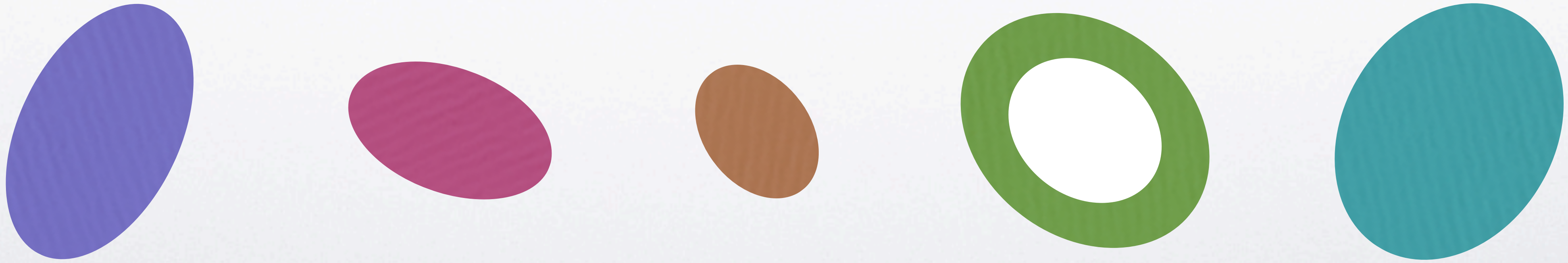

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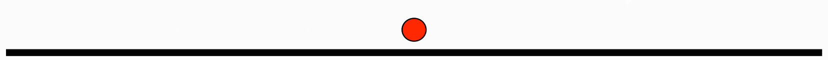
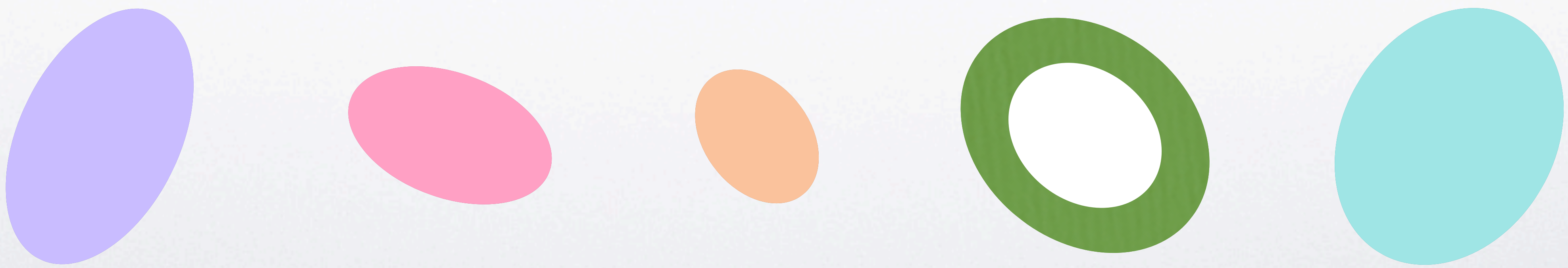
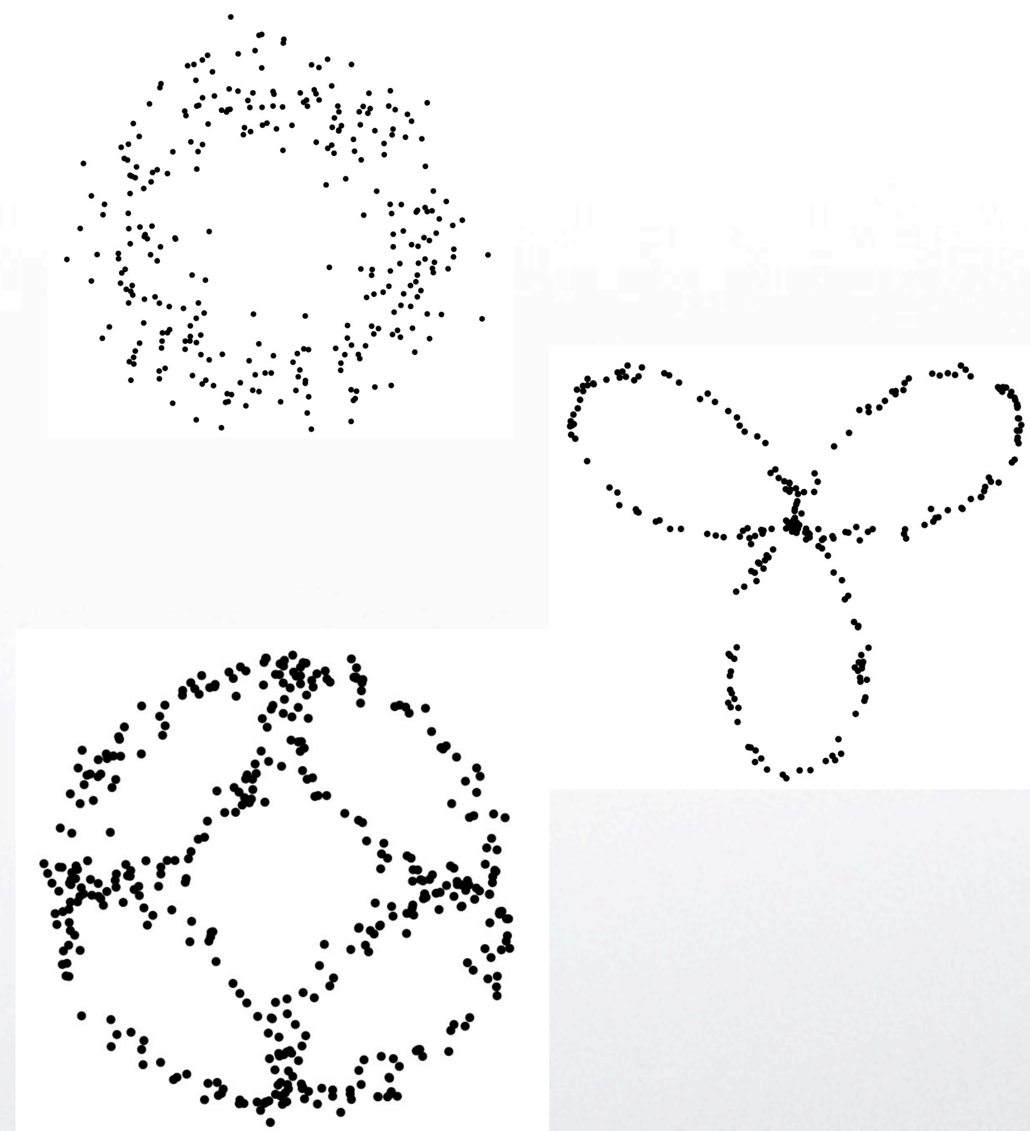




Point-cloud topology

- Topological structure in statistical data
  - density estimation and modality
  - approximation by simplicial complexes
- Assume data have been sampled from some unknown space
  - can we measure topological features of the hidden space?
  - can we assign confidence values to these measurements?
- What does "topology" mean for a cloud of data points?
  - persistent homology
  - spectral theory for point clouds



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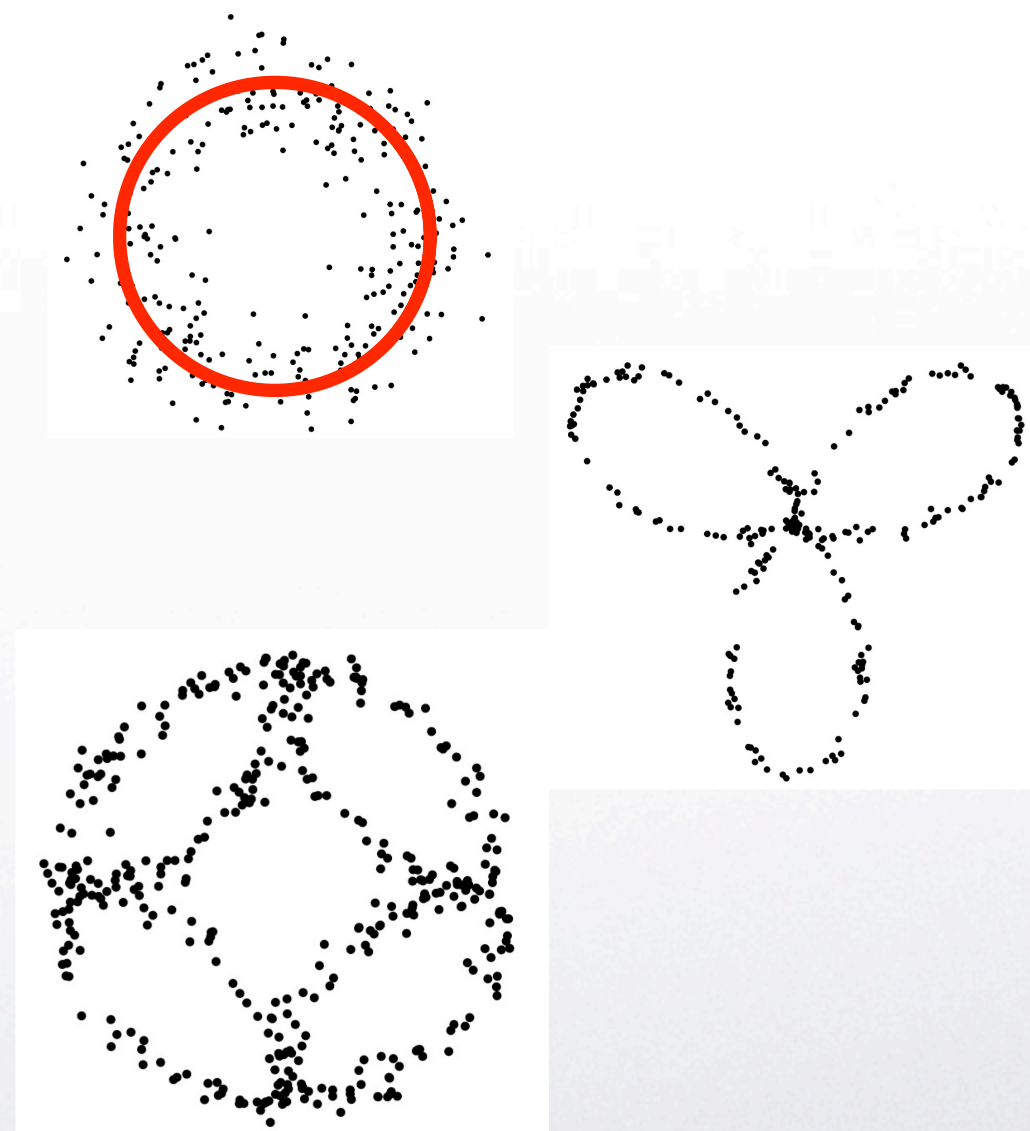
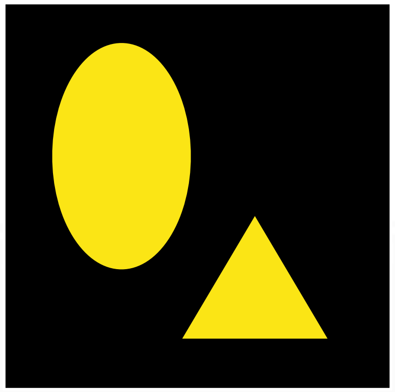


Point-cloud topology

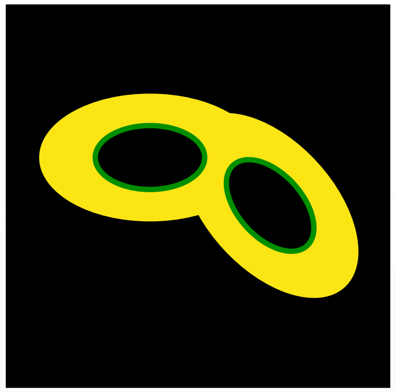
- Algebraic topology measures qualitative features of a space X
  - How many components?
  - How many tunnels/voids?
  - How do paths and loops deform within X?
- These are measured by algebraic invariants
  - fundamental group $\pi_1(X)$
  - homology groups $H_k(X)$ and Betti numbers $b_k(X)$
  - products $H_j(X) \times H_k(X) \to H_{j+k}(X)$
- Can we compute these invariants from a finite sample $Y \subset X$?

Betti numbers ↔ features

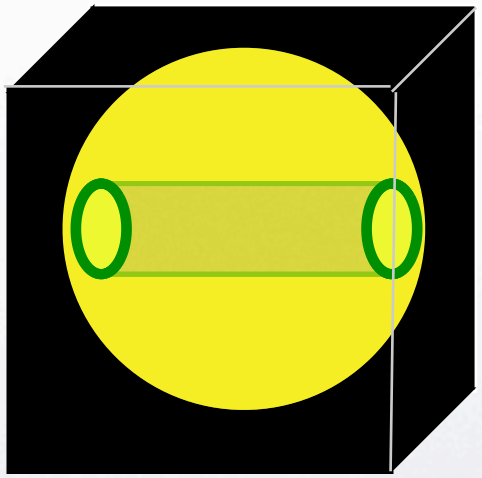
- For an object in 2D space
  - b_0 is the number of components
  - b_1 is the number of holes
- For an object in 3D space
  - b_0 is the number of components
  - b_1 is the number of tunnels or handles
  - b_2 is the number of voids
- (and so on, in higher dimensions)



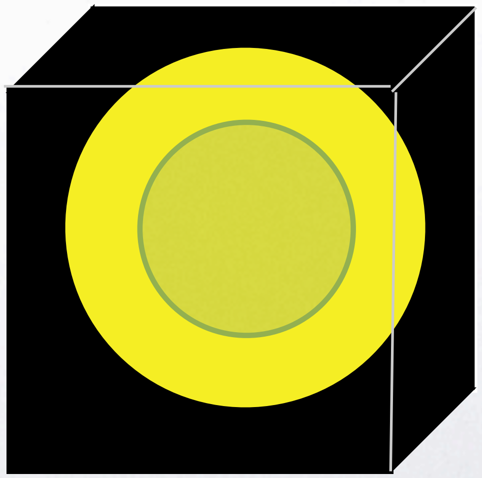
$b_0 = 2, b_1 = 0$



$b_0 = 1, b_1 = 2$



$b_0 = 1, b_1 = 1, b_2 = 0$



$b_0 = 1, b_1 = 0, b_2 = 1$

Standard Pipeline (first attempt)

hidden/unknown space X

↓

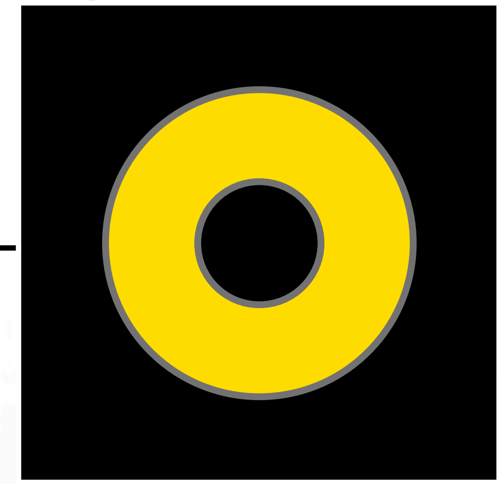
finite sample $Y \subset X$

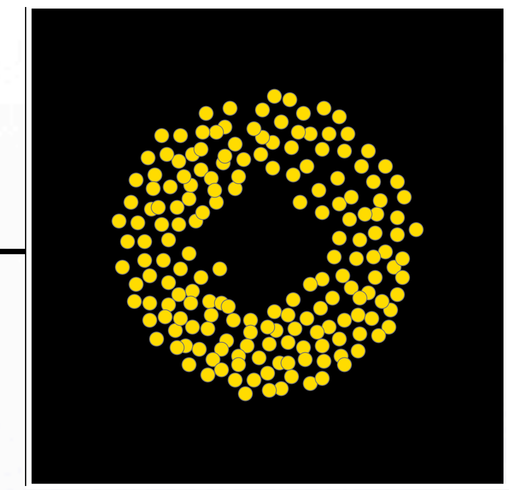
↓

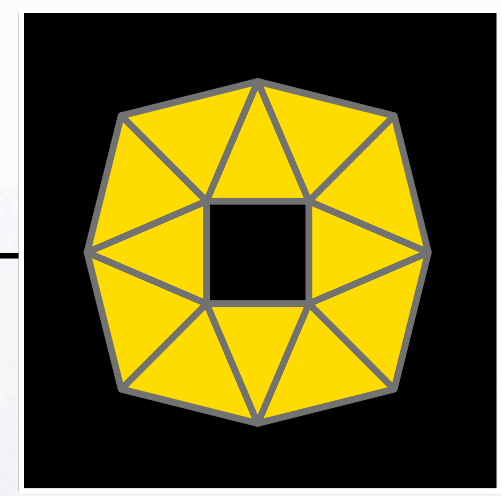
simplicial complex $S = S(Y)$

↓

homology invariants of S







$b_0 = 1$

$b_1 = 1$

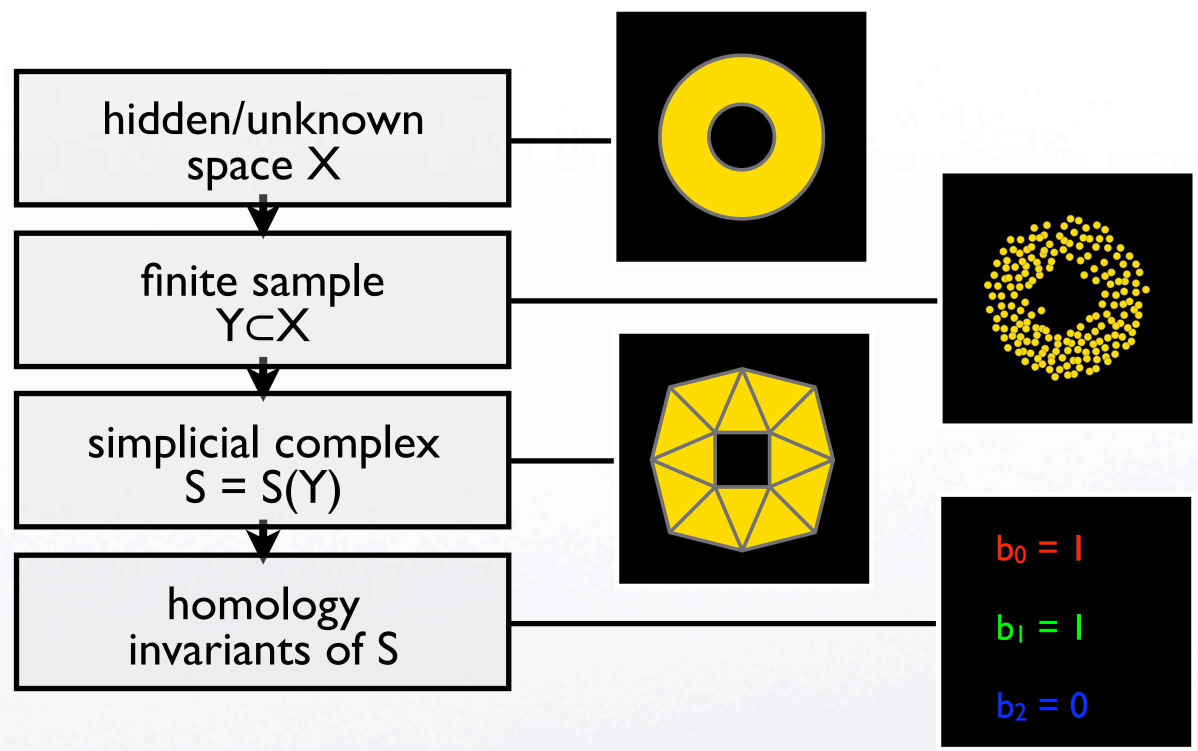
$b_2 = 0$

Reconstruction theorems

- Various constructions for S(Y)
  - Cech complex (folklore)
  - Rips–Vietoris complex (folklore)
  - α-shape complex (Edelsbrunner, Mücke)
  - witness complexes (Carlsson, dS)
- Desire theorems of the form:

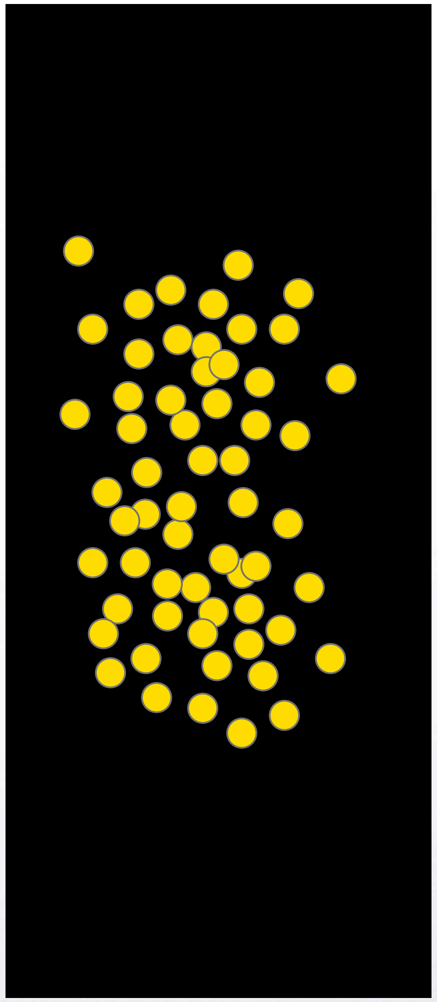
If Y is well-sampled from X then S(Y) ≈ X

- e.g. Niyogi–Smale–Weinberger (2004), Cech complex



Discrete vs continuous

- Betti numbers are discrete
- Topological spaces
  - topological spaces are continuous
  - the space of topological spaces is discrete
- Finite point-clouds
  - point-clouds are discrete
  - the space of point-clouds is continuous
- Therefore, raw Betti numbers are
  - ✔ very handy for topological spaces
  - ✘ a bit dangerous for point-clouds



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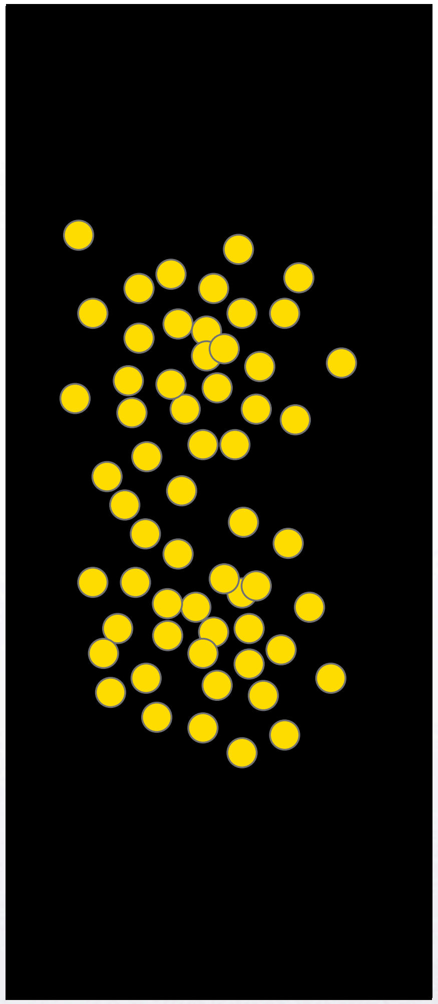

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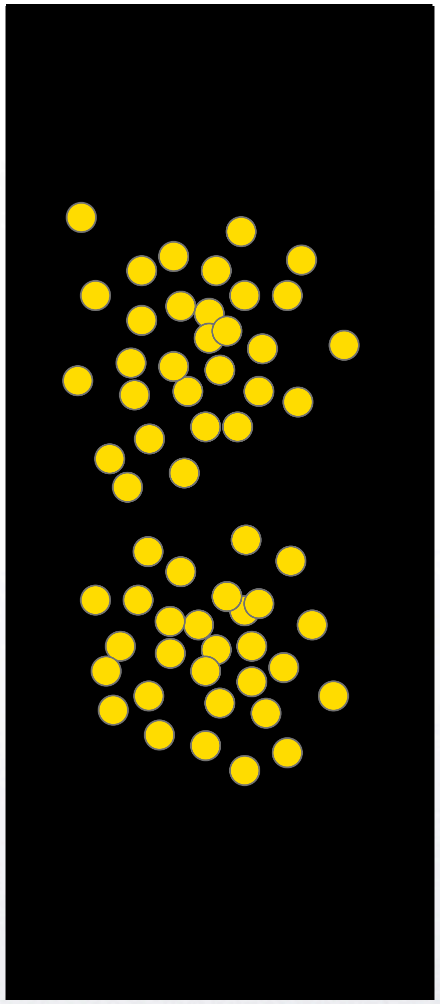

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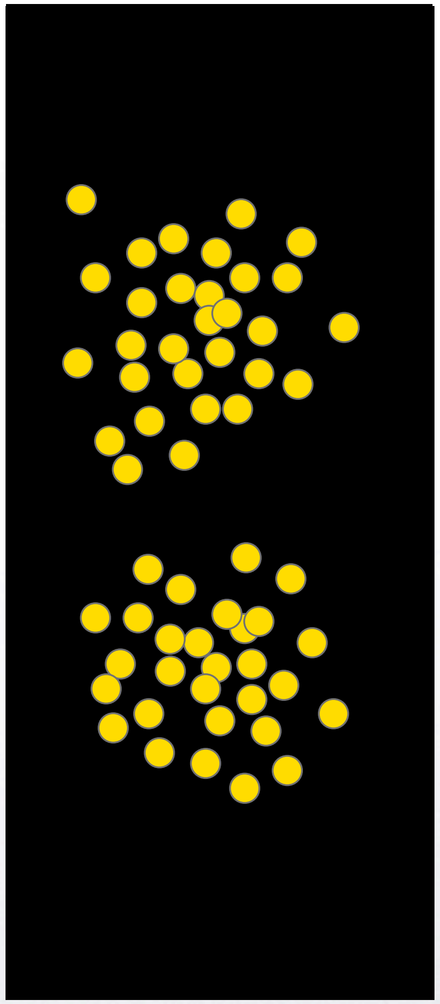

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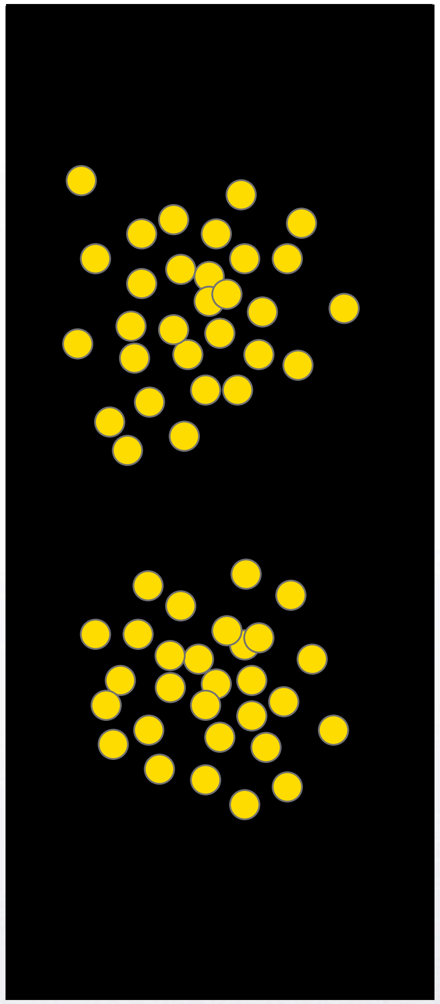

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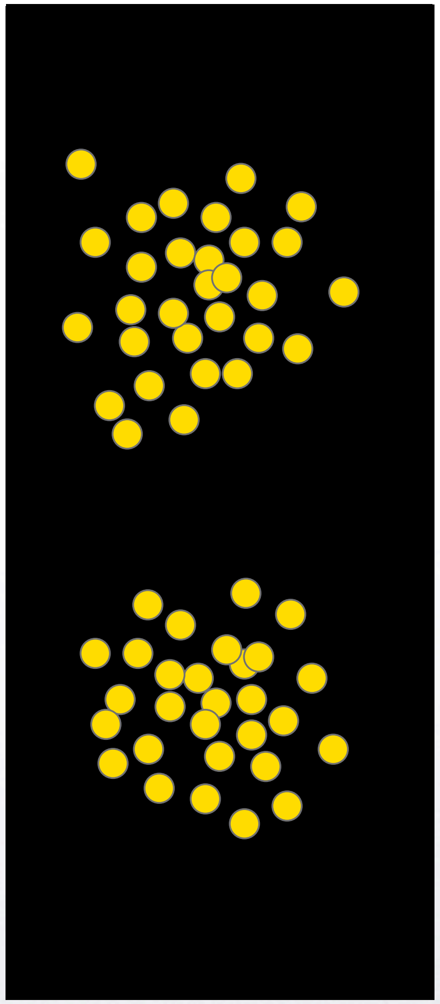


Persistence pipeline

hidden/unknown space X

↓

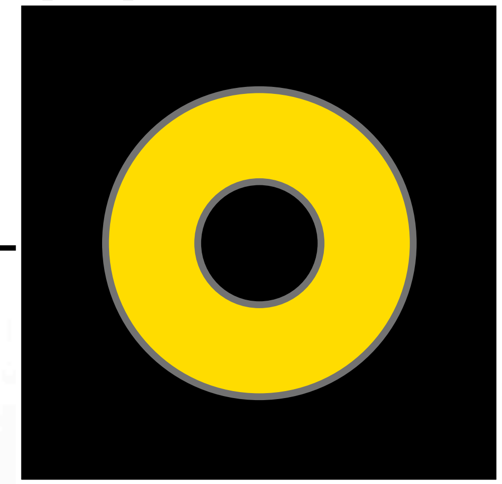
finite sample Y⊂X

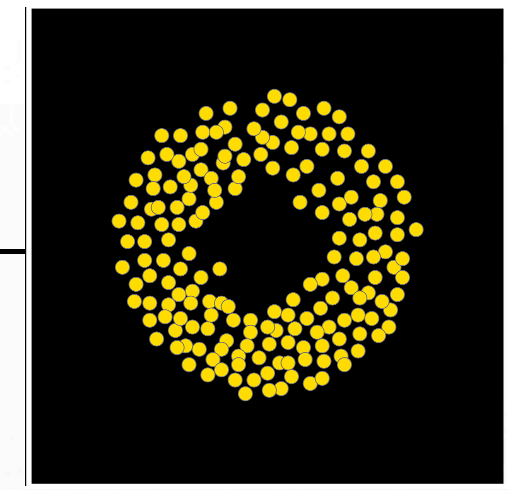
↓

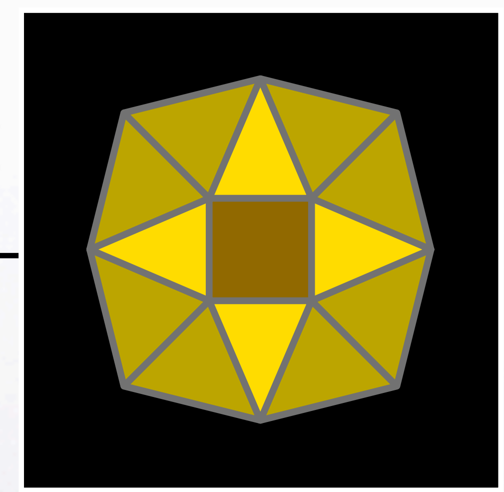
filtered complex $S(r) = S(Y,r)$

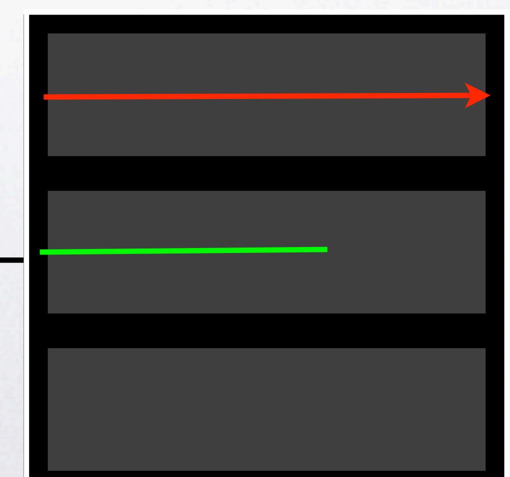
↓

persistent homology of $S(r)$









Persistence pipeline

finite sample
Y⊂X

filtered complex
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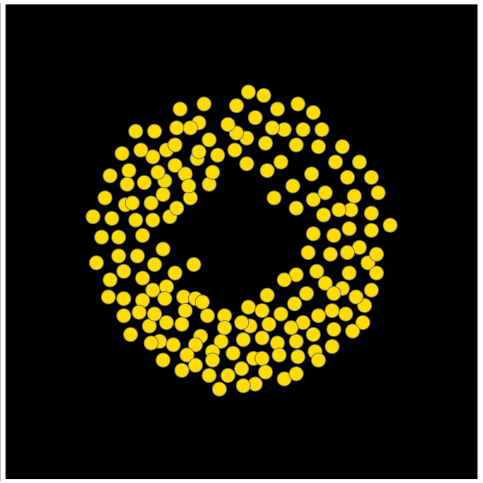
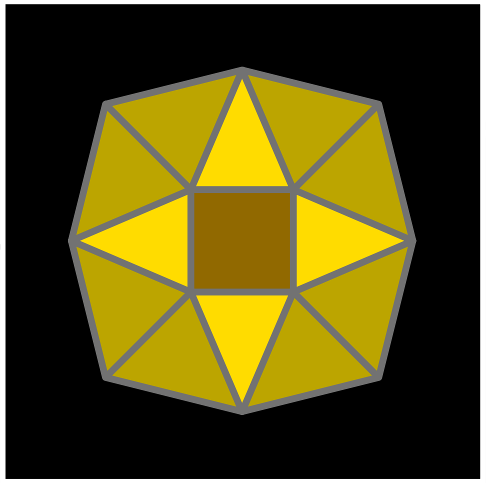




- Cech complex

$$\sigma = [a_0, \ldots, a_k] \in \check{\text{C}}\text{ech}(X, \epsilon) \Leftrightarrow \bigcap_{i=0}^{k} B_\epsilon(a_i) \neq \emptyset$$

- Rips complex

$$\sigma = [a_0, \ldots, a_k] \in \text{Rips}(X, \epsilon) \Leftrightarrow |a_i - a_j| \leq \epsilon, \forall i, j$$

And the Oscar goes to...

video by Afra Zomorodian

Persistence

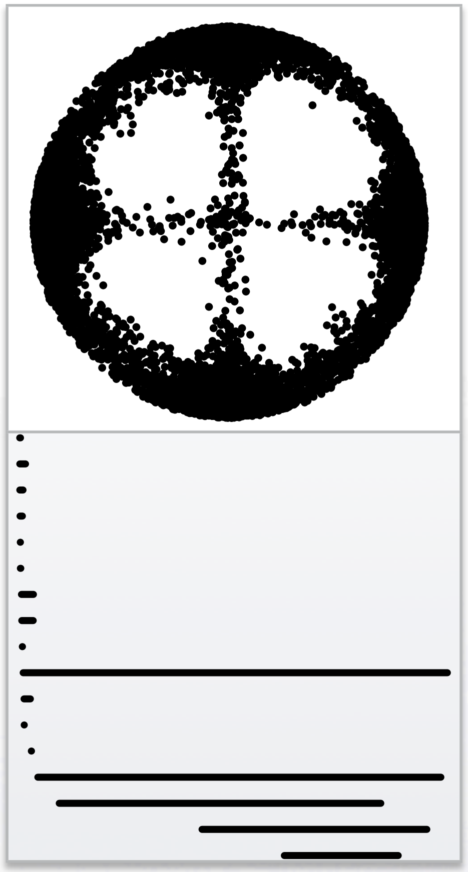
- Monotone increasing family of spaces

$$\mathbf{X} = \{X_\epsilon \mid \epsilon \geq 0\} \quad \text{such that} \quad X_\epsilon \subseteq X_{\epsilon'} \text{ if } \epsilon \leq \epsilon'$$

- Persistent homology

$$\text{rank}\,[H_*(X_\epsilon) \to H_*(X_{\epsilon'})] \quad \text{for all } \epsilon \leq \epsilon'$$

- Barcode (Edelsbrunner, Letscher, Zomorodian '00)
  - finite collection of intervals $[b_i,d_i)$
  - [b,d) indicates feature born at time b, dies at time d
- Stability theorem (Cohen-Steiner, Edelsbrunner, Harer '07)
  - barcode depends continuously on the underlying data
  - see also Chazal, Cohen-Steiner, Glisse, Guibas, Oudot '09
- Continuous measurements (interval length) coupled to discrete information (number of intervals)



Persistence module theory

- Spaces

$$X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n$$

- Persistent homology

$$H_*(X_1) \longrightarrow H_*(X_2) \longrightarrow \cdots \longrightarrow H_*(X_n)$$

- Barcode algebra (Carlsson, Zomorodian '05)
  - H(**X**) is naturally a module over polynomial ring k[t]
  - ⇒ has unique representation as a sum of indecomposables
  - indecomposable summands depicted as barcode intervals
  - calculate decomposition using linear algebra over k[t]

Decomposition into summands

- Examples of indecomposable summands

$$0 \longrightarrow 0 \longrightarrow k \xrightarrow{\mathrm{Id}} k \xrightarrow{\mathrm{Id}} k \longrightarrow 0$$

$$0 \longrightarrow k \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

- Example of decomposable system

$$0 \longrightarrow k \xrightarrow{f} k^2 \xrightarrow{g} k \longrightarrow 0 \longrightarrow 0$$

- (decomposition depends on f,g)

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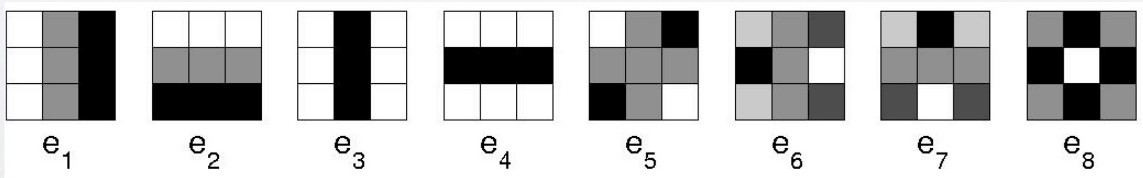
Visual Image Patches

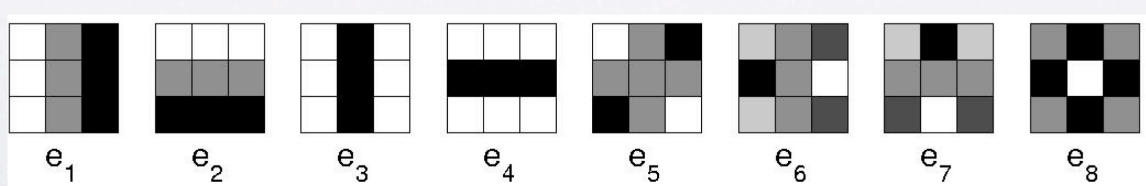
Visual image patches

- Lee, Pedersen, Mumford (2003) studied the local statistical properties of natural images (from Van Hateren's database)
- 3-by-3 pixel patches with high contrast between pixels: are some patches more likely than others?
- Carlsson, VdS, Ishkhanov, Zomorodian (2004,2008): topological properties of high-density regions in pixel-patch space



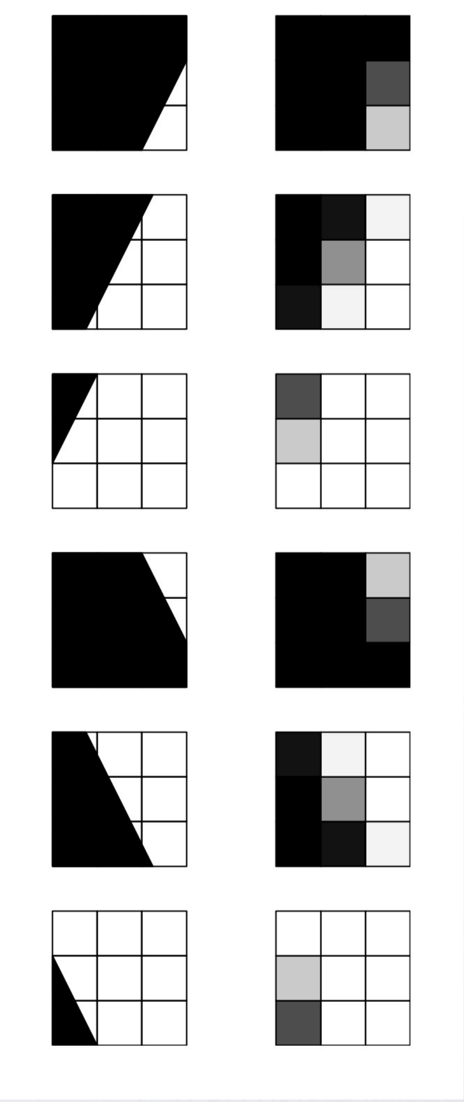
The space of image patches

- ~4.2 million high-contrast 3-by-3 patches selected randomly from images in database.
- Normalise each patch twice: subtract mean intensity, then rescale to unit norm.
- Normalised patches live on a unit 7-sphere in 8-dimensional space with the following basis:



High-density regions

- LPM2003 found that the distribution of patches is dense in the 7-sphere.
- There are high-density regions:
  - edge features
- Can we describe the structure of the high-density regions?
  - threshold by k-nearest-neighbour density estimator



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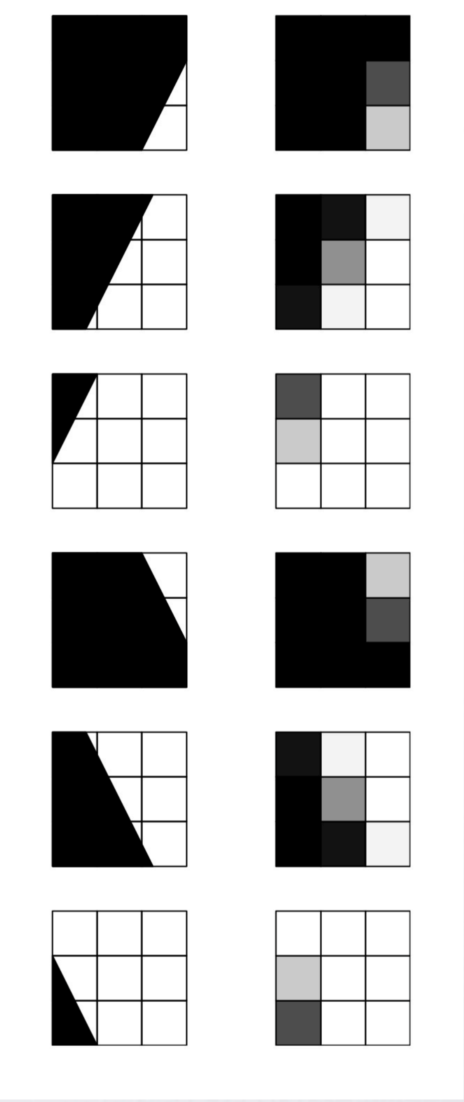
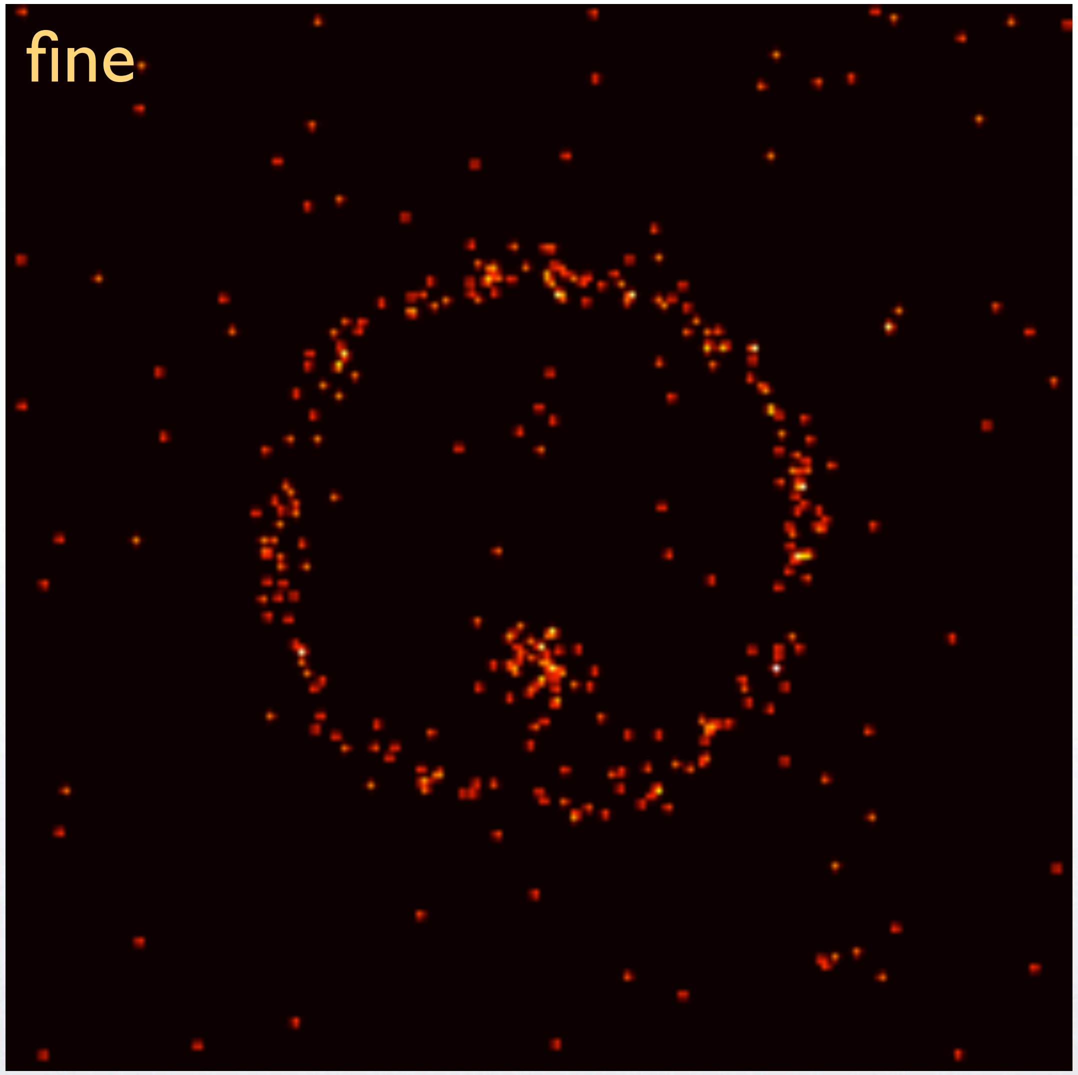


Straining a data soup

Varying the density parameter

(toy example)



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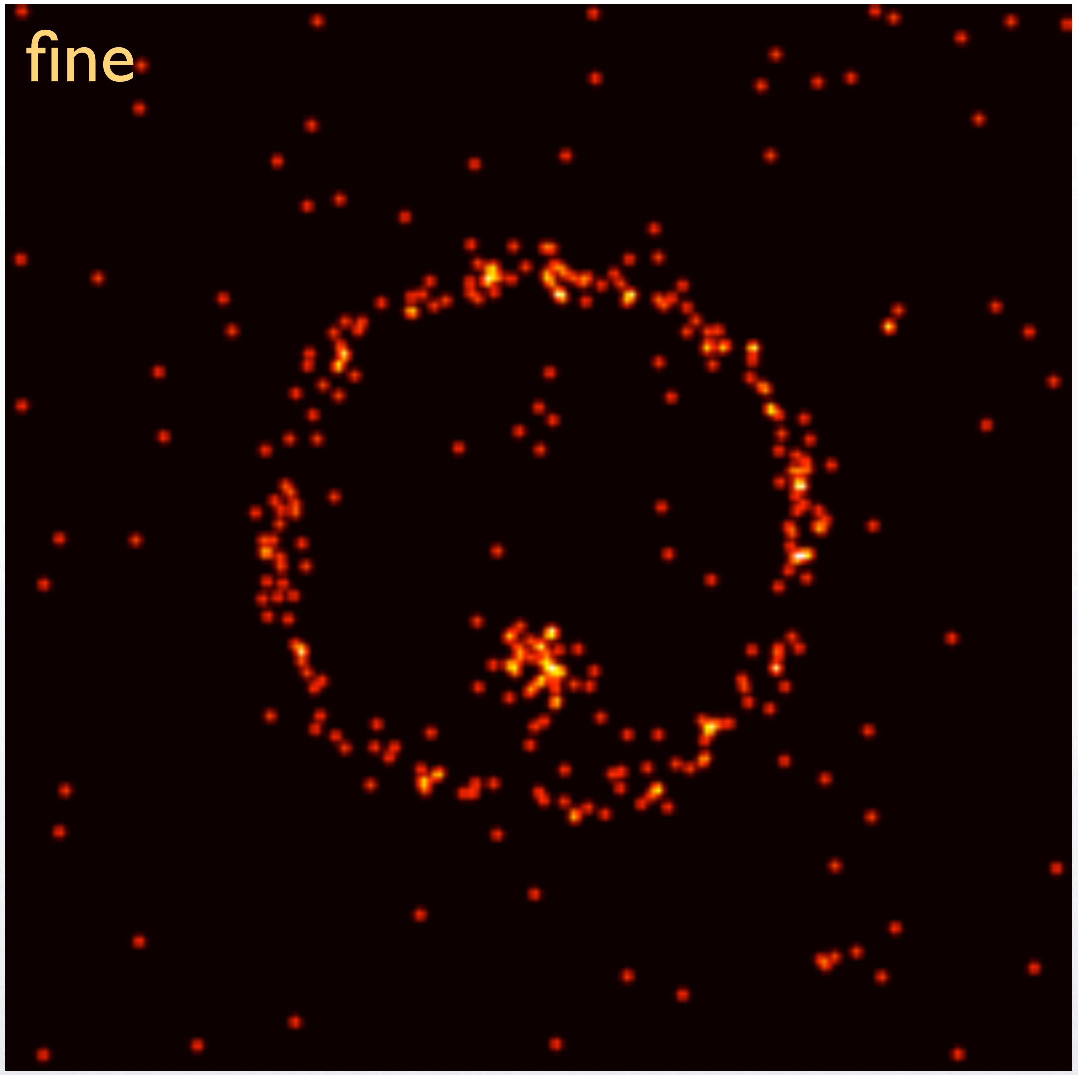

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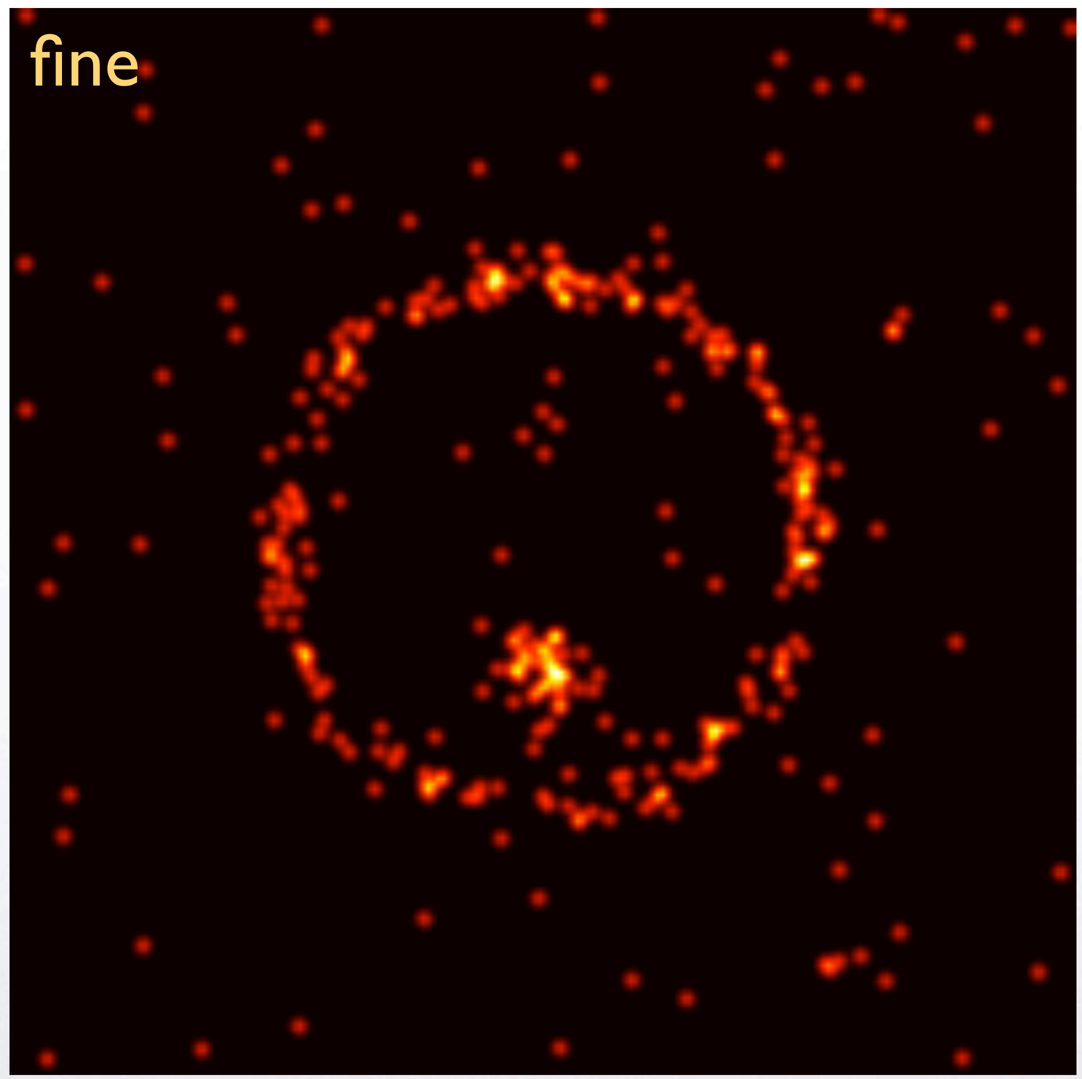

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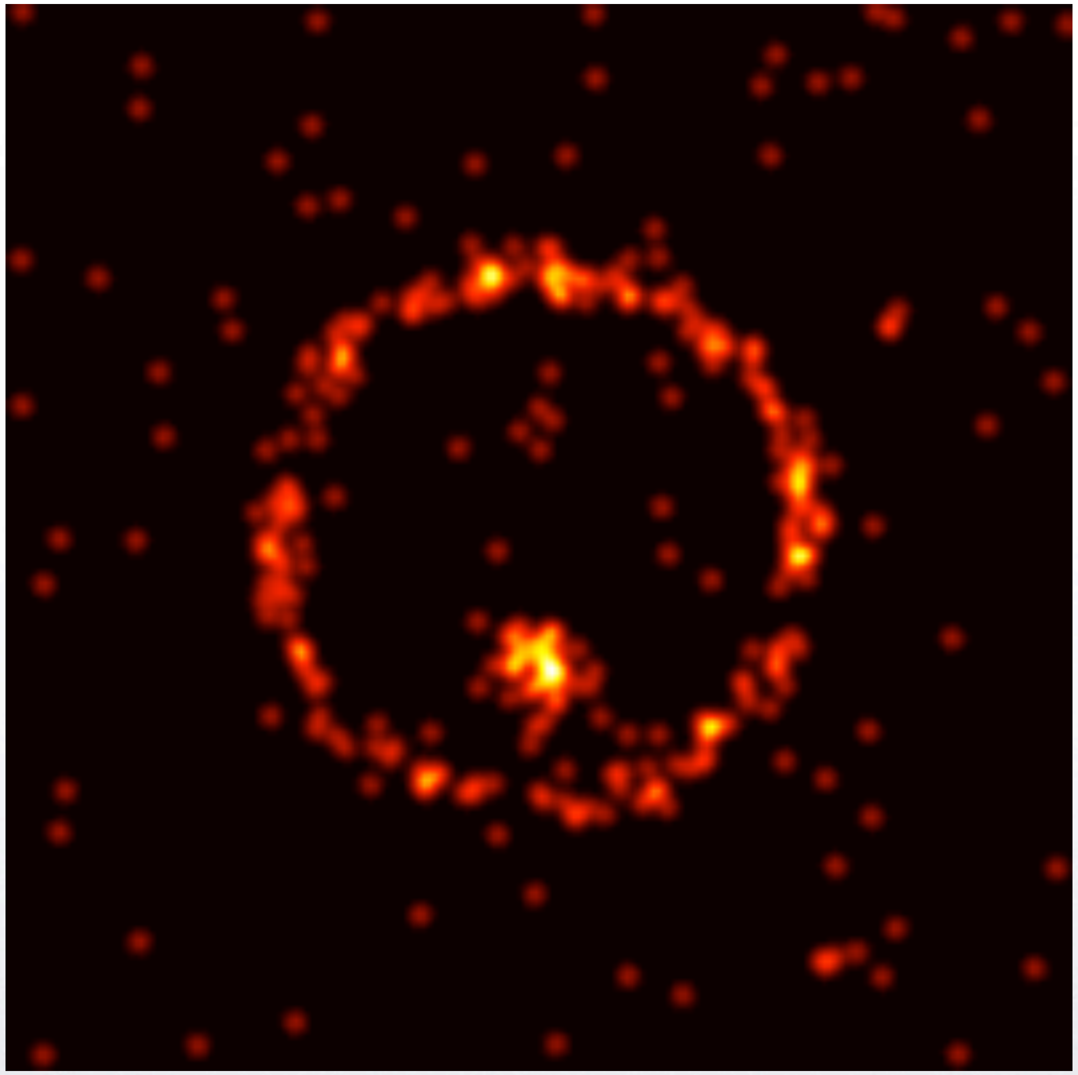

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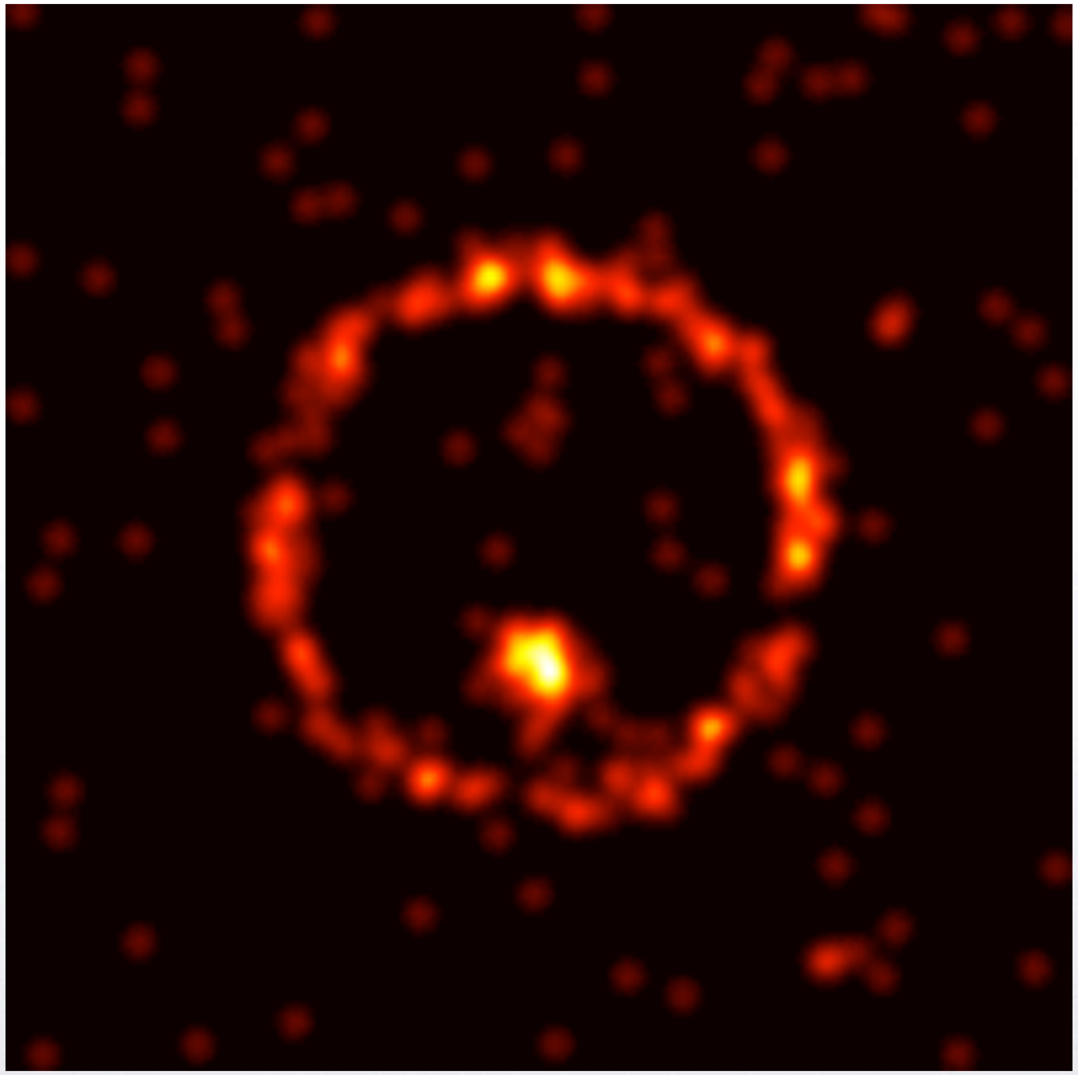

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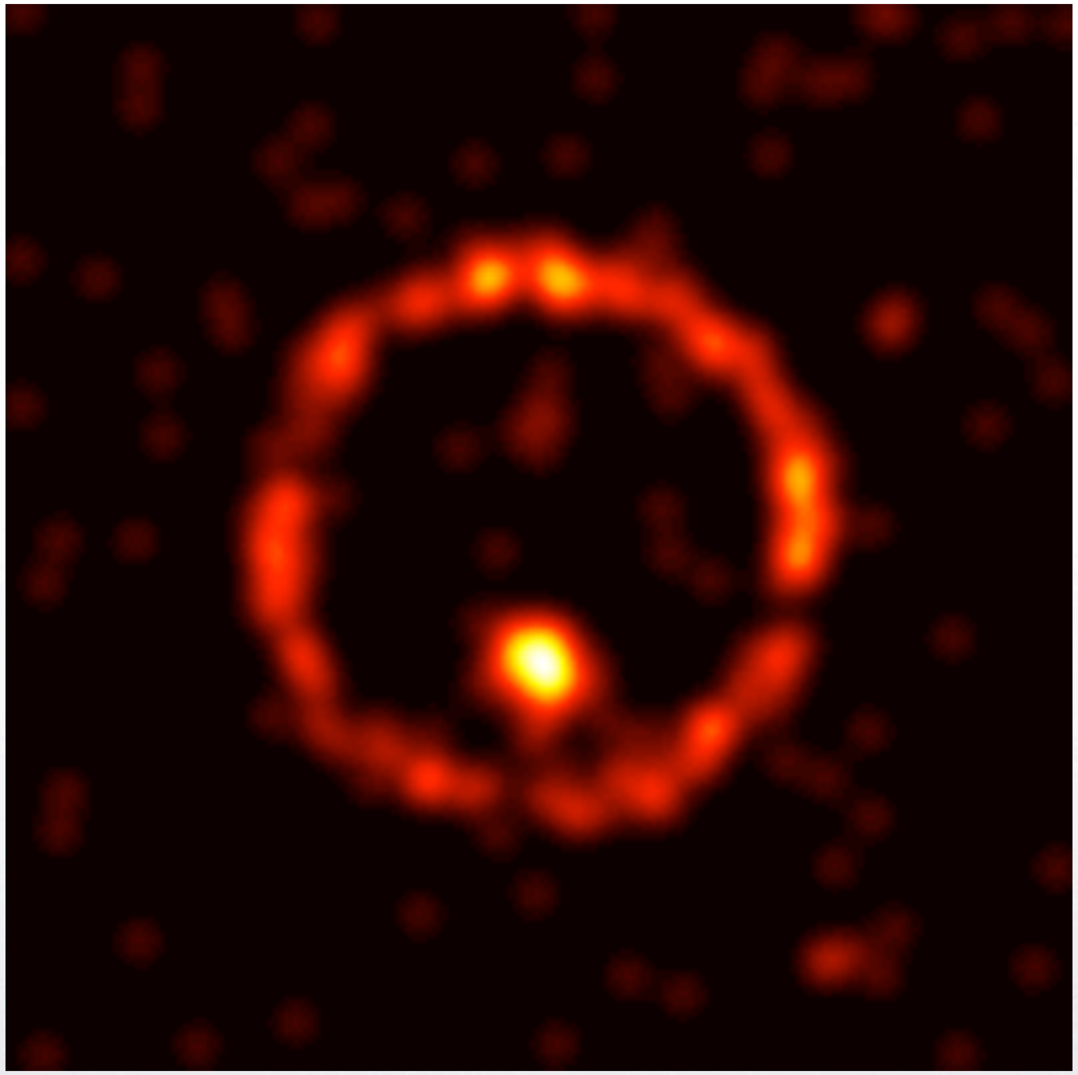

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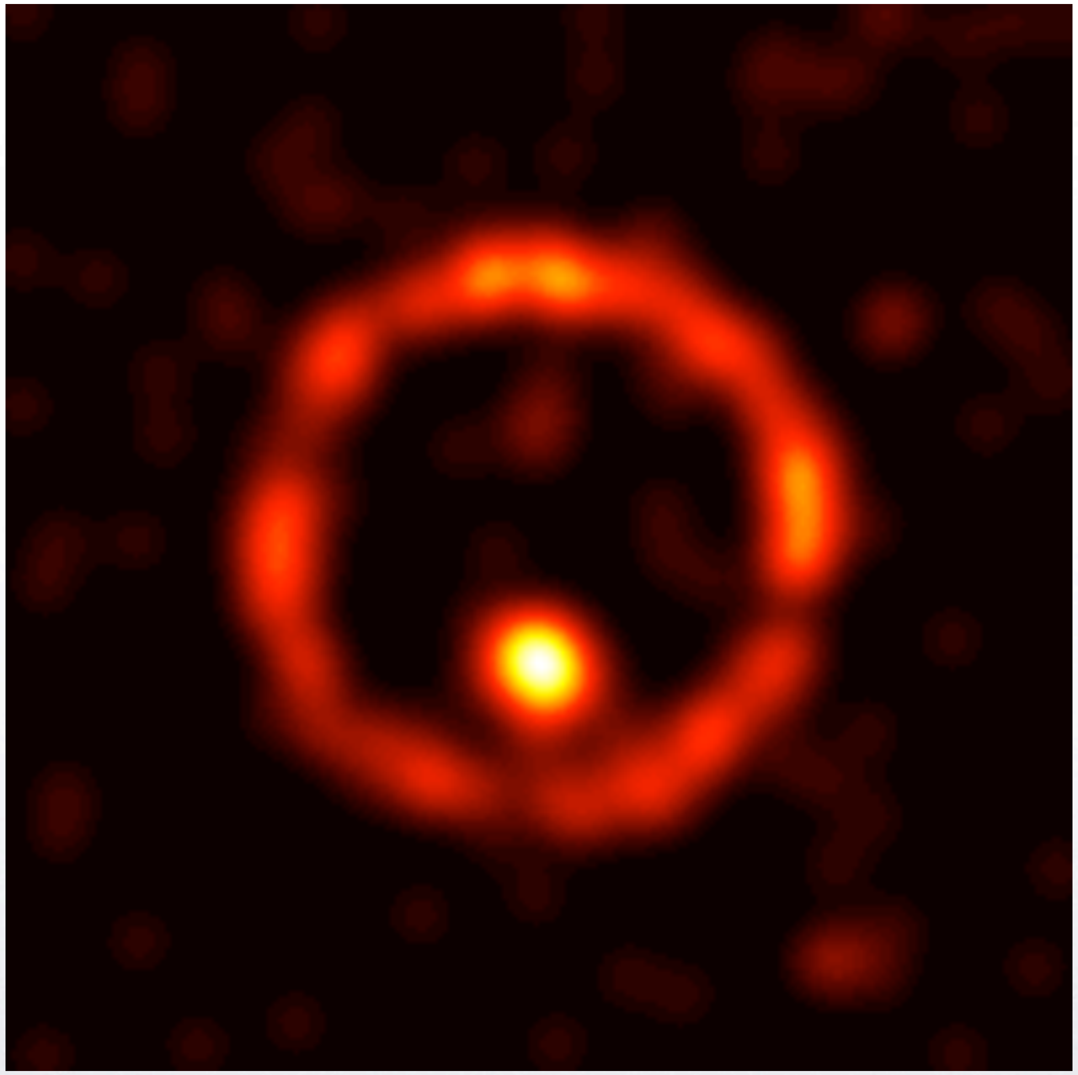

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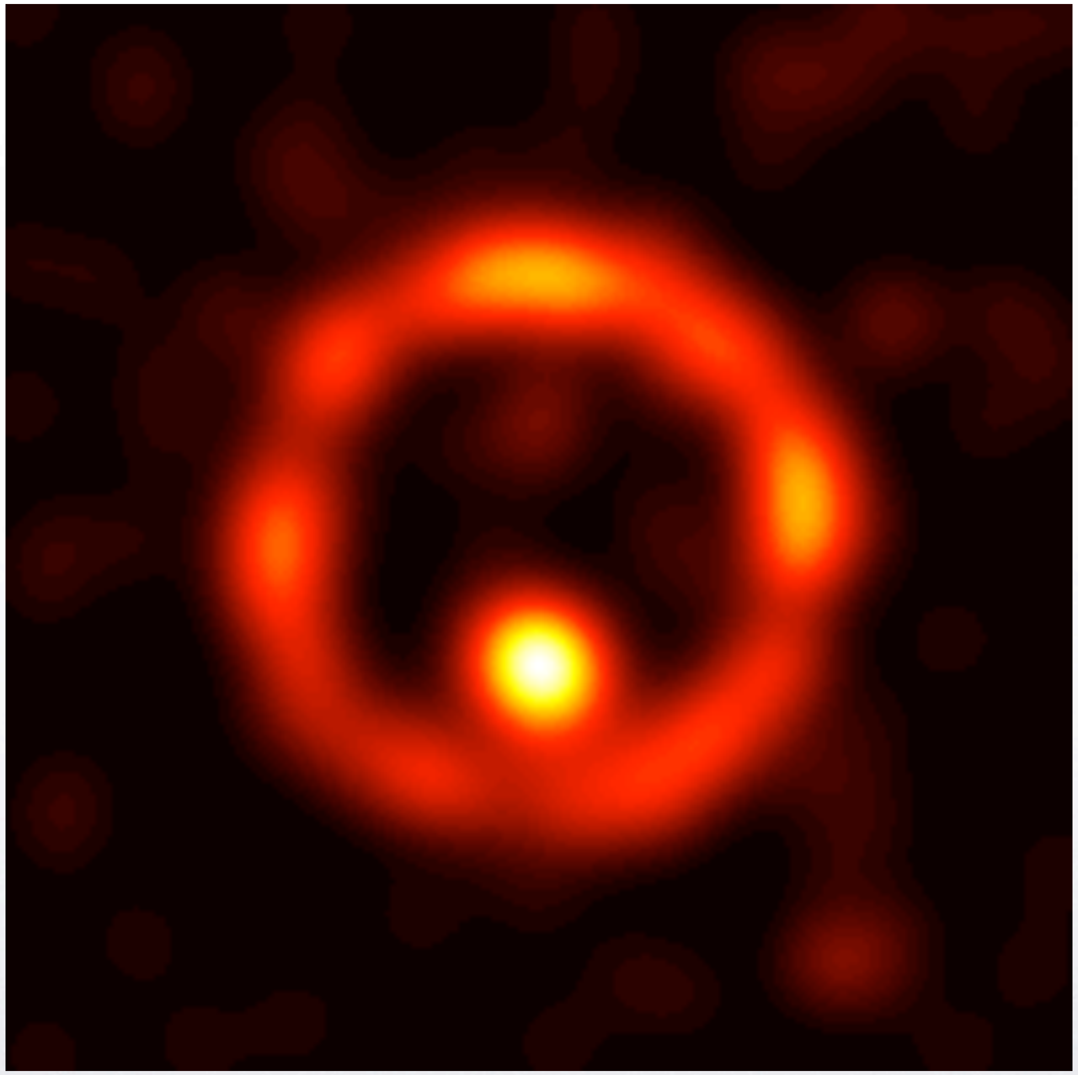

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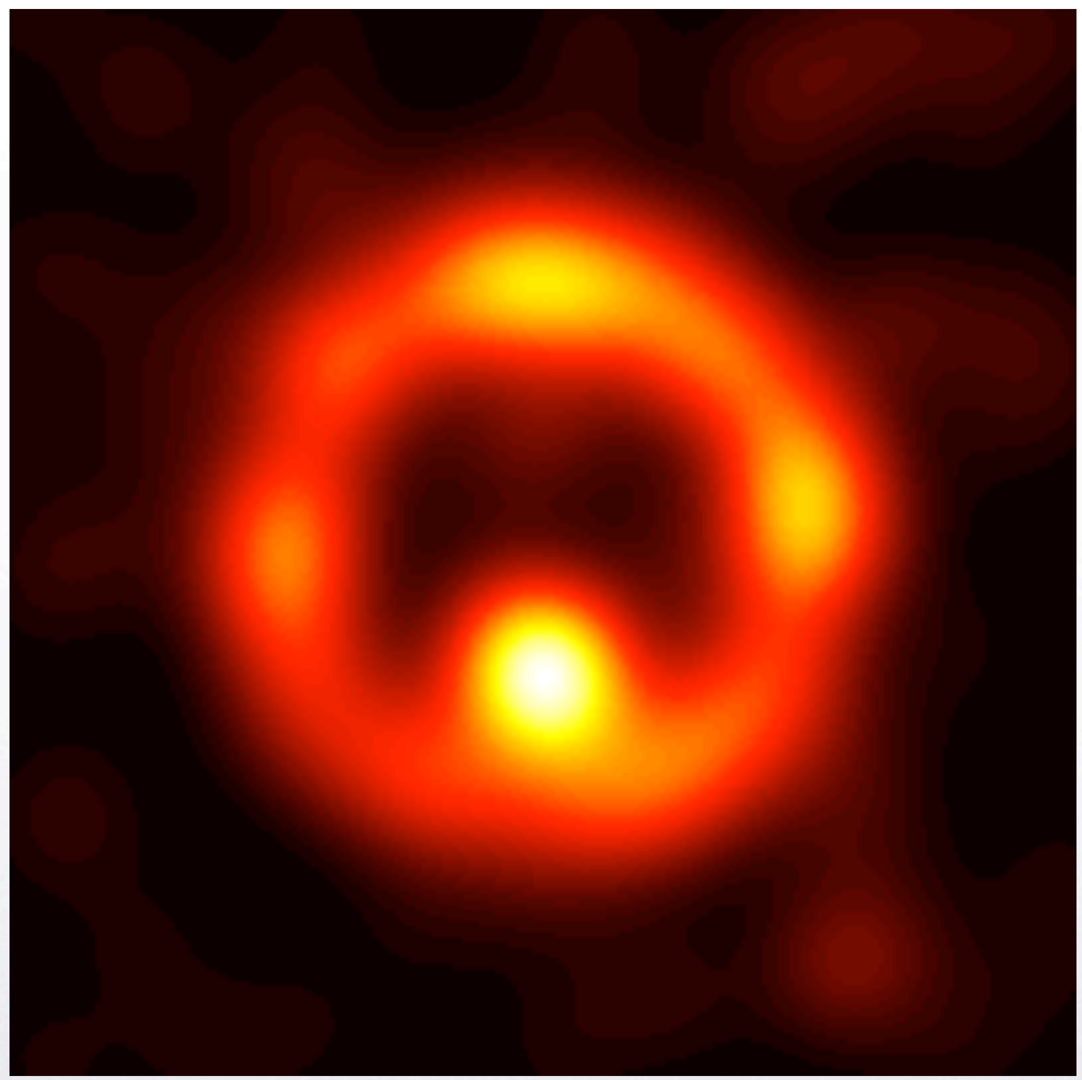

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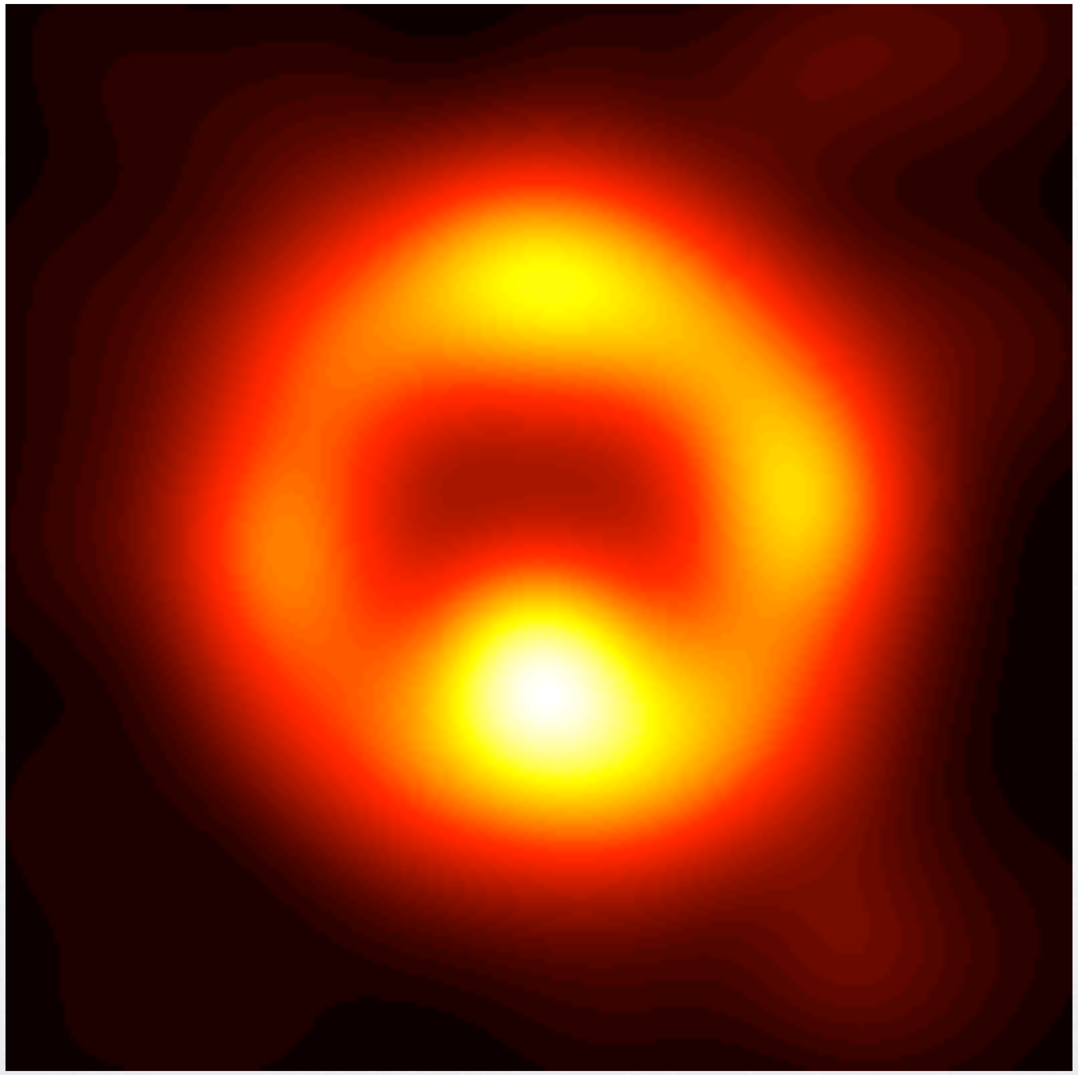

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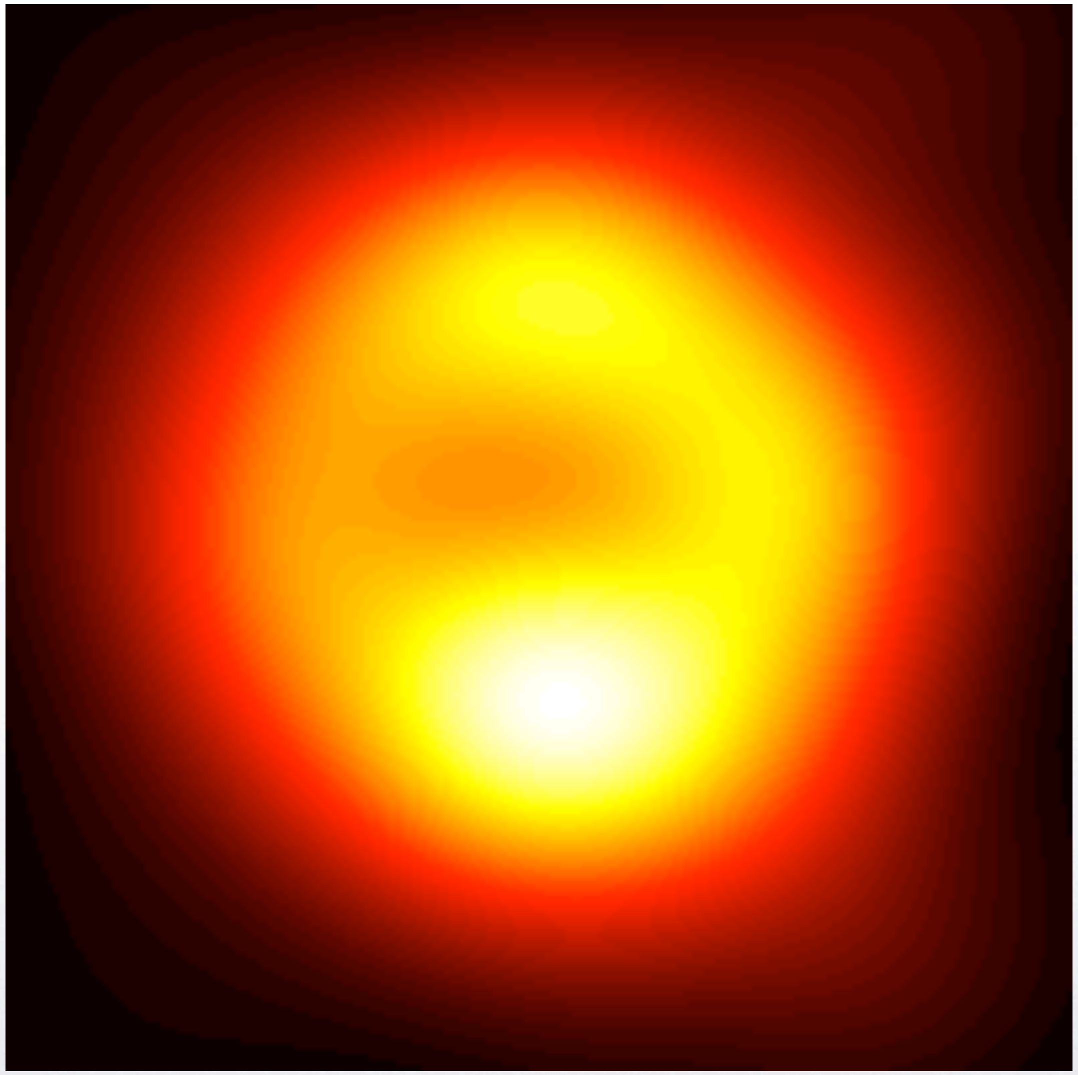

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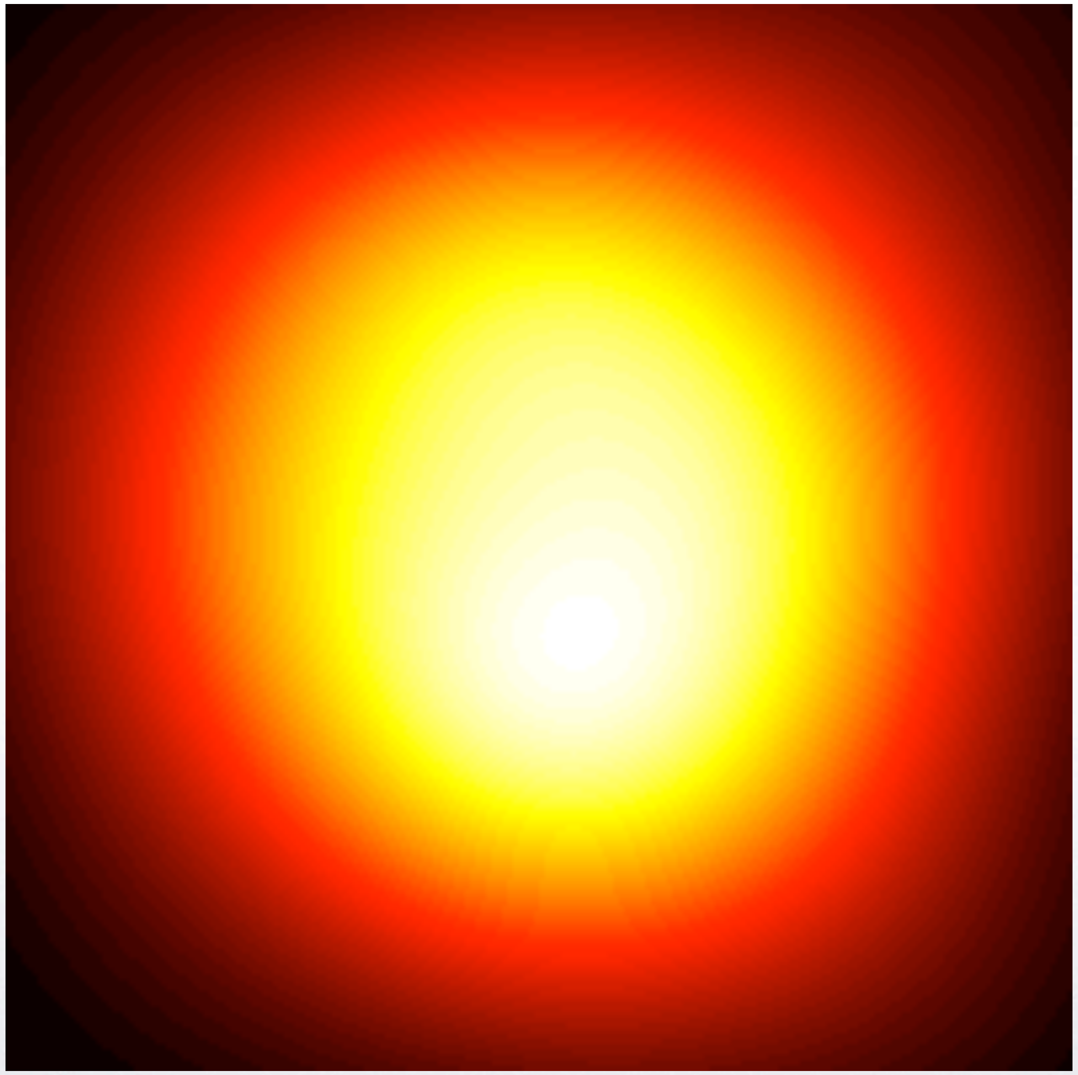

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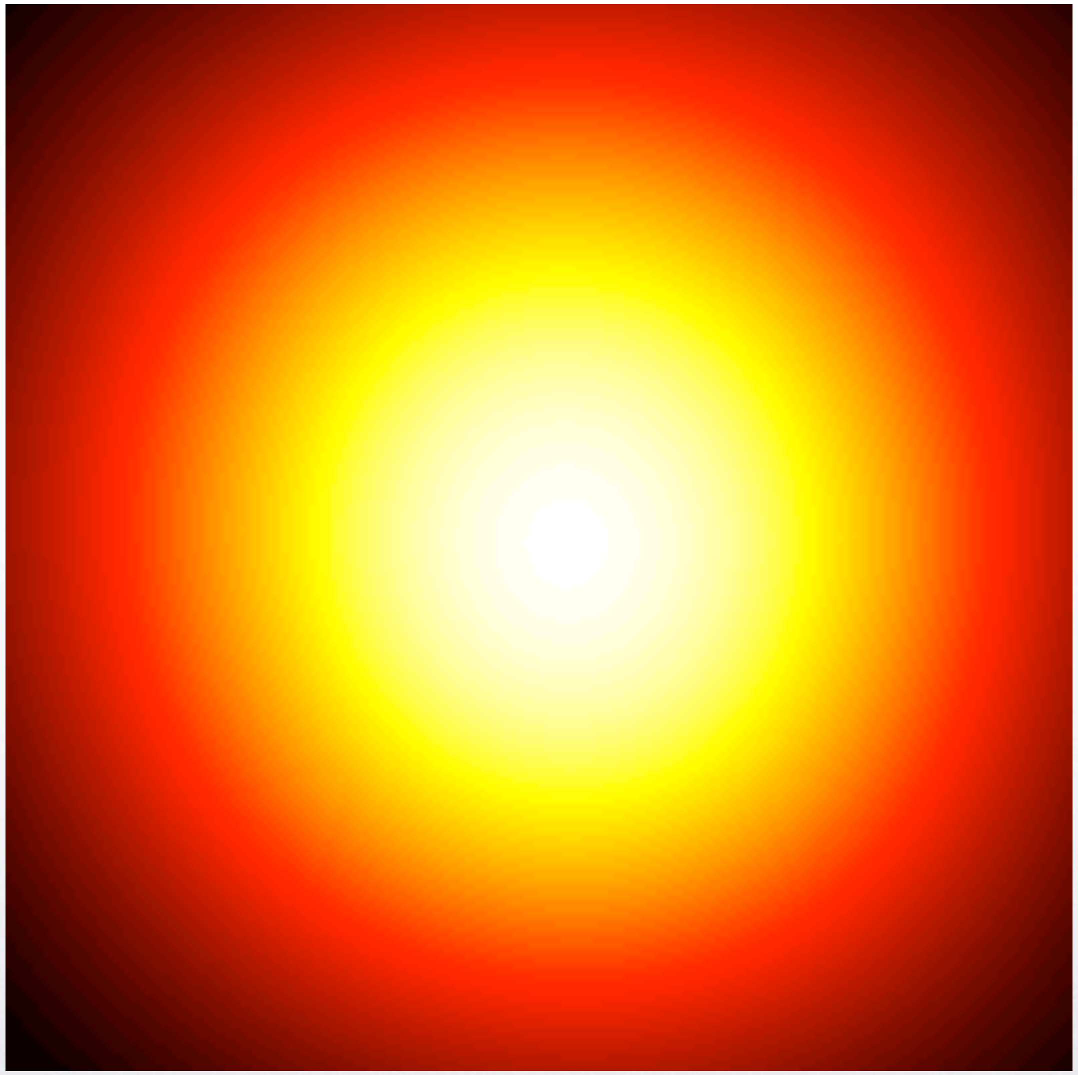

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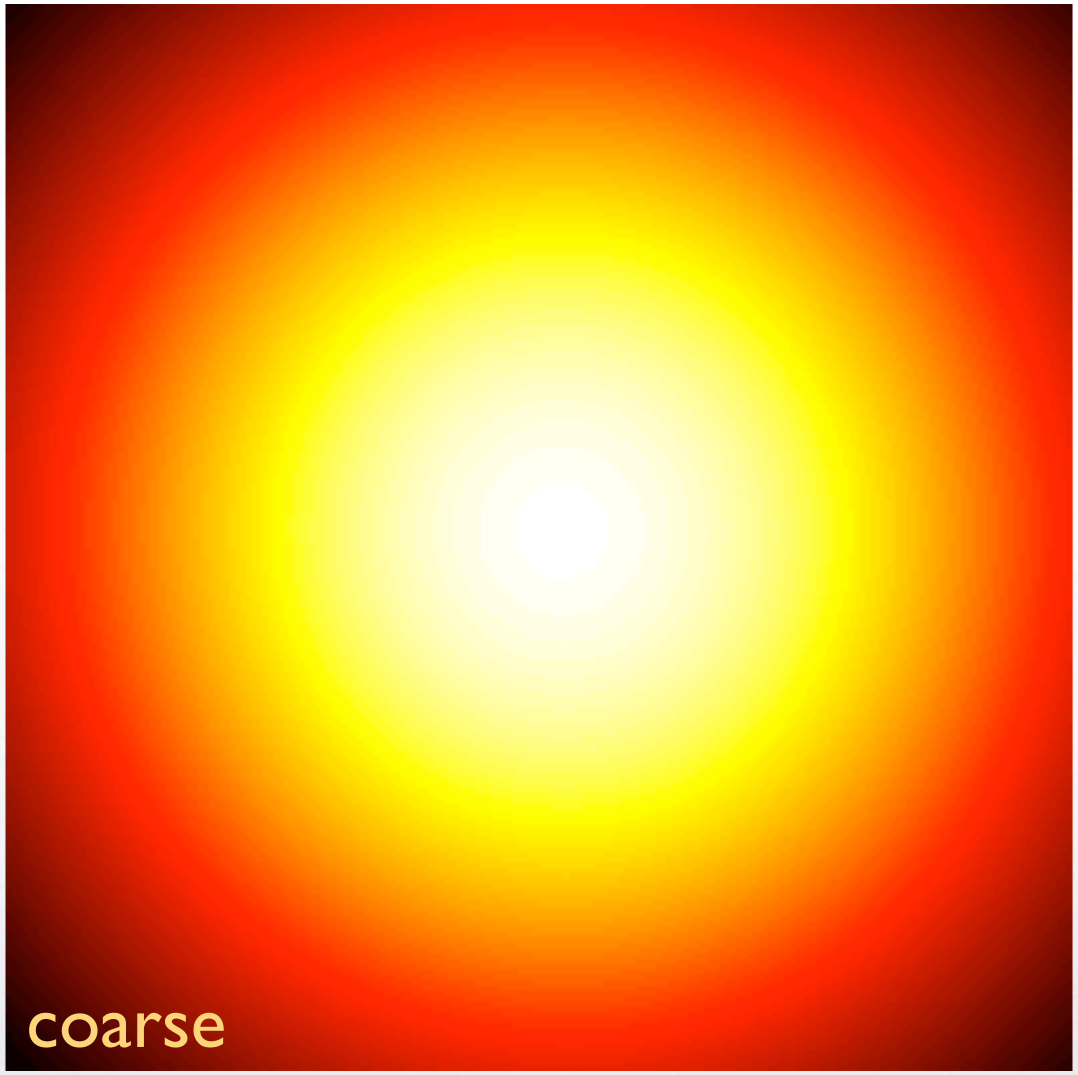

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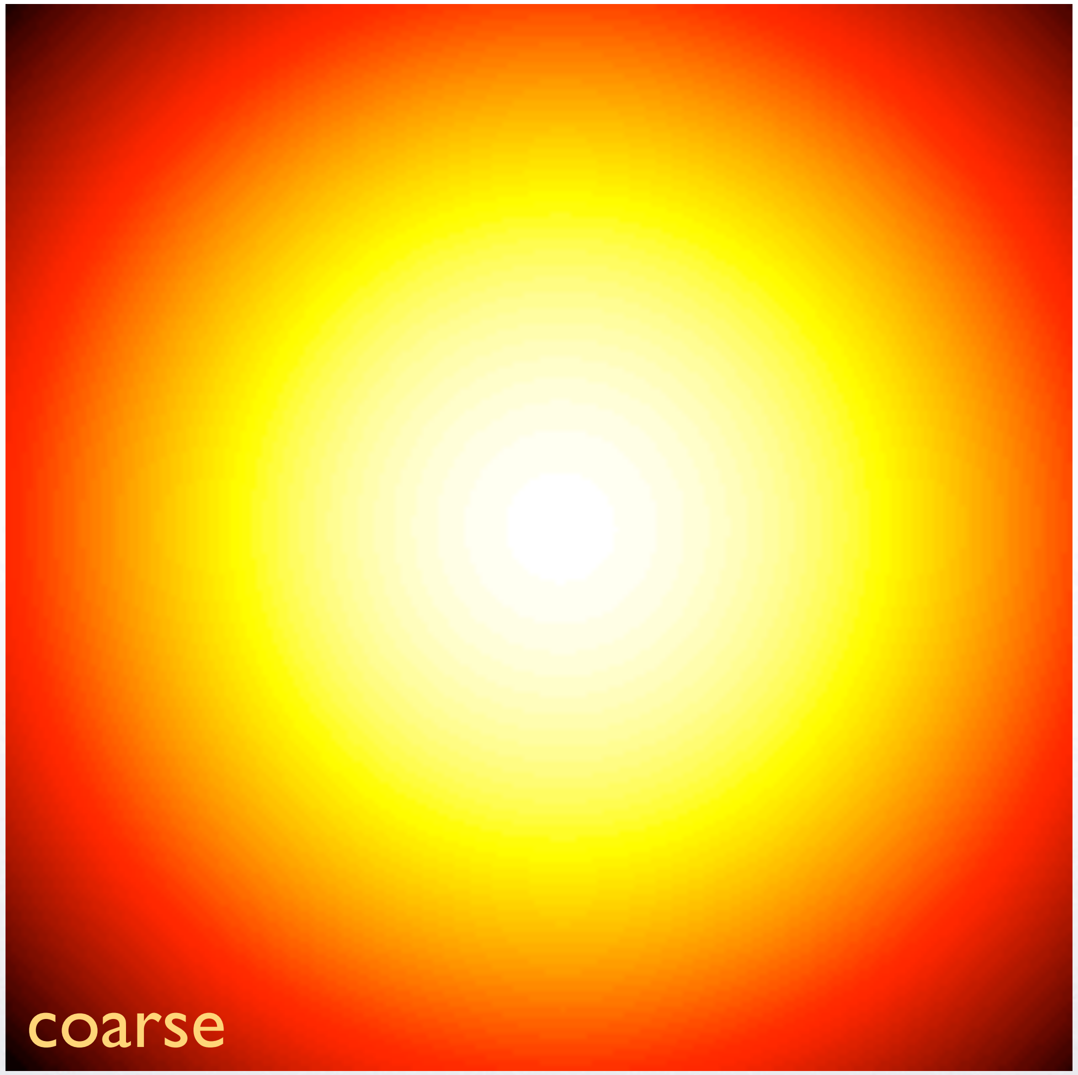
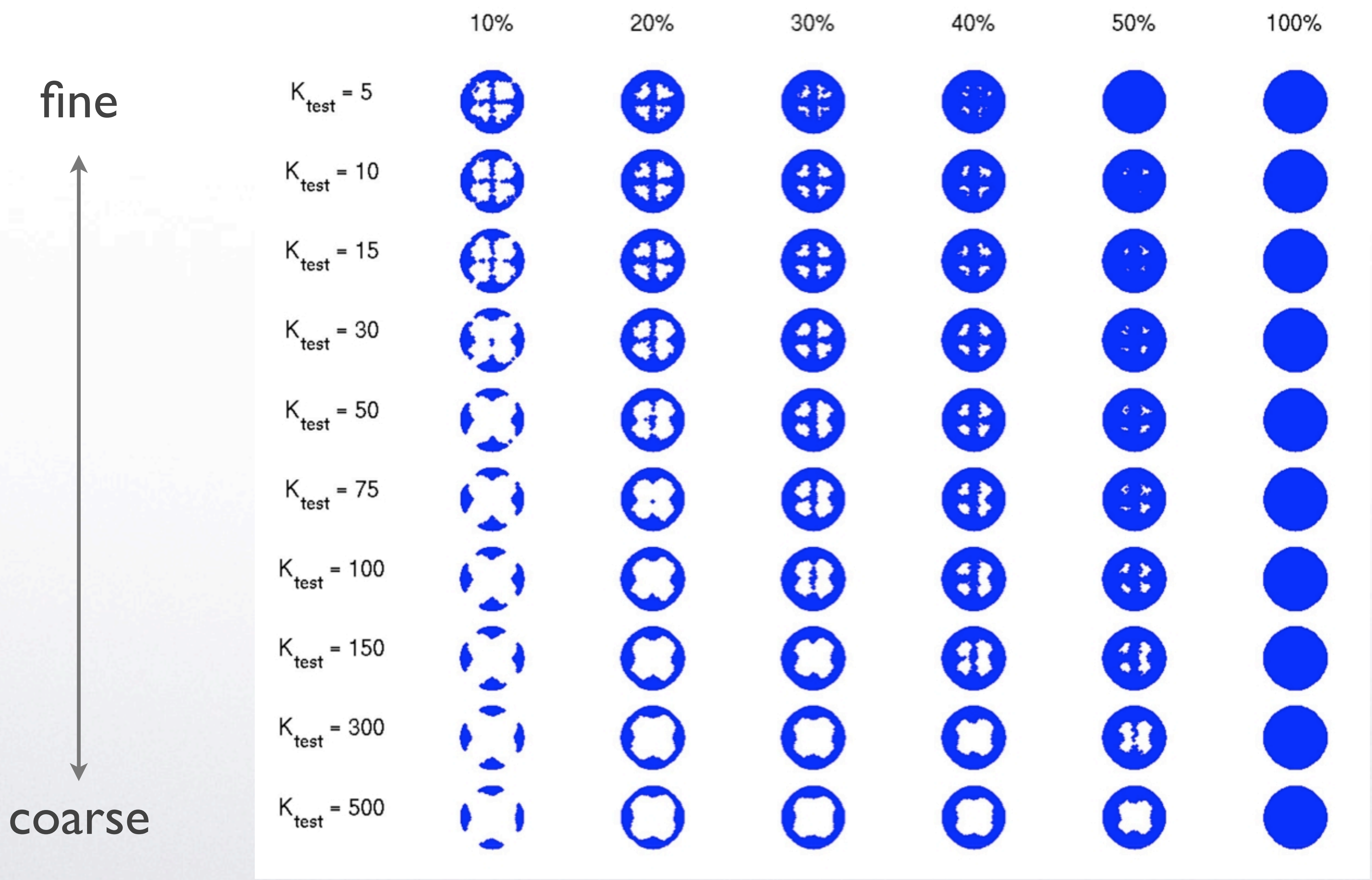
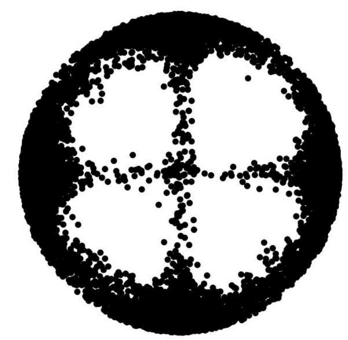
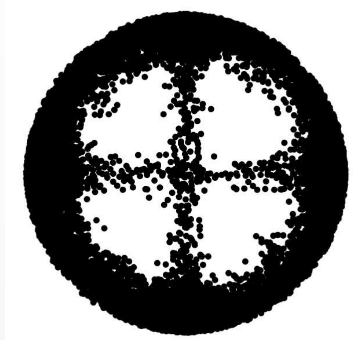
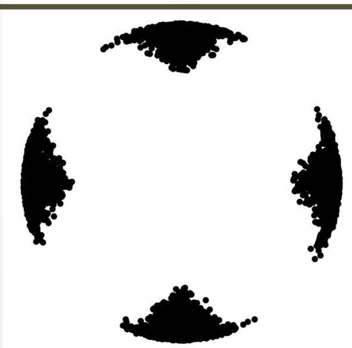
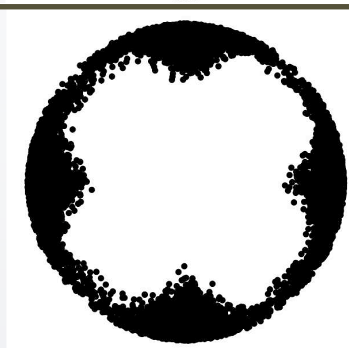
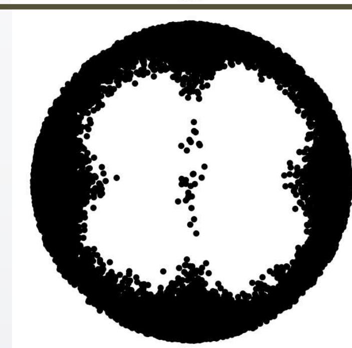
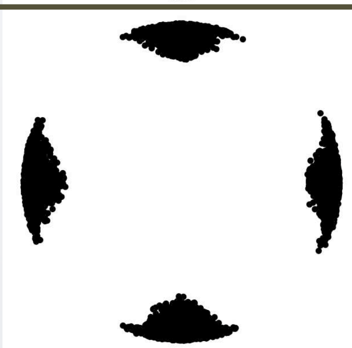
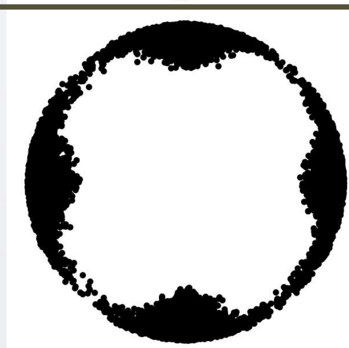
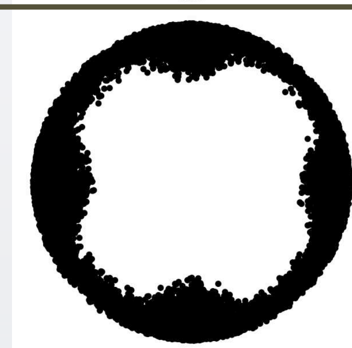


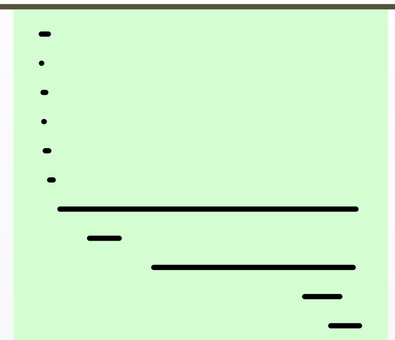
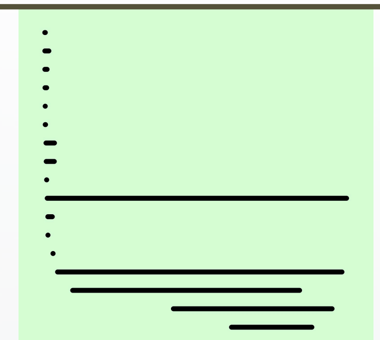
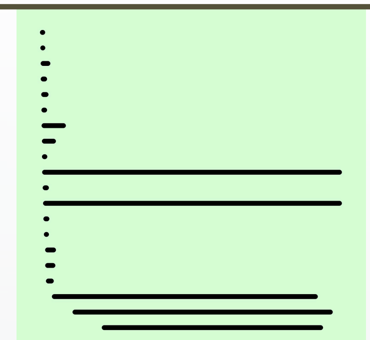
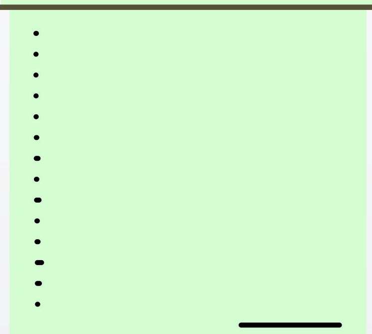
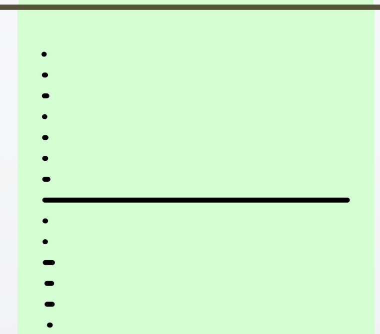
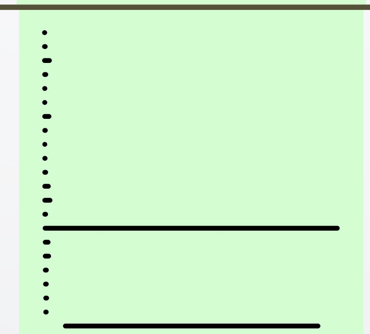
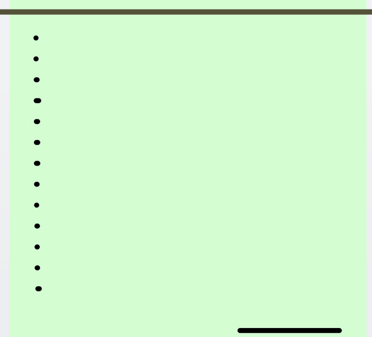
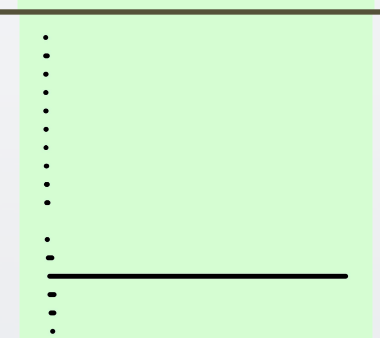
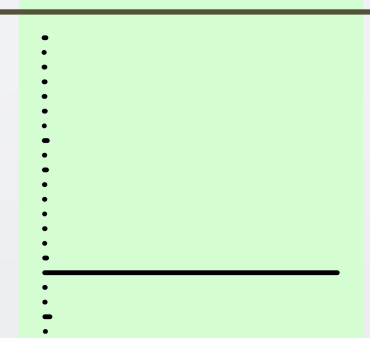
Straining a data soup



A small platter of cuts

	10%	20%	30%
K=15			
K=100			
K=300			

A small platter of cuts

	10%	20%	30%
K=15			
K=100			
K=300			

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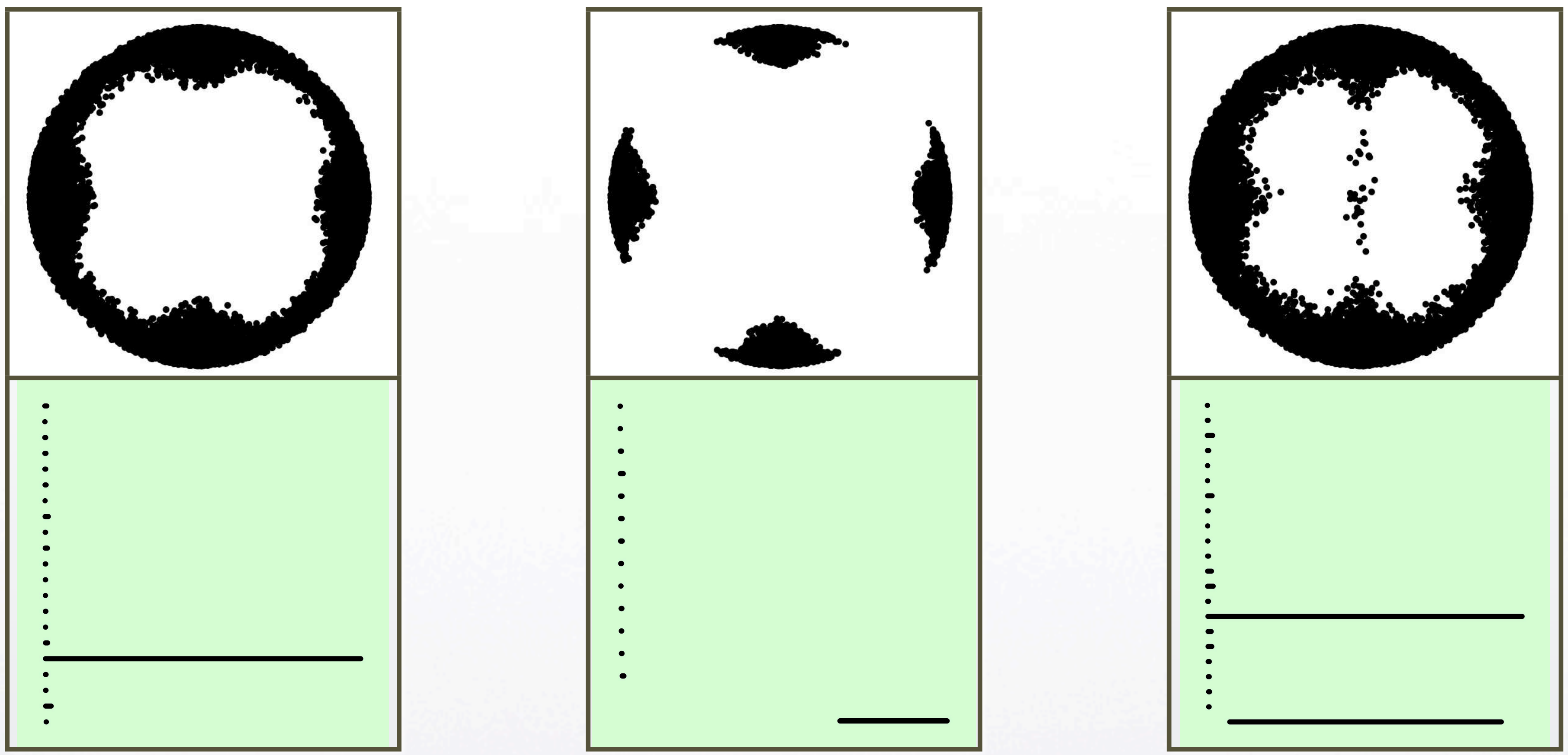
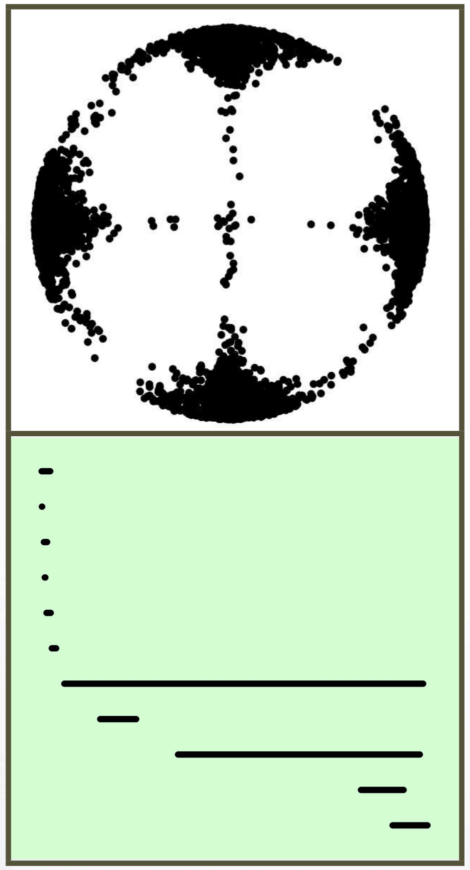
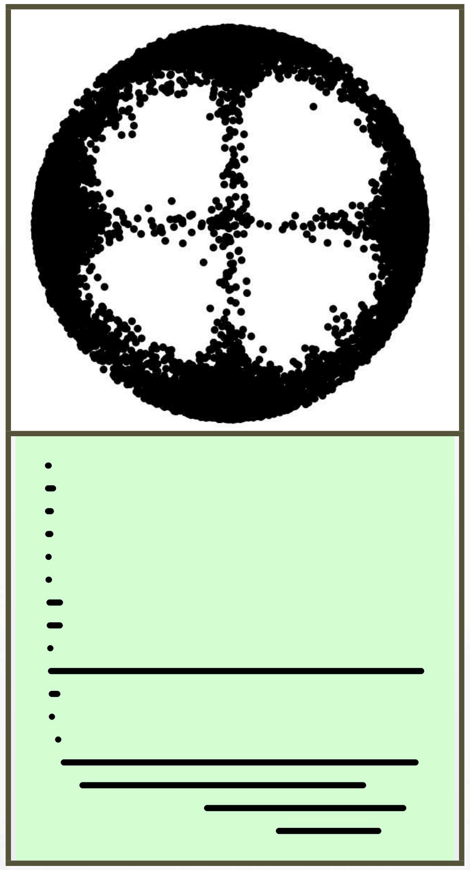
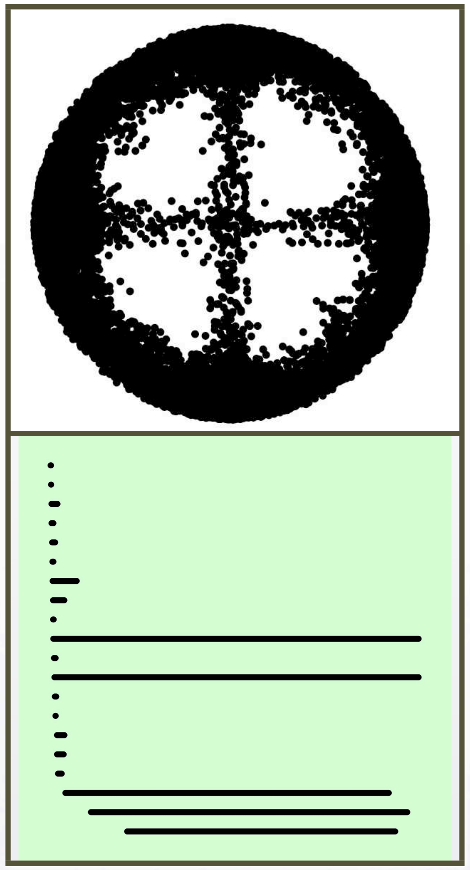
(8-dimensional data)



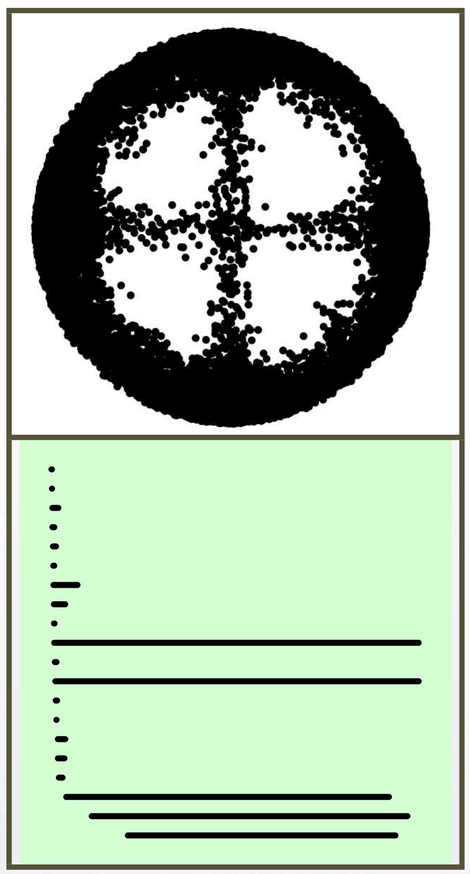
(8-dimensional data)

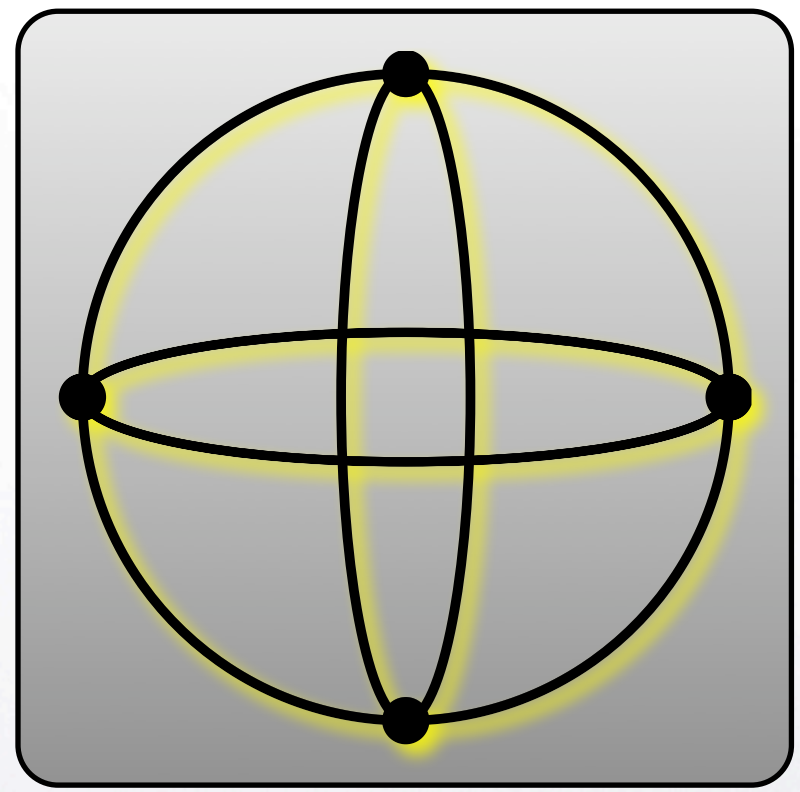




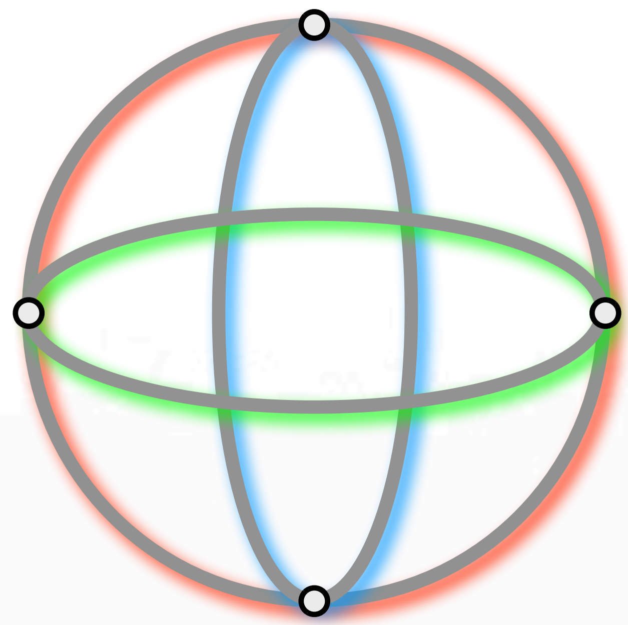


3-circles model

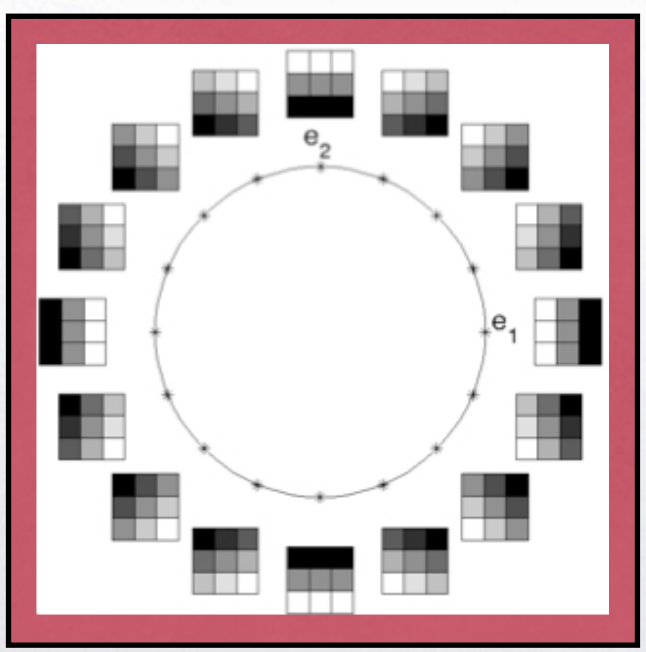




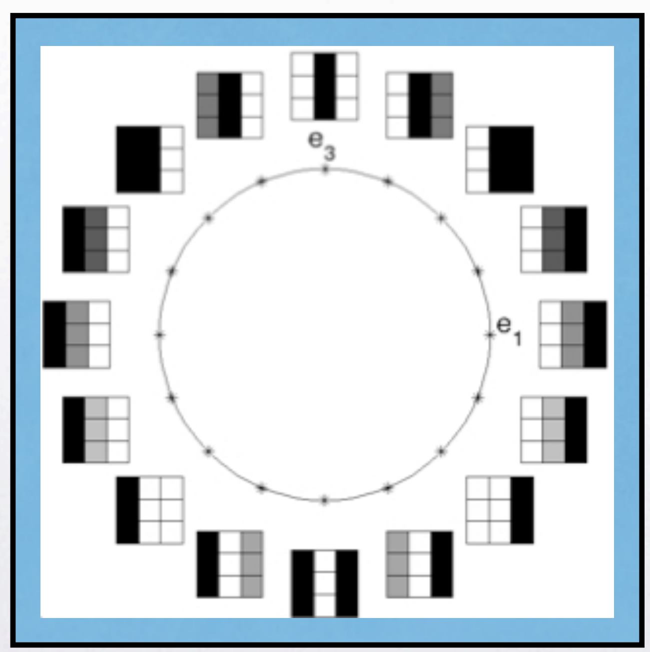
3 circles explained



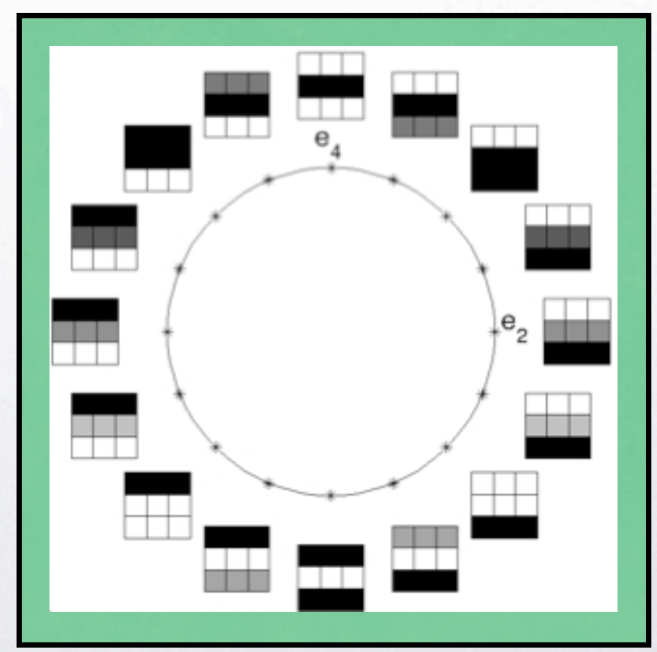
linear gradients



vertical features



horizontal features



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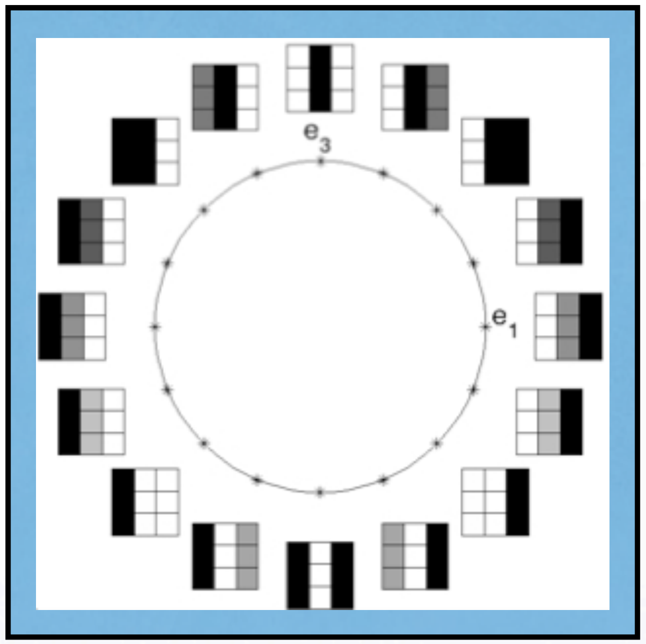
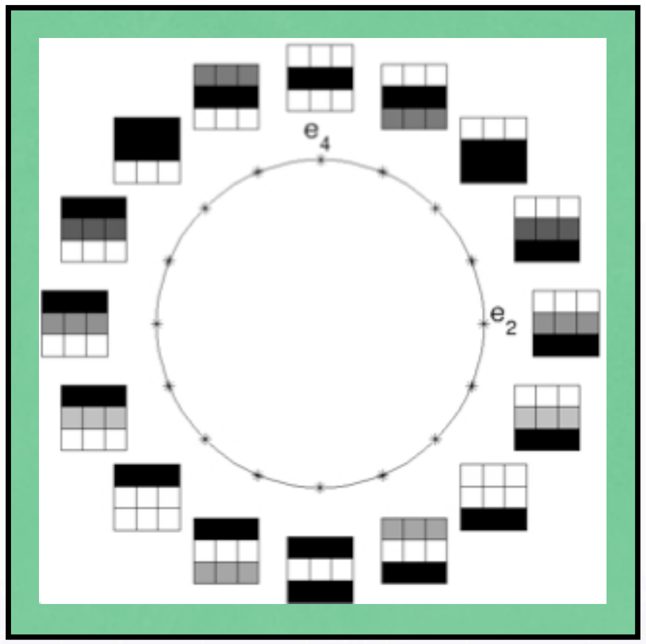
The secondary circles

vertical features



horizontal features



Why is there a predominance of vertical/horizontal local features?

Artefact of the square patch shape?
Artefact of the natural world?

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Multiparameter diagrams

Three parameters

Three parameters

vary scale epsilon
fix density parameter
fix percentage

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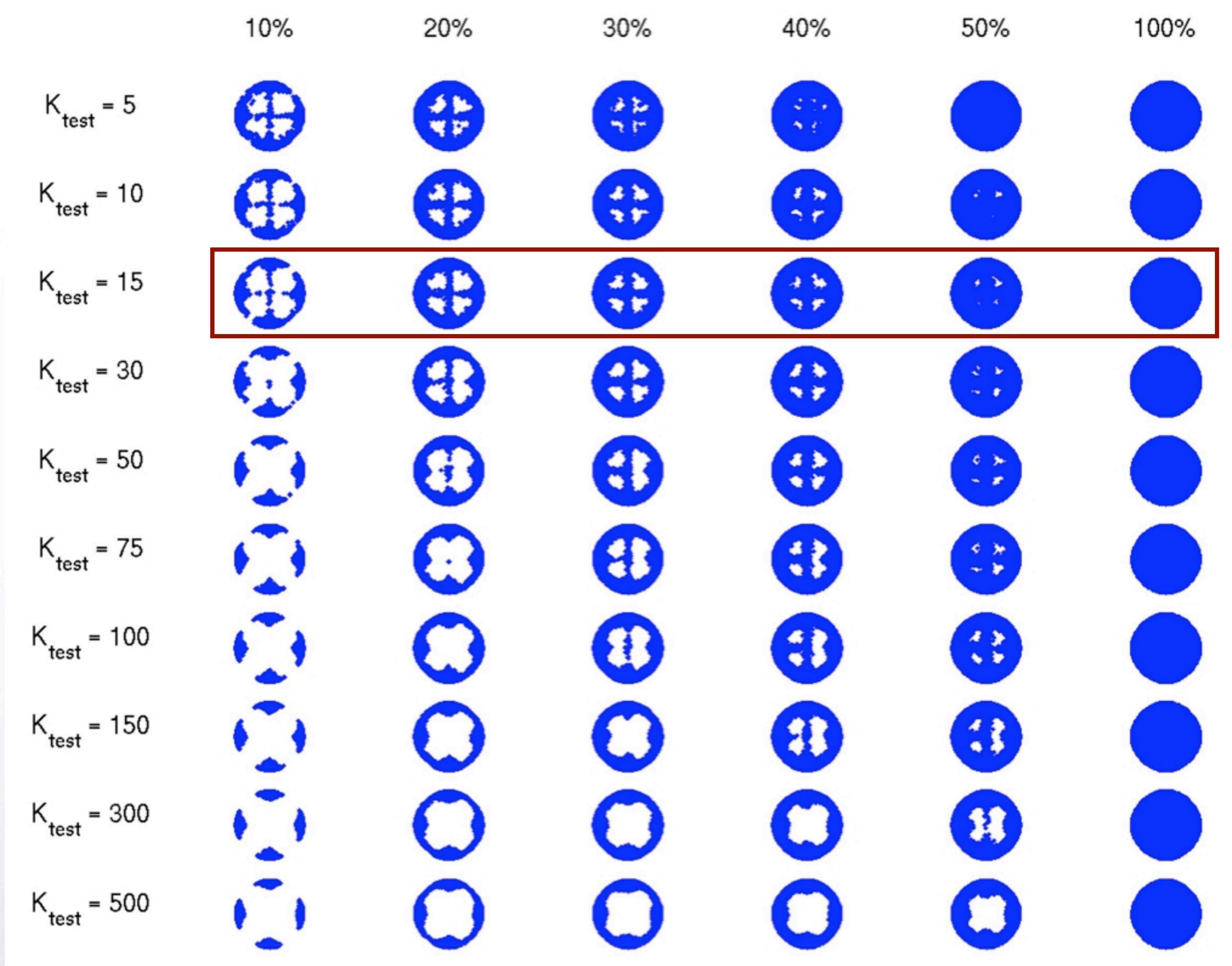
Three parameters

fix scale epsilon
fix density parameter
vary percentage



monotone

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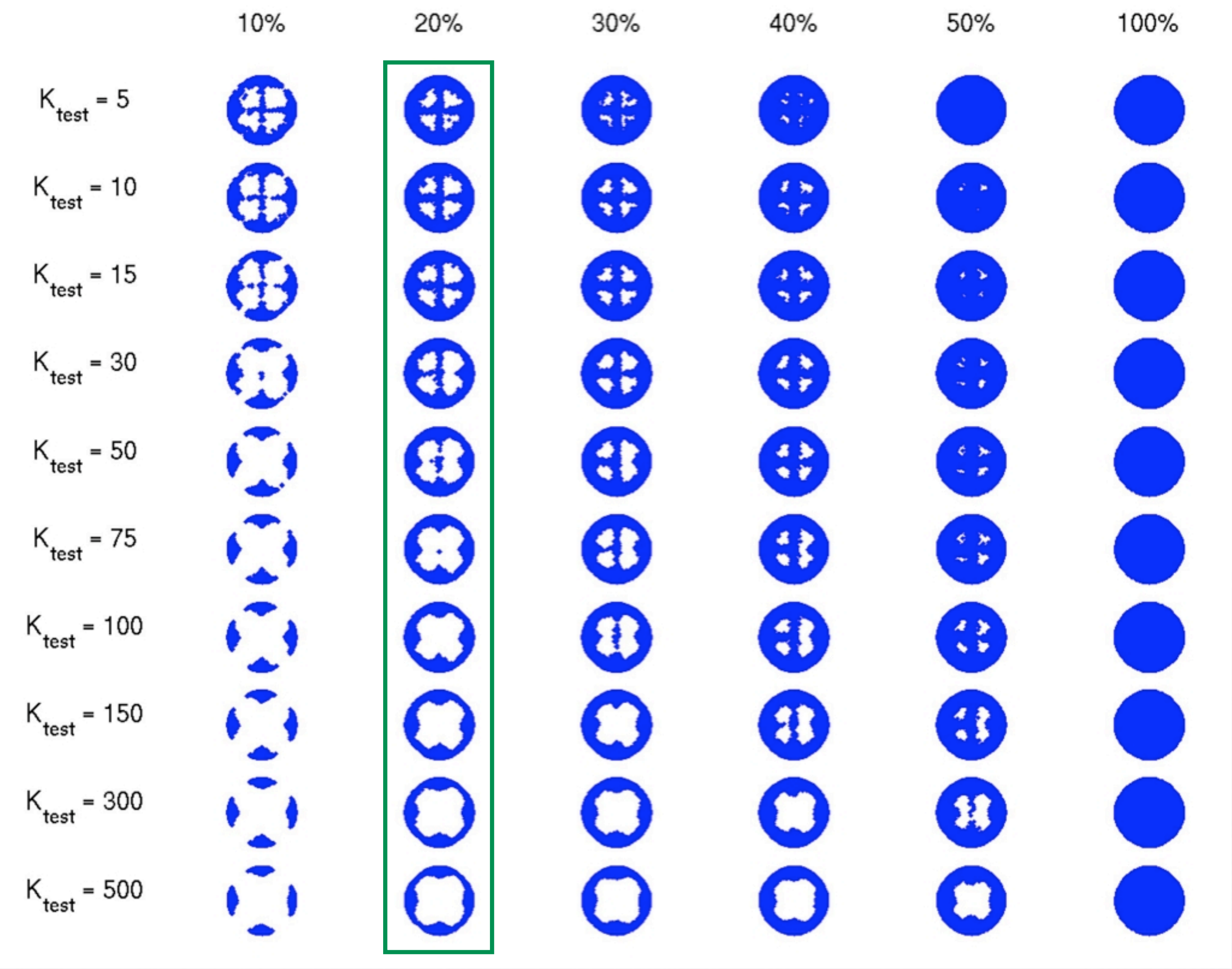
Three parameters

fix scale epsilon
vary density parameter
fix percentage



non-monotone

Non-monotone families

Non-monotone families

non-monotone

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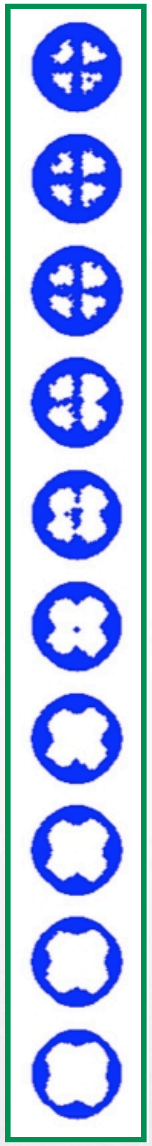
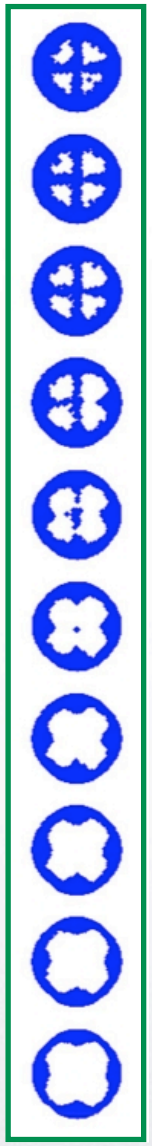
Non-monotone families



non-monotone

Non-monotone families



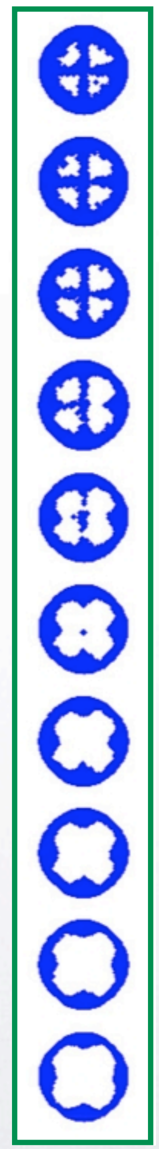
X_0

X_1

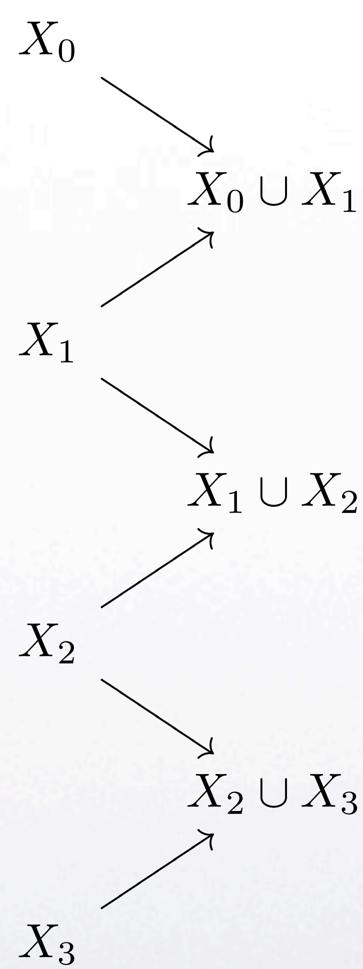
X_2

X_3

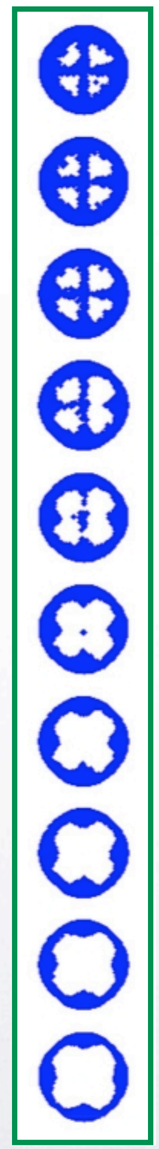
Non-monotone families



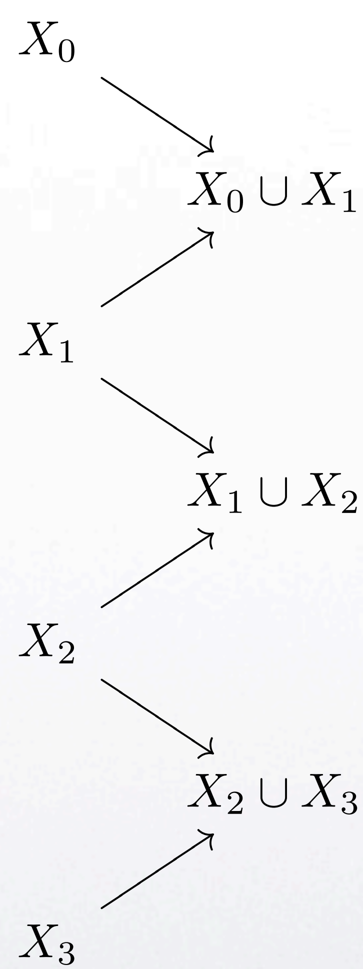
non-monotone



Non-monotone families



non-monotone



Persistence over zigzag diagrams?

Examples of general zigzags

- Example: time-varying complex $X = \{X_t \mid t \text{ real}\}$.
- Example: 30% strained data soup, varying the smoothing parameter.
- Example: witness complex with fixed vertex set, varying the set of witnesses.
- How do the features of X change as t varies?
  - New cell appears $X \longrightarrow X \cup \sigma$
  - Old cell disappears $X \longleftarrow X \setminus \tau$
  - Inclusion map directions vary arbitrarily, e.g.

$$\cdots \longrightarrow X_{i-1} \longleftarrow X_i \longrightarrow X_{i+1} \longrightarrow \cdots$$

- Can we do non-monotone persistence?

Various diagram structures

standard persistence

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow$$

standard persistence (dual)

$$X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow X_3 \longleftarrow$$

zigzag persistence

$$\begin{array}{ccccccccc} & & Y_1 & & & & Y_2 & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow \\ X_0 & & & & X_1 & & & & X_2 & \end{array}$$

2-parameter persistence

$$\begin{array}{ccccccccc} \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ X_{3,0} & \longrightarrow & X_{3,1} & \longrightarrow & X_{3,2} & \longrightarrow & X_{3,3} & \longrightarrow & \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ X_{2,0} & \longrightarrow & X_{2,1} & \longrightarrow & X_{2,2} & \longrightarrow & X_{2,3} & \longrightarrow & \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ X_{1,0} & \longrightarrow & X_{1,1} & \longrightarrow & X_{1,2} & \longrightarrow & X_{1,3} & \longrightarrow & \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ X_{0,0} & \longrightarrow & X_{0,1} & \longrightarrow & X_{0,2} & \longrightarrow & X_{0,3} & \longrightarrow & \end{array}$$

Various diagram structures

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$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow$$

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$$X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow X_3 \longleftarrow$$

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Persistence over general diagrams?

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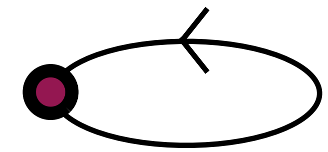
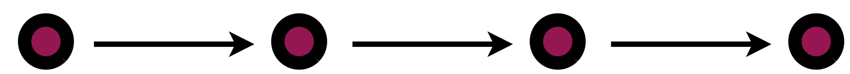
Quiver representations

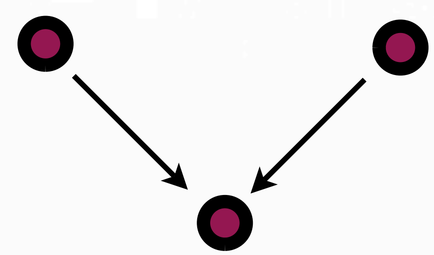
Quivers

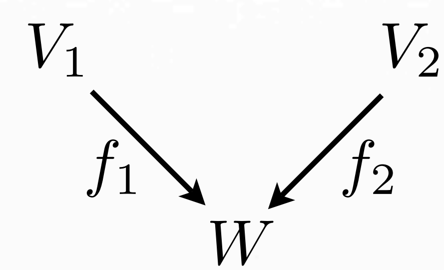


- A **quiver** is a directed (multi-)graph:
  - nodes
  - arrows



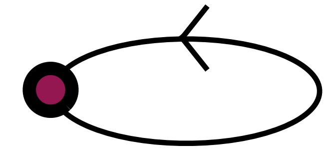
- A **representation** of a quiver Q has:
  - a vector space for every node
  - a linear map for every arrow





- **General question**: classify representations of a given quiver Q.
- What would be the ideal answer?
  - unique decomposition into indecomposable representations
  - + explicit list of indecomposables
  - + algorithm to determine decomposition type

Quivers



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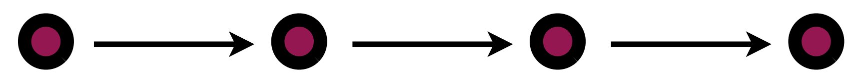

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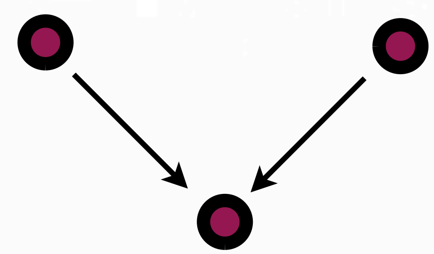

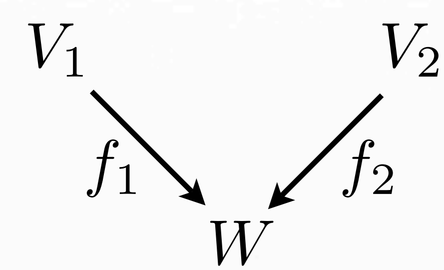

- **General question**: classify representations of a given quiver Q.
- What would be the ideal answer?
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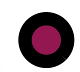
  - + explicit list of indecomposables
  - + algorithm to determine decomposition type

Gabriel (1972): this is very rarely possible!
Carlsson, Zomorodian (2007): 2-parameter persistence is "hard"

Quivers

- Example 

- Typical representation V

- Indecomposable representations (over complex numbers)

$$\mathbb{C}$$

- Classifying invariant $\dim(V)$

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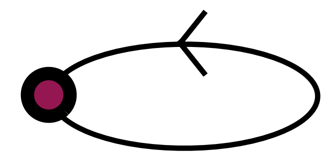
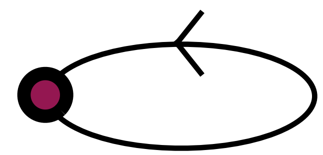
Quivers

- Example 
- Typical representation $V \xrightarrow{f} V$
- Indecomposable representations (over complex numbers)
- Classifying invariant

generalised eigenspectrum of f

Quivers

- Example



- Typical representation $V \xrightarrow{f} V$

- Indecomposable representations (over complex numbers)

$$\text{Jordan blocks} \begin{bmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & \lambda \end{bmatrix}$$

- Classifying invariant

generalised eigenspectrum of f

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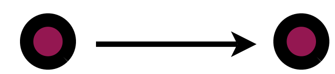
Quivers

- Example



- Typical representation

$$V \xrightarrow{f} W$$

- Indecomposable representations (over complex numbers)

$$\mathbb{C} \xrightarrow{1} \mathbb{C}$$

$$\mathbb{C} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{C}$$

- Classifying invariants

$$\mathrm{rank}(f), \mathrm{nullity}(f), \mathrm{nullity}(f^*)$$

Quivers

- Example 
- Typical representation $V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} V_2 \xrightarrow{f_3} V_3$
- Indecomposable representations (over complex numbers)

- Classifying invariants

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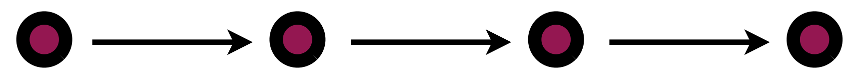
Quivers

- Example 
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- Indecomposable representations (over complex numbers)

$$[b, d], \quad 0 \leq b \leq d \leq 3$$

- Classifying invariants

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Quivers

- Example 
- Typical representation $V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} V_2 \xrightarrow{f_3} V_3$
- Indecomposable representations (over complex numbers)

$$[b, d], \quad 0 \leq b \leq d \leq 3$$

$[0,0]$:	$\mathbb{C} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$
$[1,1]$:	$0 \longrightarrow \mathbb{C} \longrightarrow 0 \longrightarrow 0$
$[2,2]$:	$0 \longrightarrow 0 \longrightarrow \mathbb{C} \longrightarrow 0$
$[3,3]$:	$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{C}$
$[0,1]$:	$\mathbb{C} \xrightarrow{\text{Id}} \mathbb{C} \longrightarrow 0 \longrightarrow 0$
$[1,2]$:	$0 \longrightarrow \mathbb{C} \xrightarrow{\text{Id}} \mathbb{C} \longrightarrow 0$
$[2,3]$:	$0 \longrightarrow 0 \longrightarrow \mathbb{C} \xrightarrow{\text{Id}} \mathbb{C}$
$[0,2]$:	$\mathbb{C} \xrightarrow{\text{Id}} \mathbb{C} \xrightarrow{\text{Id}} \mathbb{C} \longrightarrow 0$
$[1,3]$:	$0 \longrightarrow \mathbb{C} \xrightarrow{\text{Id}} \mathbb{C} \xrightarrow{\text{Id}} \mathbb{C}$
$[0,3]$:	$\mathbb{C} \xrightarrow{\text{Id}} \mathbb{C} \xrightarrow{\text{Id}} \mathbb{C} \xrightarrow{\text{Id}} \mathbb{C}$

- Classifying invariants

Quivers

- Example 
- Typical representation $V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} V_2 \xrightarrow{f_3} V_3$
- Indecomposable representations (over complex numbers)

$$[b,d], \quad 0 \le b \le d \le 3$$

$$\begin{array}{lccccccc}
[0,0]: & \mathbb{C} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
[1,1]: & 0 & \longrightarrow & \mathbb{C} & \longrightarrow & 0 & \longrightarrow & 0 \\
[2,2]: & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\
[3,3]: & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{C} \\
\\
[0,1]: & \mathbb{C} & \xrightarrow{\mathrm{Id}} & \mathbb{C} & \longrightarrow & 0 & \longrightarrow & 0 \\
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\\
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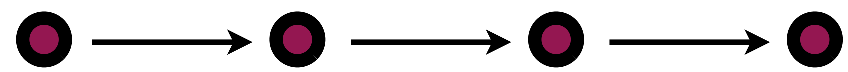
\end{array}$$

- Classifying invariants $\operatorname{rank}(V_i \to V_j)\,, \quad 0 \le i \le j \le 3$

Quivers

- Example 
- Typical representation $V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} V_2 \xrightarrow{f_3} V_3$
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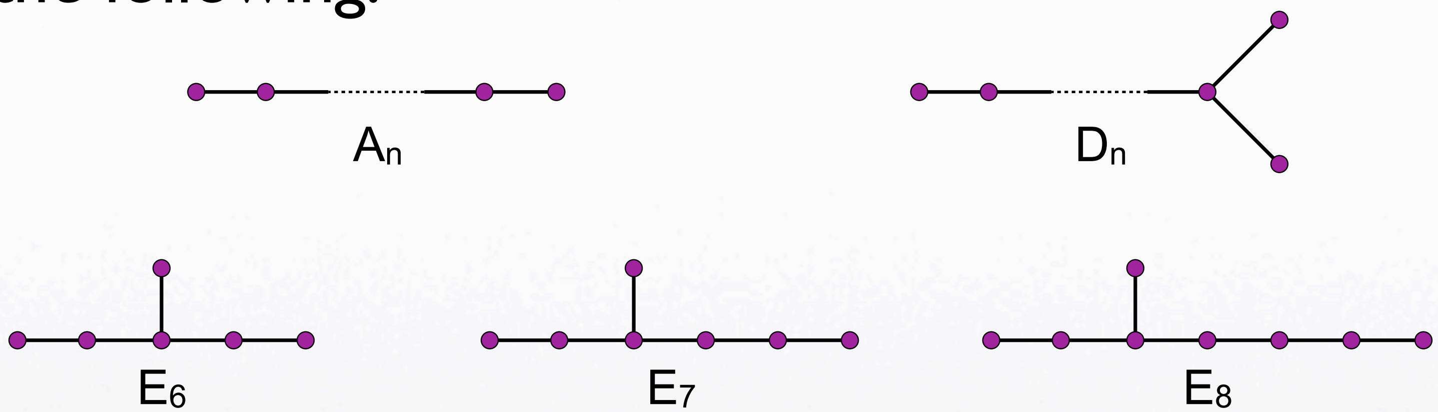
\end{array}$$

- Classifying invariants $\mathrm{rank}\,(V_i \to V_j), \quad 0 \le i \le j \le 3$

relation: $\mathrm{rank}\,[V_i \to V_j] = \sum_{[b,d]\supseteq[i,j]} \text{multiplicity of } [b,d]$

Quivers

- A quiver Q is of finite type if there is a unique decomposition theorem with a finite list of indecomposables.
- Gabriel (1972): Q is of finite type iff its underlying undirected graph is one of the following:

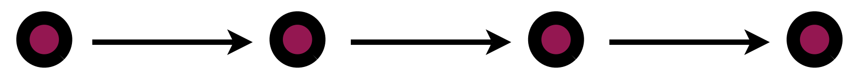


- The dimension vectors of the indecomposables are given by the positive roots of the associated Dynkin lattices.

Corollary: interval decomposition for all quivers of type A_n

⇒ zigzag persistent homology!

Quivers

- Example 
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- Classifying invariants $\mathrm{rank}\,(V_i \to V_j)\,, \quad 0 \le i \le j \le 3$

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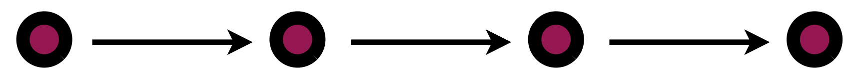
Quivers

- Example 
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 & & & & & & & \\
[0,3]: & \mathbb{C} & \xrightarrow{\mathrm{Id}} & \mathbb{C} & \xleftarrow{\mathrm{Id}} & \mathbb{C} & \xrightarrow{\mathrm{Id}} & \mathbb{C}
\end{array}$$

- Classifying invariants (not so obvious)

Calculating zigzag persistence

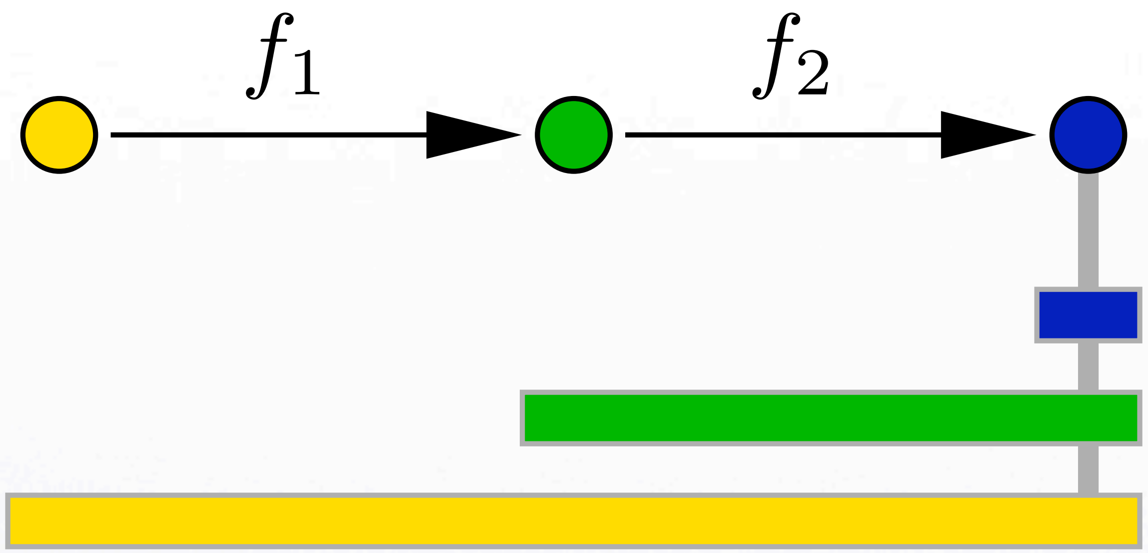
Calculating the zig-zag barcode

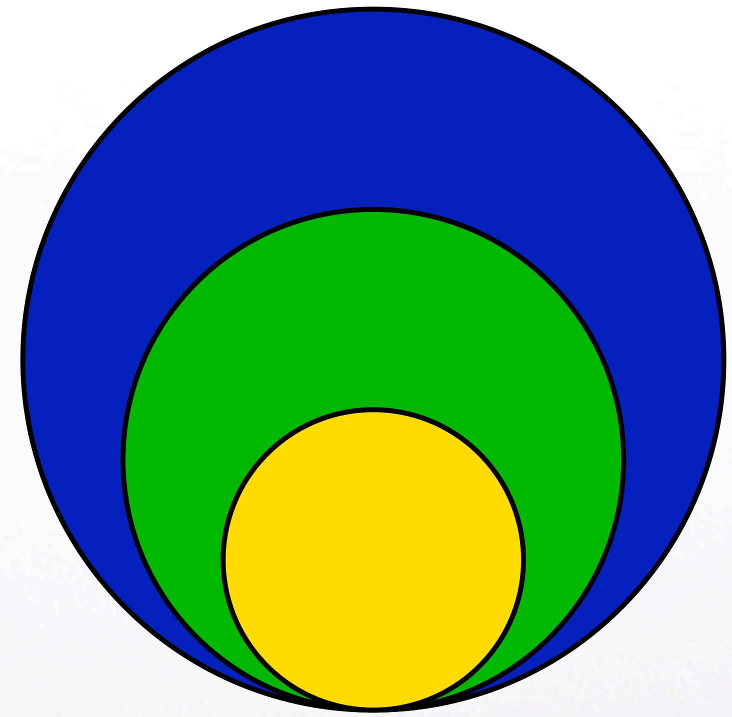
- Construct barcode inductively (left to right).

- At step k, we have
  - a filtration of vector space V_k $\quad 0 = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_k = V_k$
  - a birth-time index vector $\quad \mathbf{b} = [b_1, b_2, \ldots, b_k]$
  - interpretation: vectors in $(R_i \setminus R_{i-1})$ are born at time b_i.

- Update step:
  - compute the filtration on V_{k+1}.
  - output list of intervals which terminate at V_k.

Three-term sequences





$$f_2(f_1(V_0)) \subseteq f_2(V_1) \subseteq V_2$$

* * *

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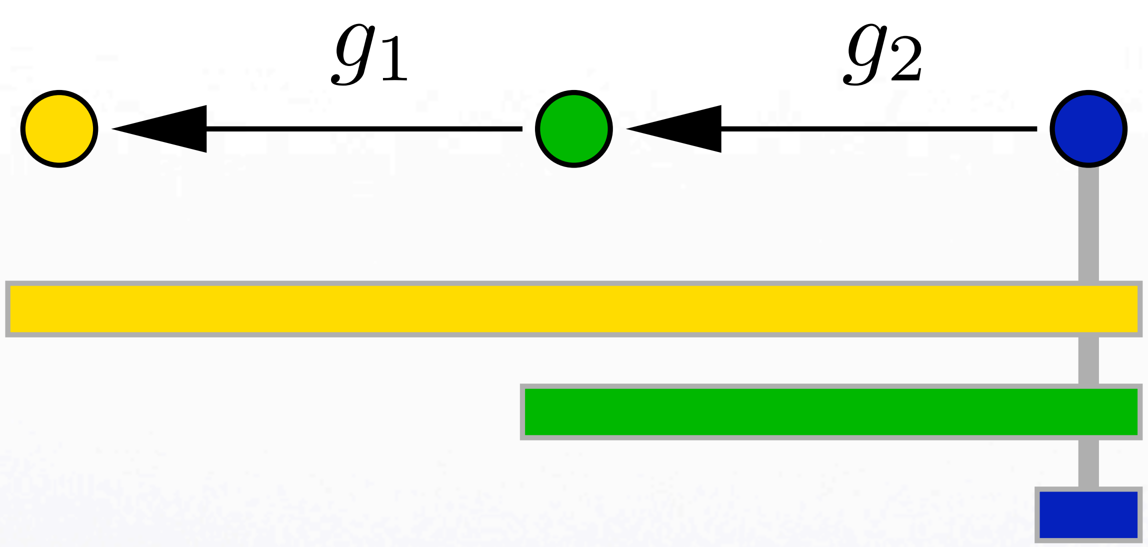
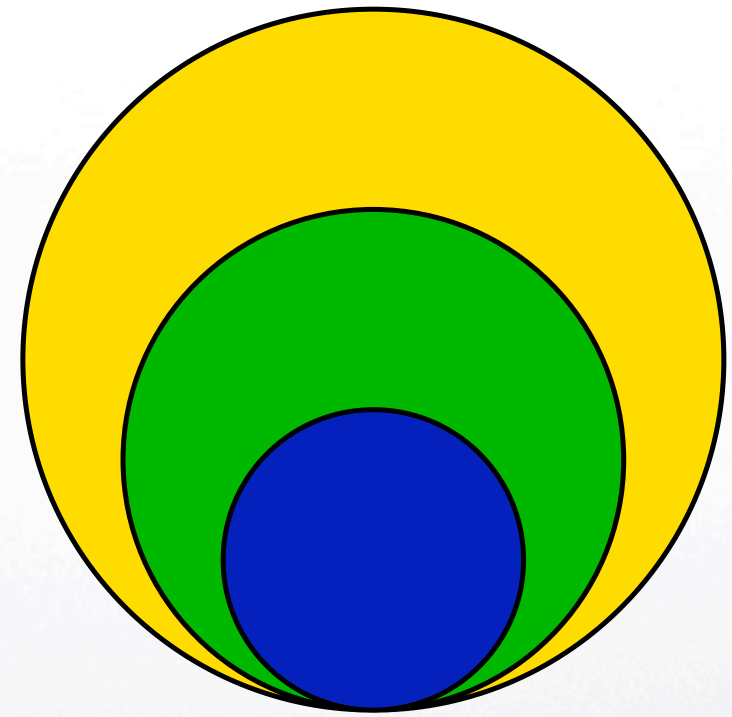
Three-term sequences

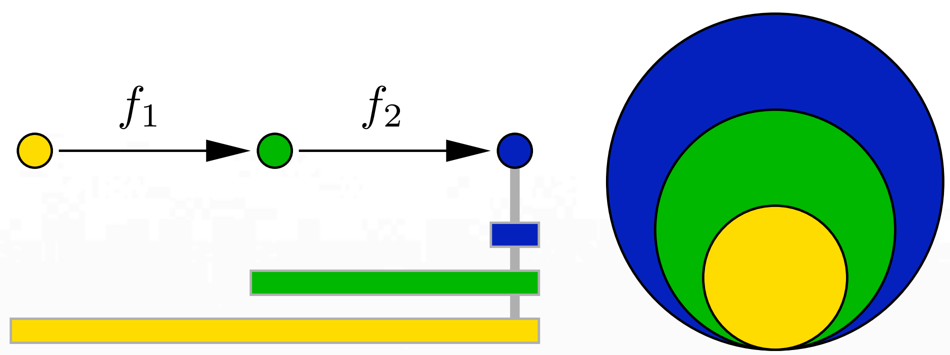




$$g_2^{-1}(0) \subseteq g_2^{-1}(g_1^{-1}(0)) \subseteq V_2$$

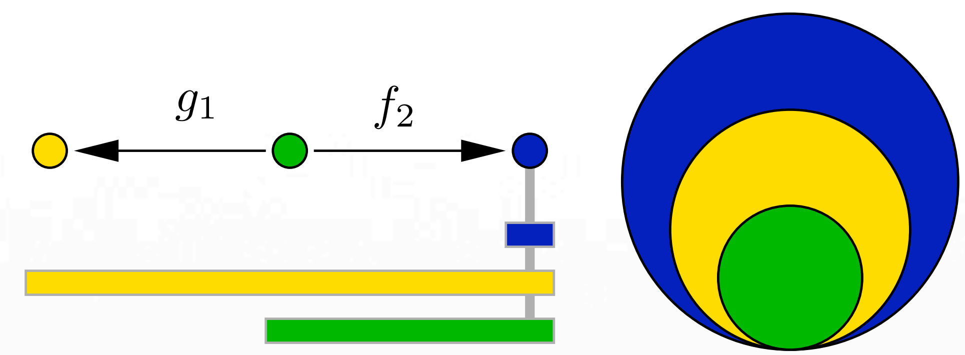
* * *

Three-term sequences



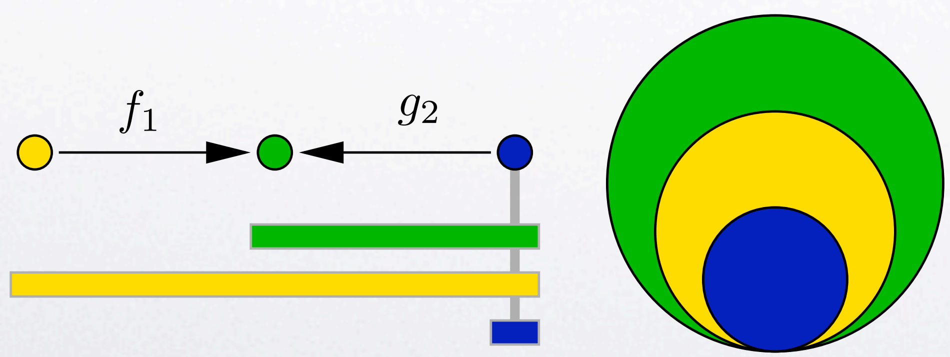
$f_2(f_1(V_0)) \subseteq f_2(V_1) \subseteq V_2$

*   *   *



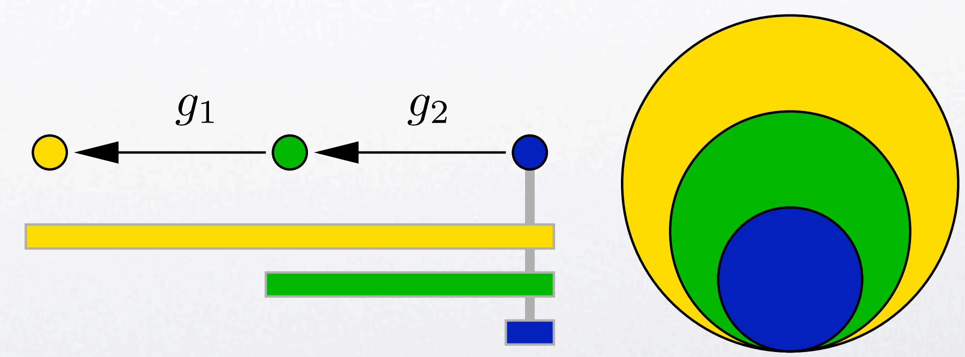
$f_2(g_1^{-1}(0)) \subseteq f_2(V_1) \subseteq V_2$

*   *   *



$g_2^{-1}(0) \subseteq g_2^{-1}(f_1(V_0)) \subseteq V_2$

*   *   *



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*   *   *

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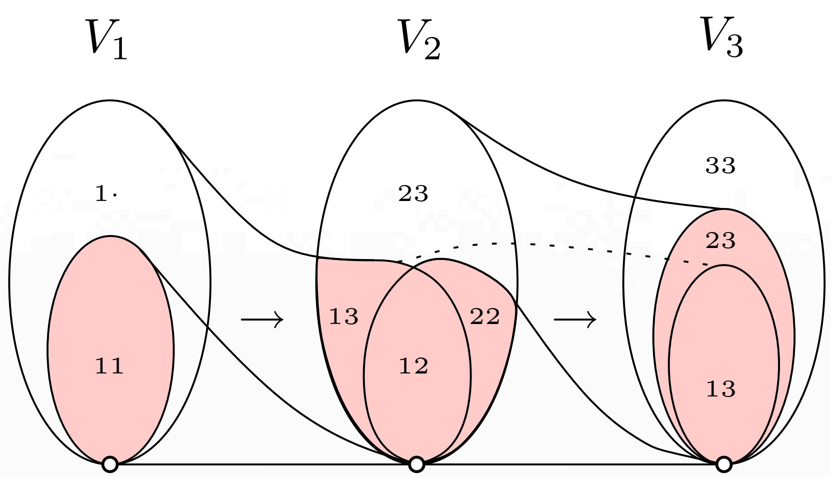
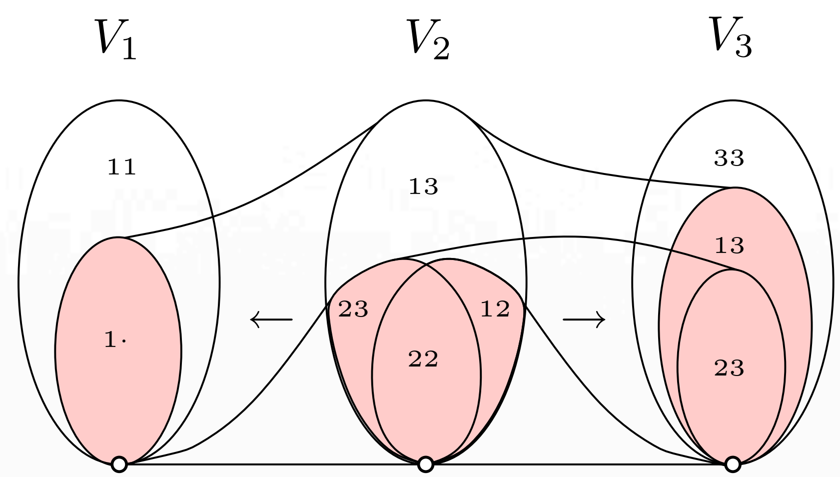
Three-term sequences

$V_1 \to V_2 \to V_3$

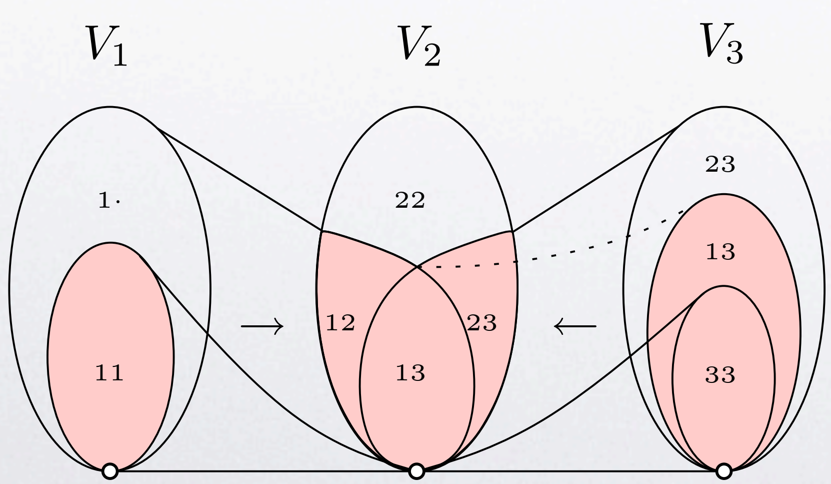


$V_1 \leftarrow V_2 \to V_3$

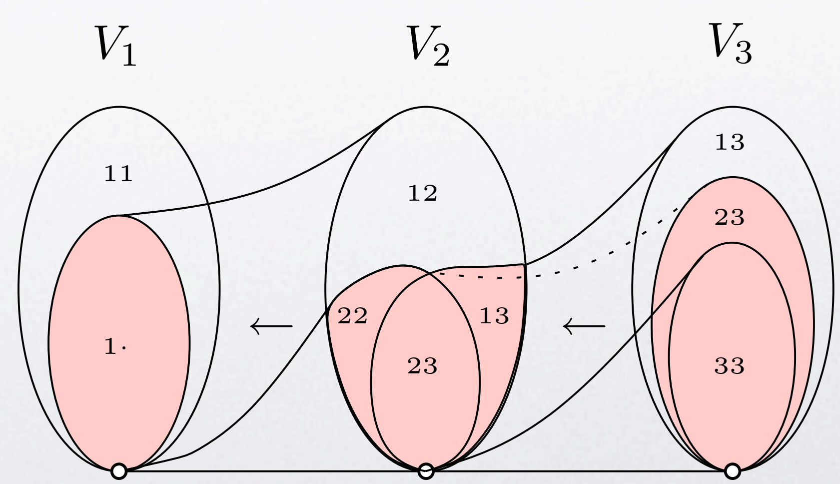


(figures by Dmitriy)

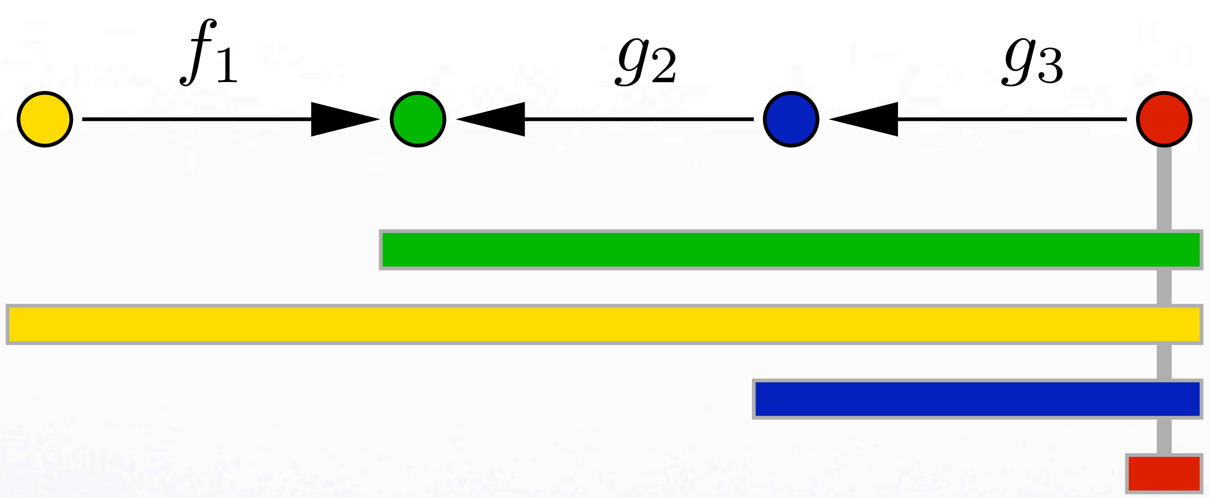
$V_1 \to V_2 \leftarrow V_3$

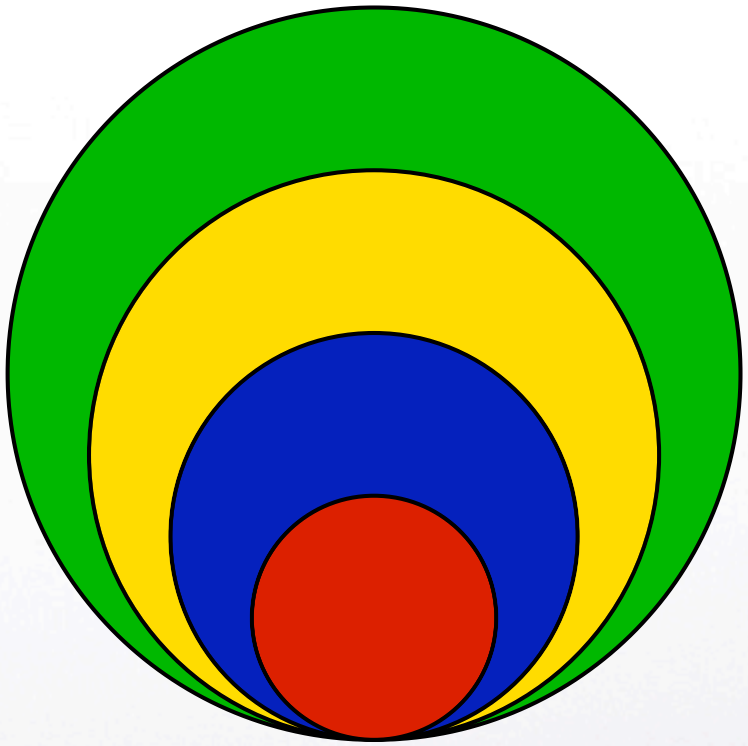


$V_1 \leftarrow V_2 \leftarrow V_3$



Four-term sequence

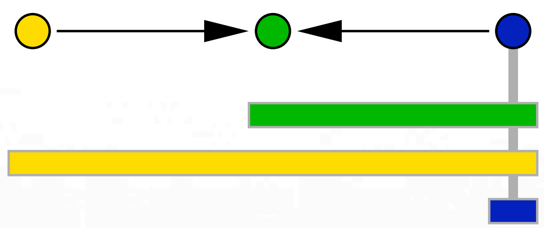




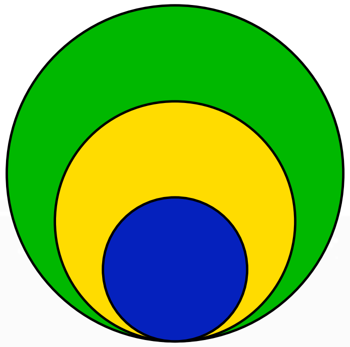
$$g_3^{-1}(0) \subseteq g_3^{-1}(g_2^{-1}(0)) \subseteq g_3^{-1}(g_2^{-1}(f_1(V_0))) \subseteq V_3$$

*　　*　　*　　*

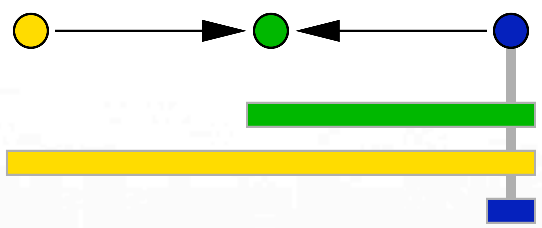
Inductive step



$g_2^{-1}(0) \subseteq g_2^{-1}(f_1(V_0)) \subseteq V_2$



Inductive step



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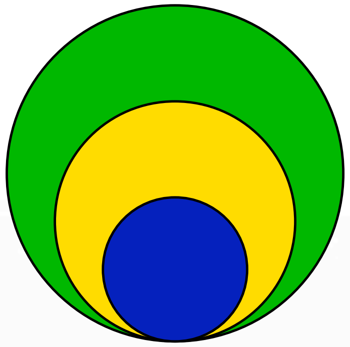

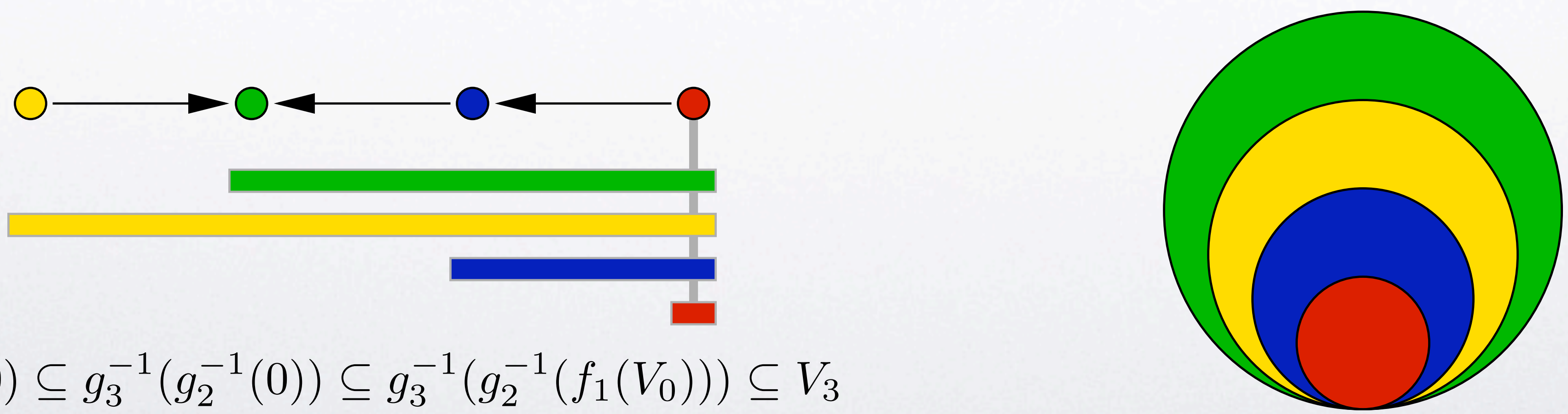

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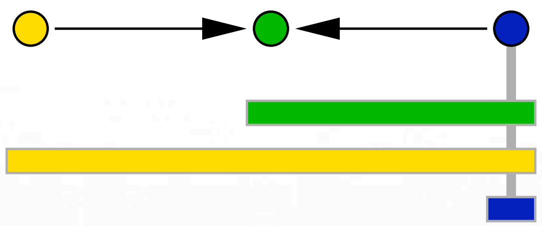
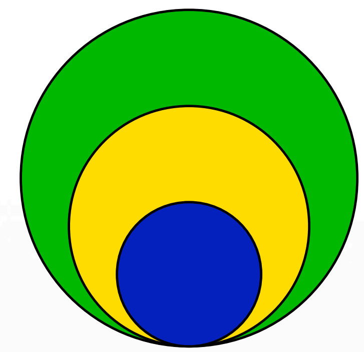
Inductive step





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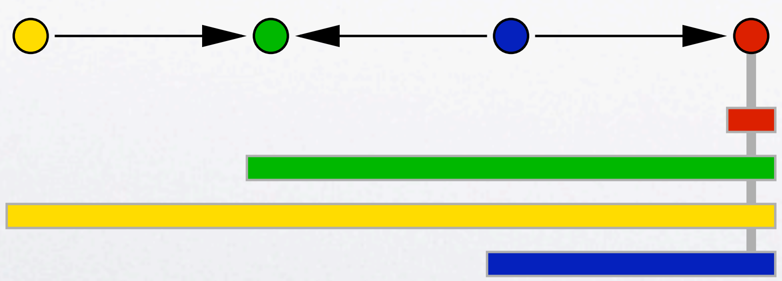

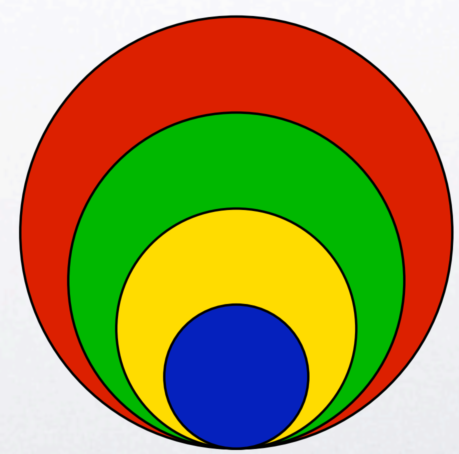

$f_3(g_2^{-1}(0)) \subseteq f_3(g_2^{-1}(f_1(V_0))) \subseteq f_3(V_2) \subseteq V_3$

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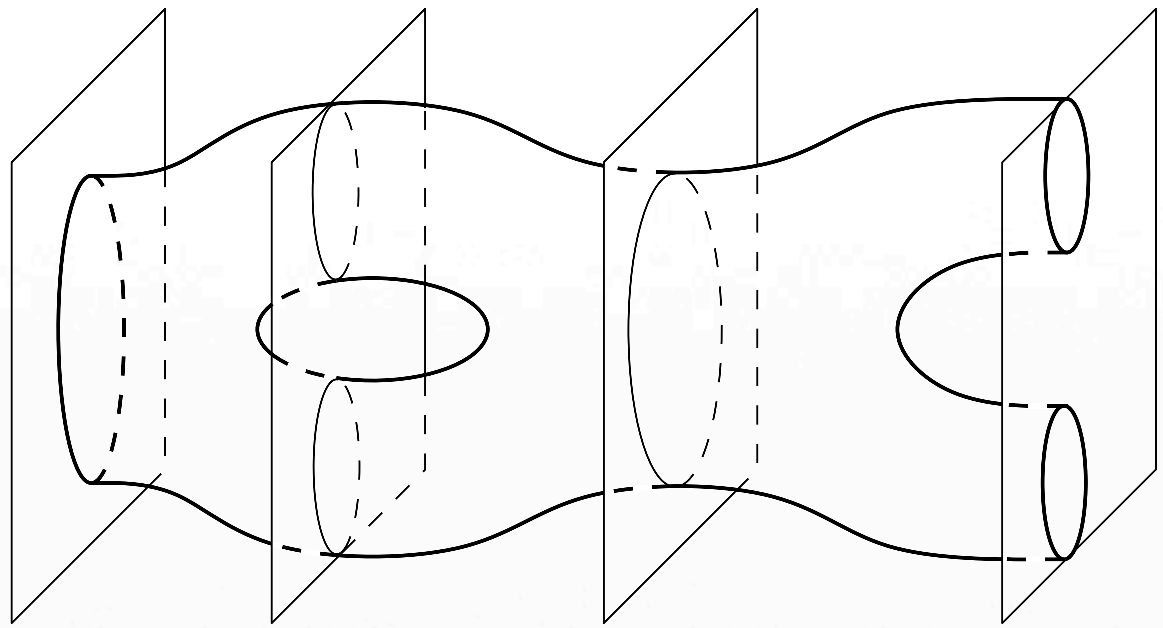
Levelset zigzag persistence

Levelset zigzag persistence

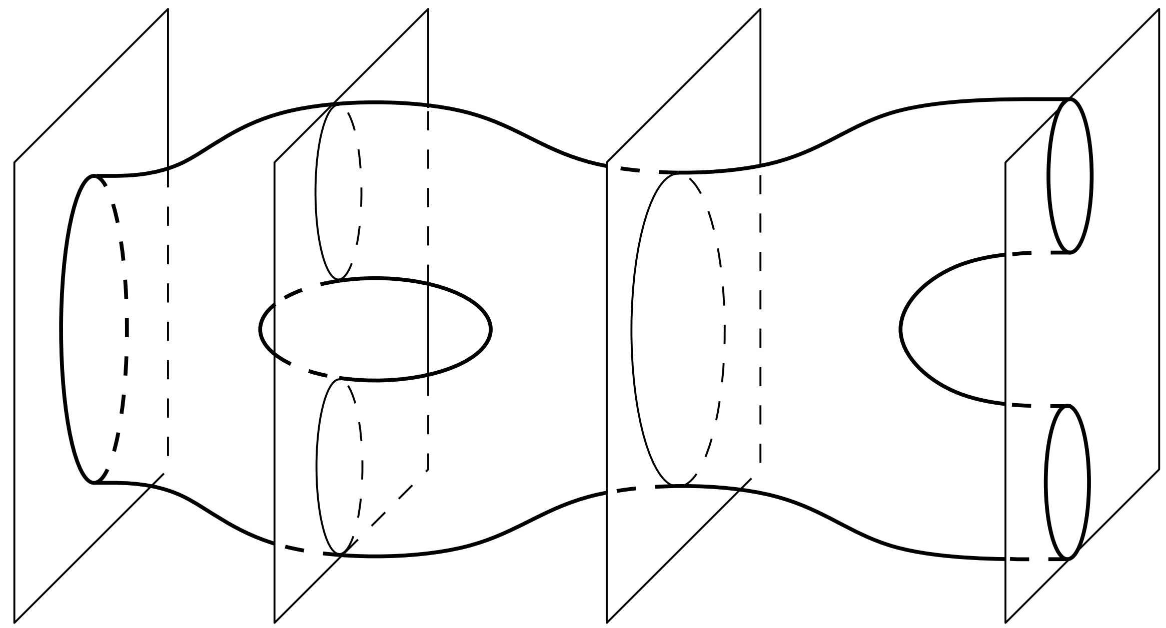


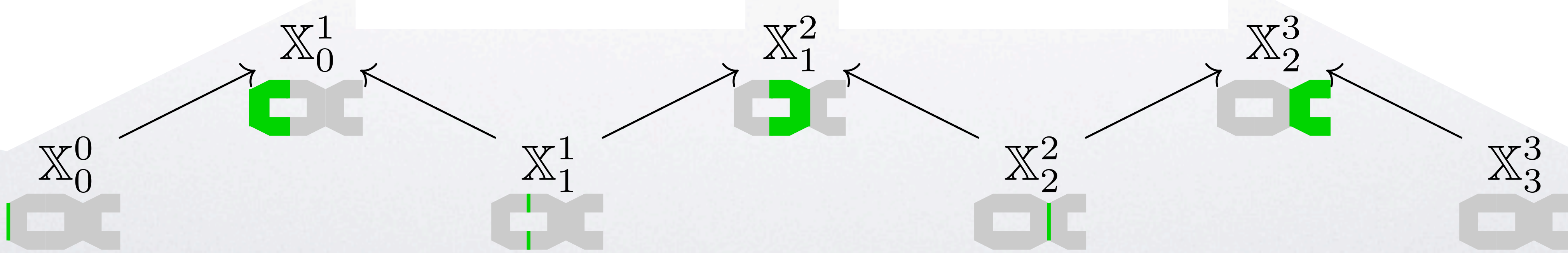
- X any space with a (tame) real-valued function f.
- Define the levelset zigzag of (X,f):

$$\mathbb{X}_0^0 \rightarrow \mathbb{X}_0^1 \leftarrow \mathbb{X}_1^1 \rightarrow \mathbb{X}_1^2 \leftarrow \mathbb{X}_2^2 \rightarrow \ldots \leftarrow \mathbb{X}_{n-1}^{n-1} \rightarrow \mathbb{X}_{n-1}^{n} \leftarrow \mathbb{X}_n^n,$$

$$\mathbb{X}_i^j = f^{-1}[i,j]$$

Levelset zigzag persistence





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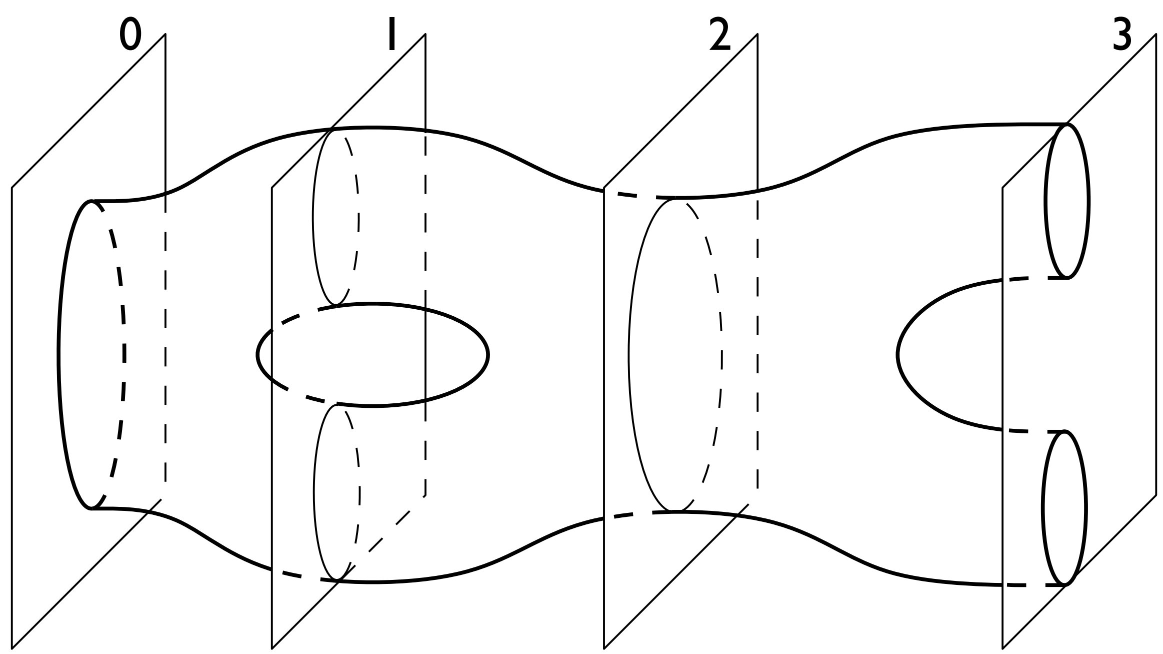

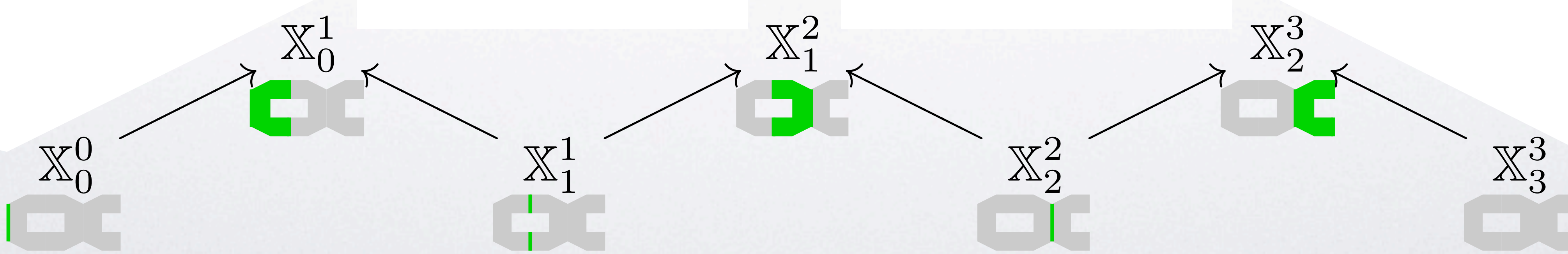

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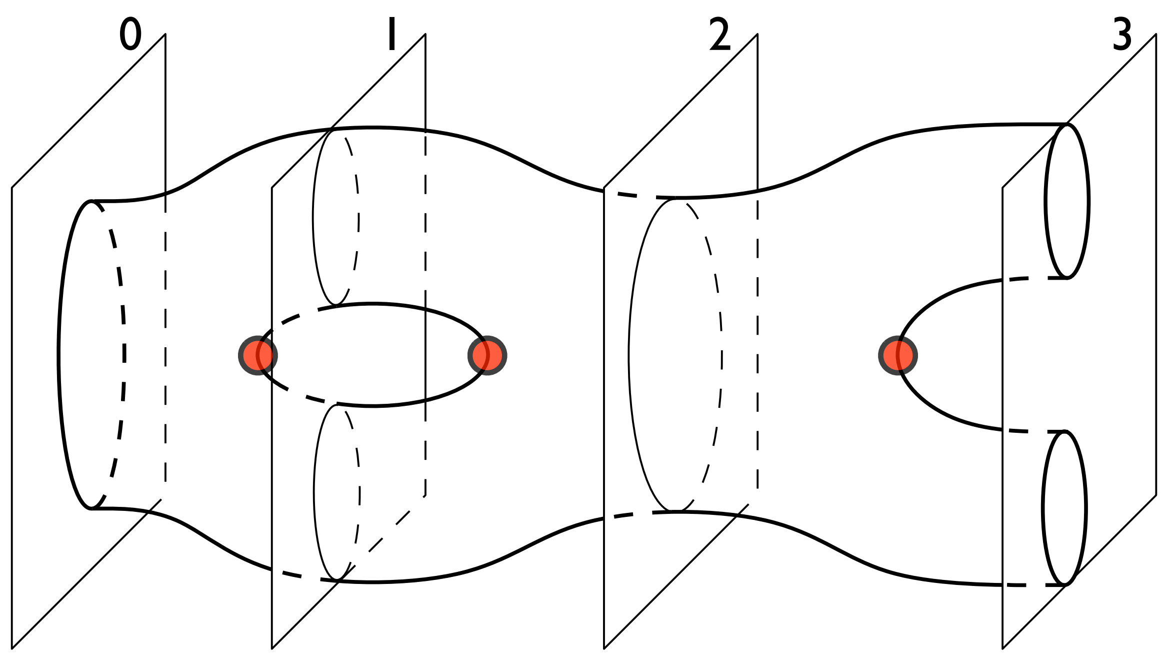

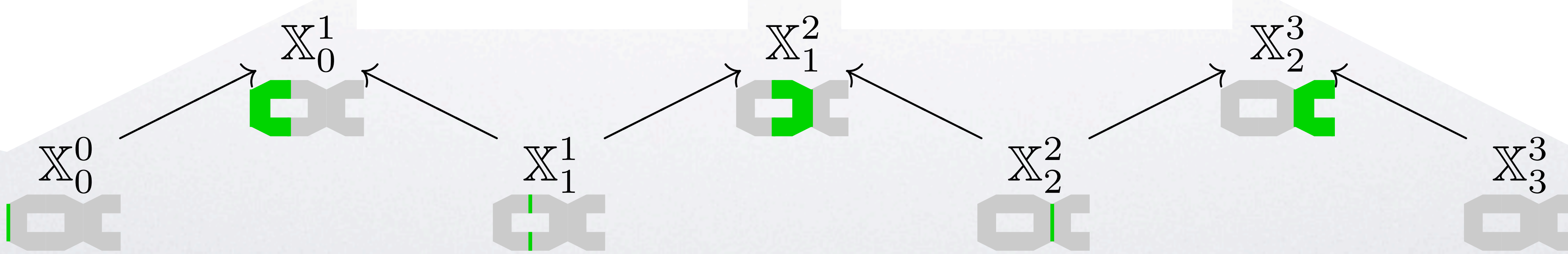

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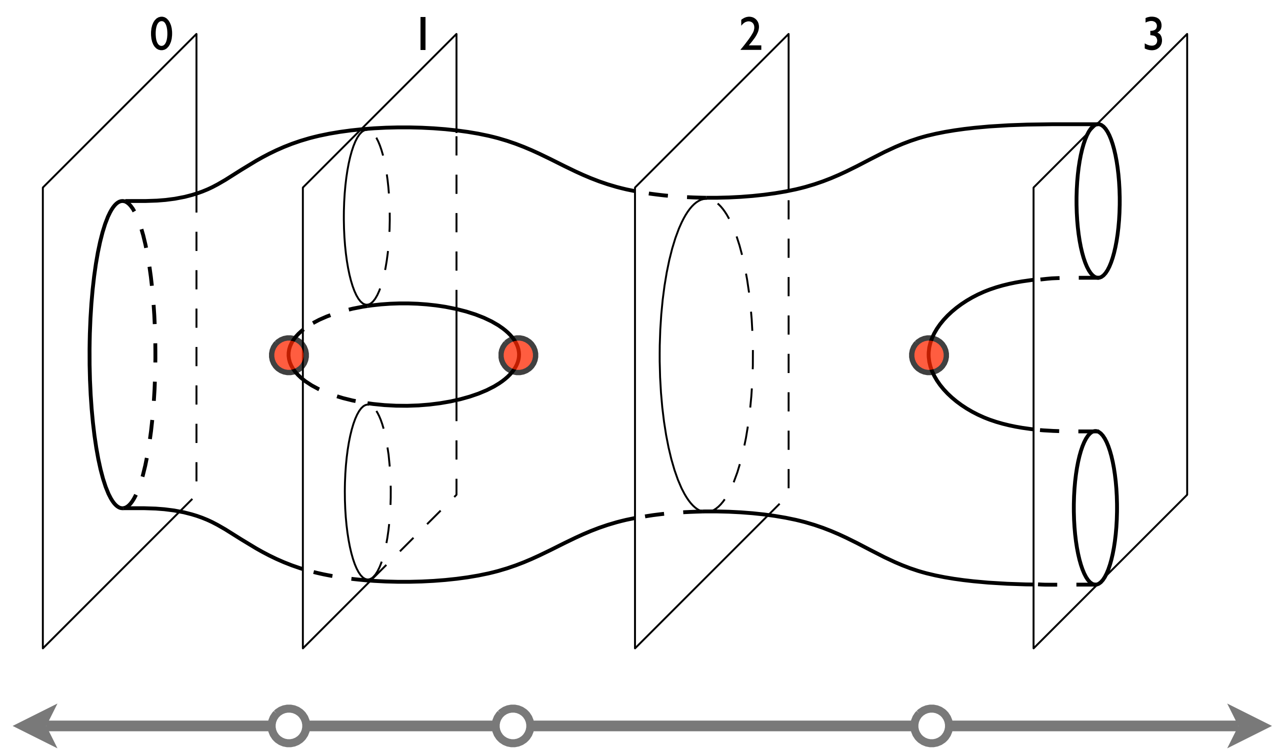

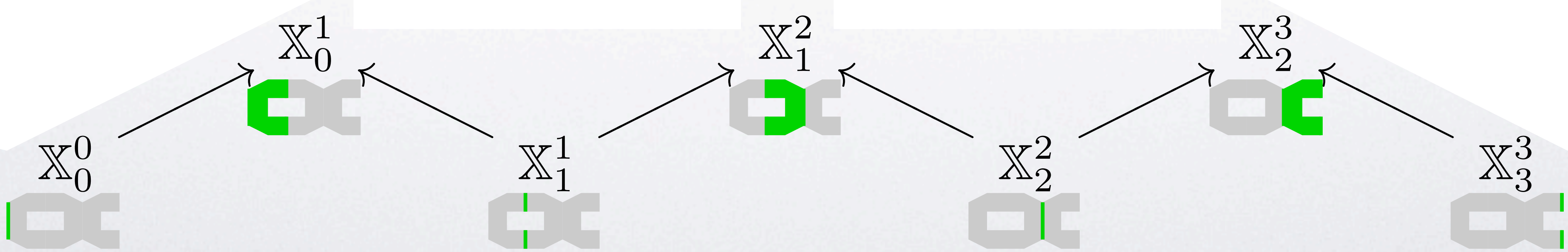

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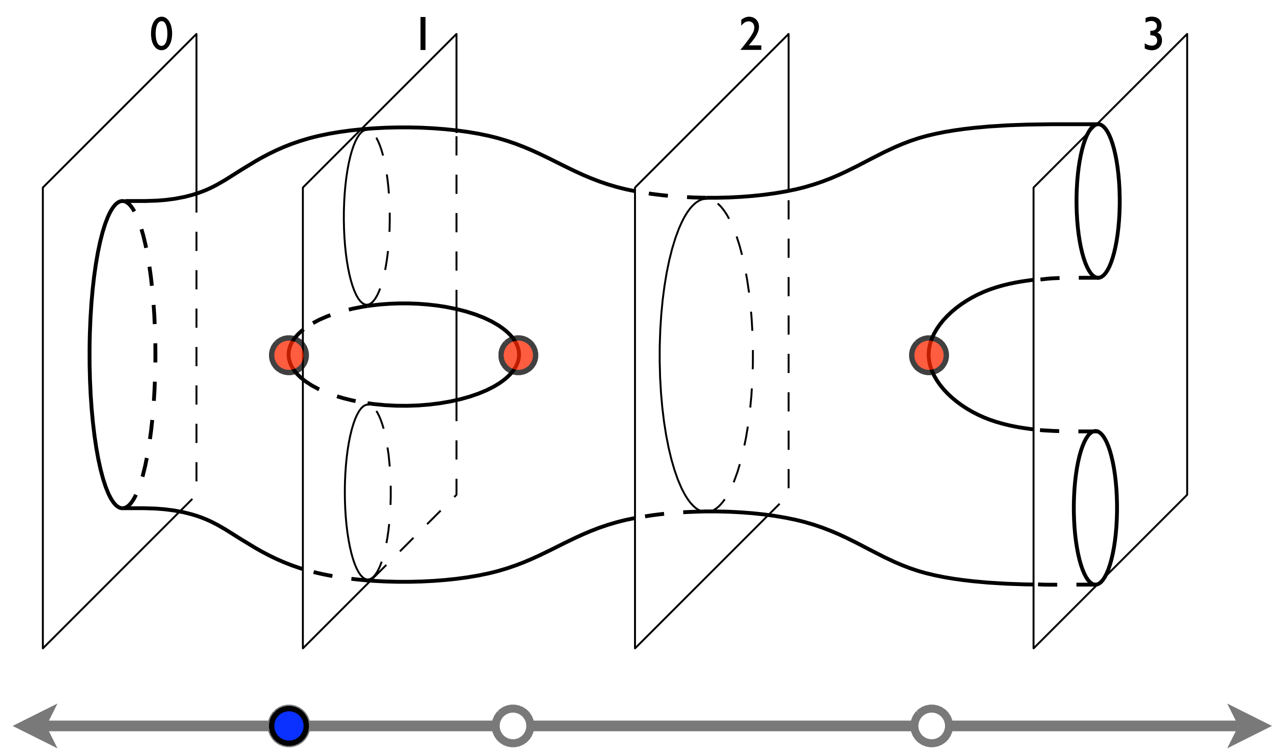

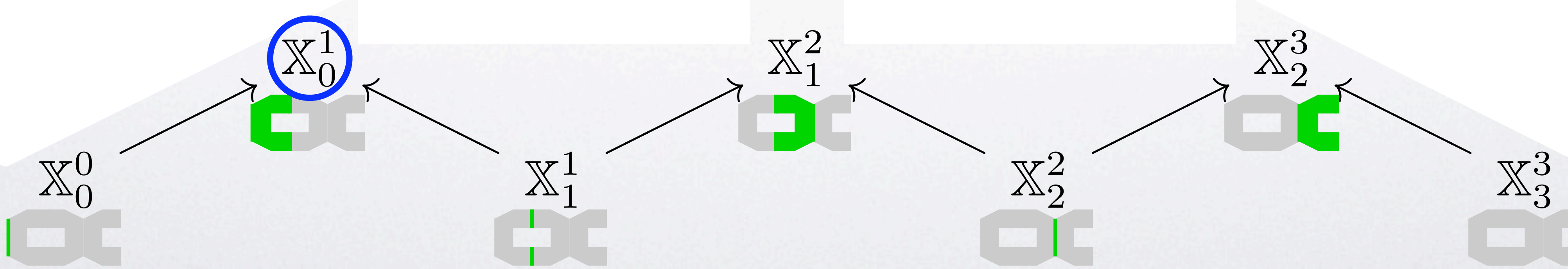

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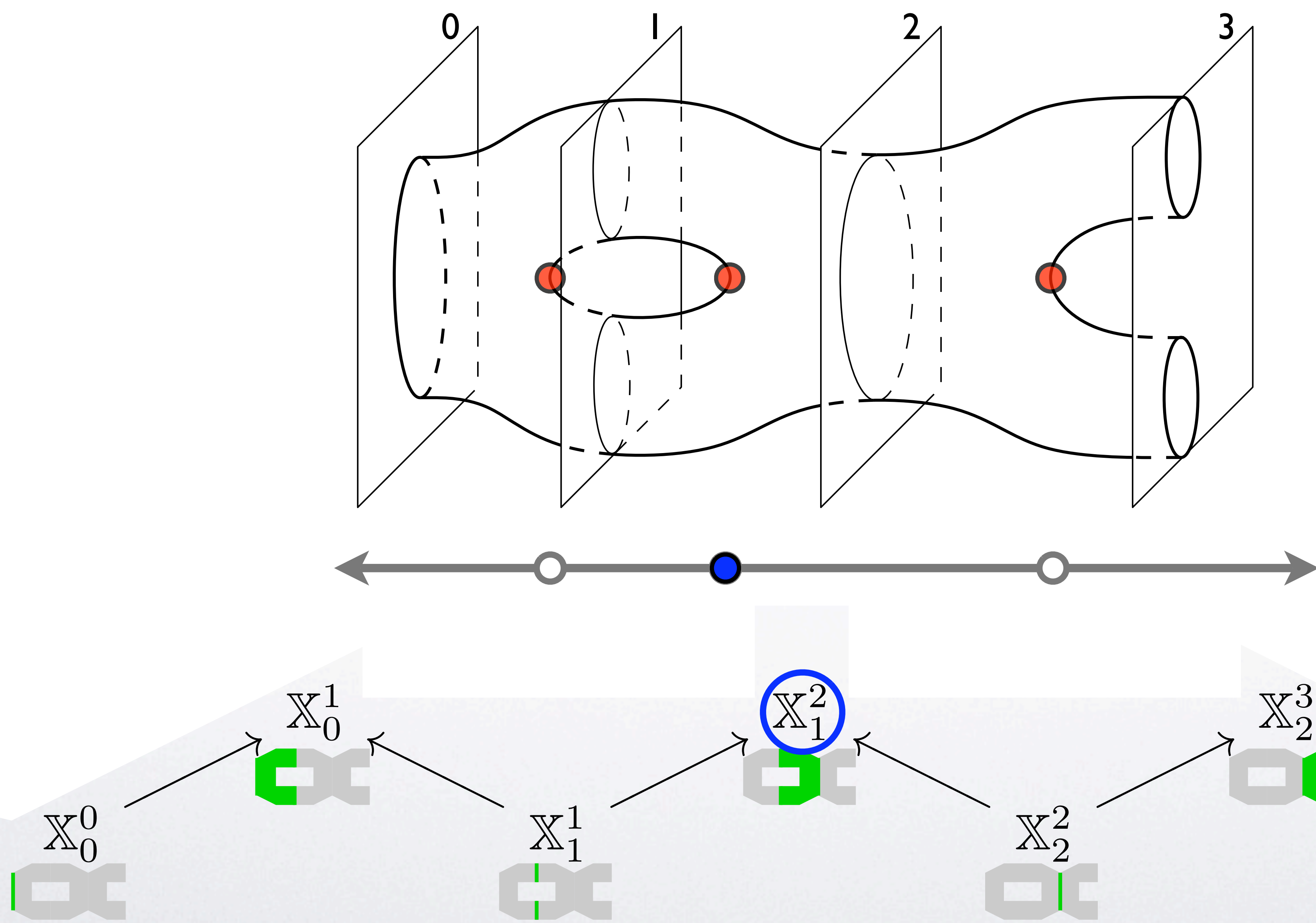

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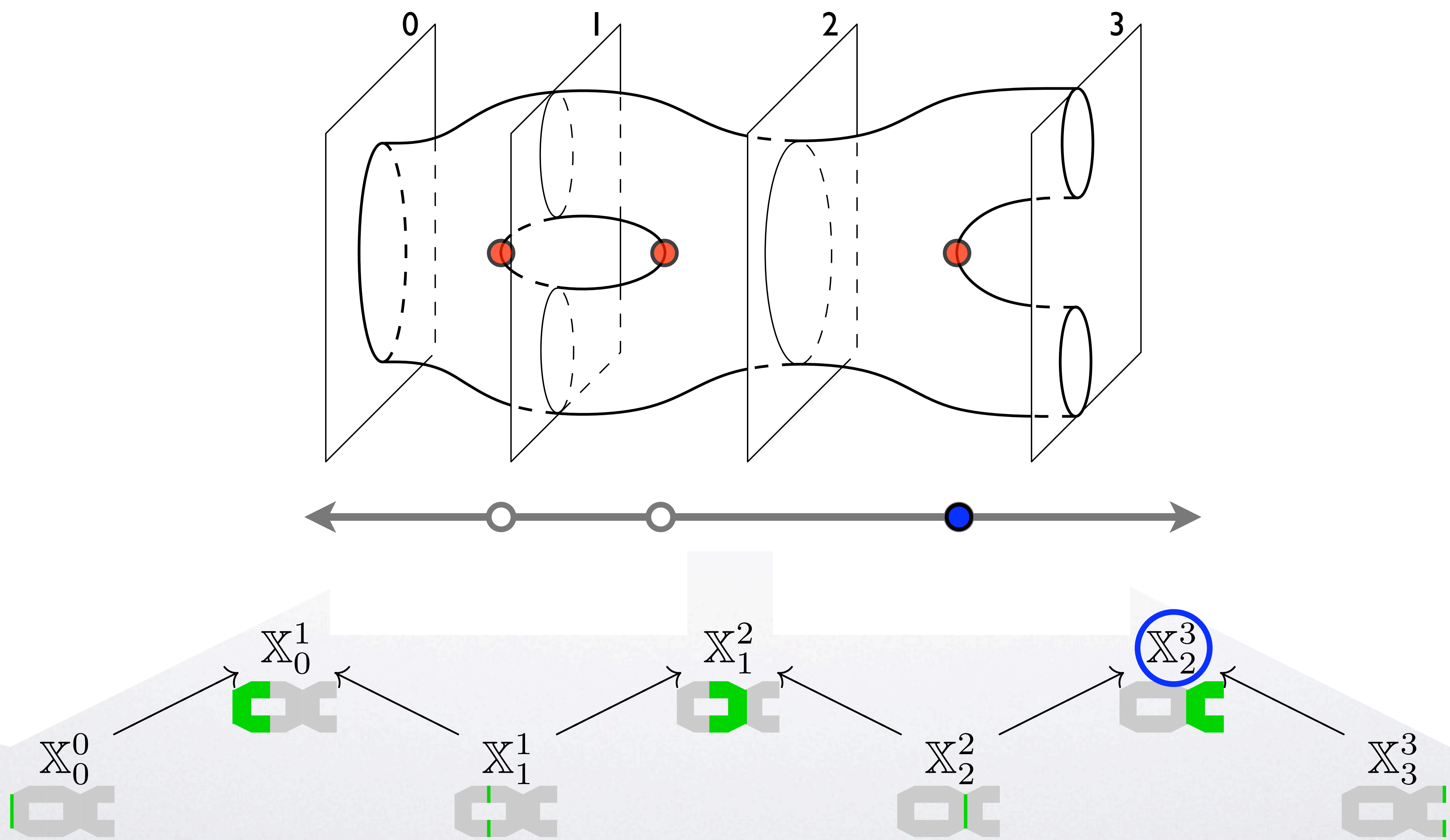
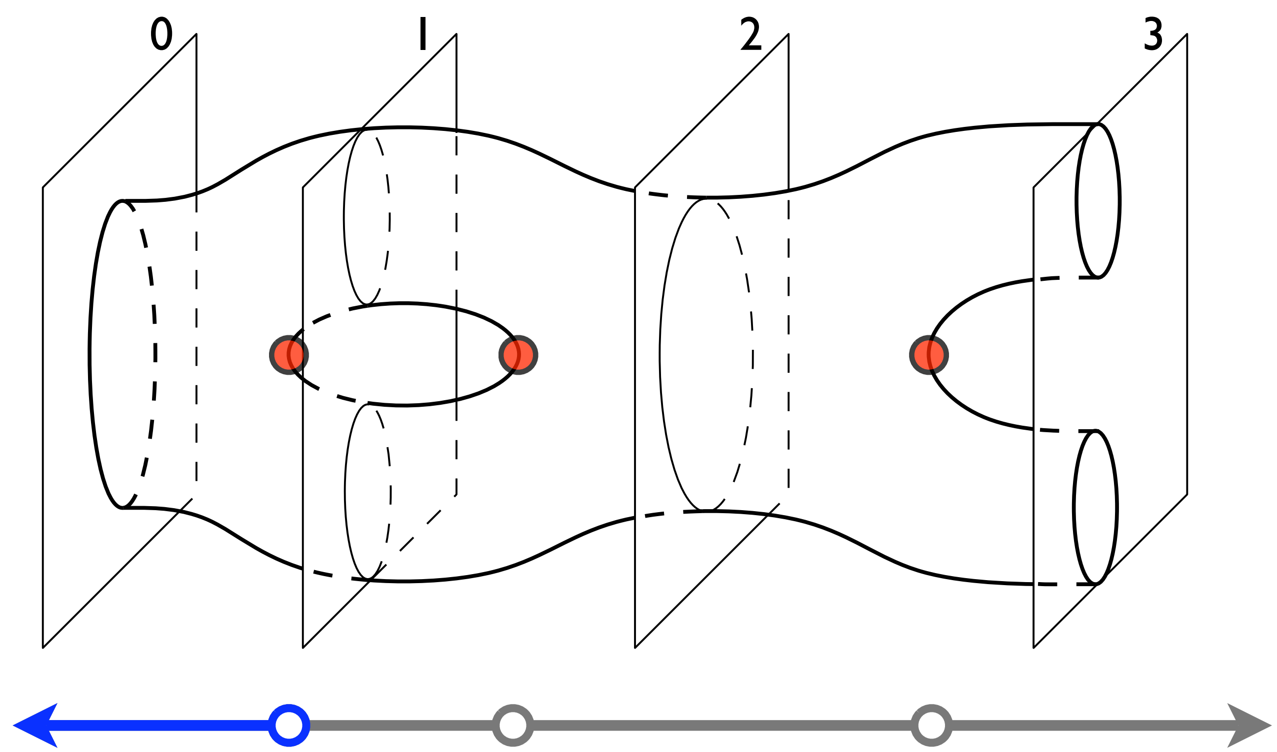
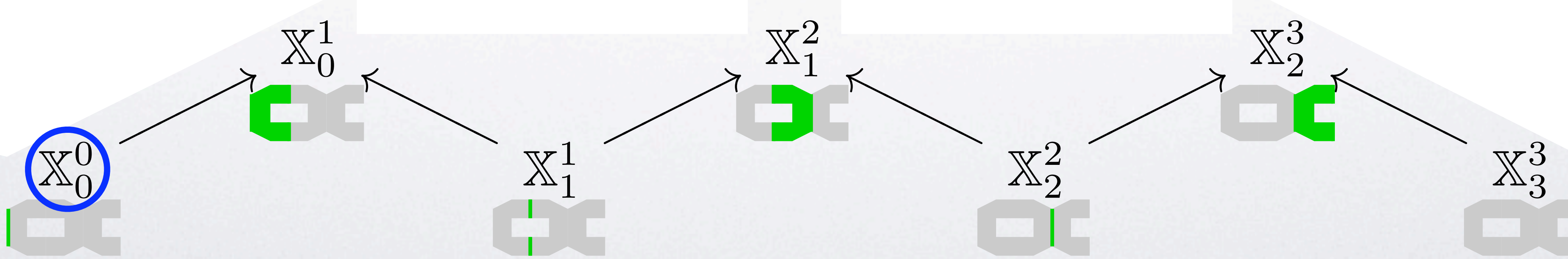


Levelset zigzag persistence





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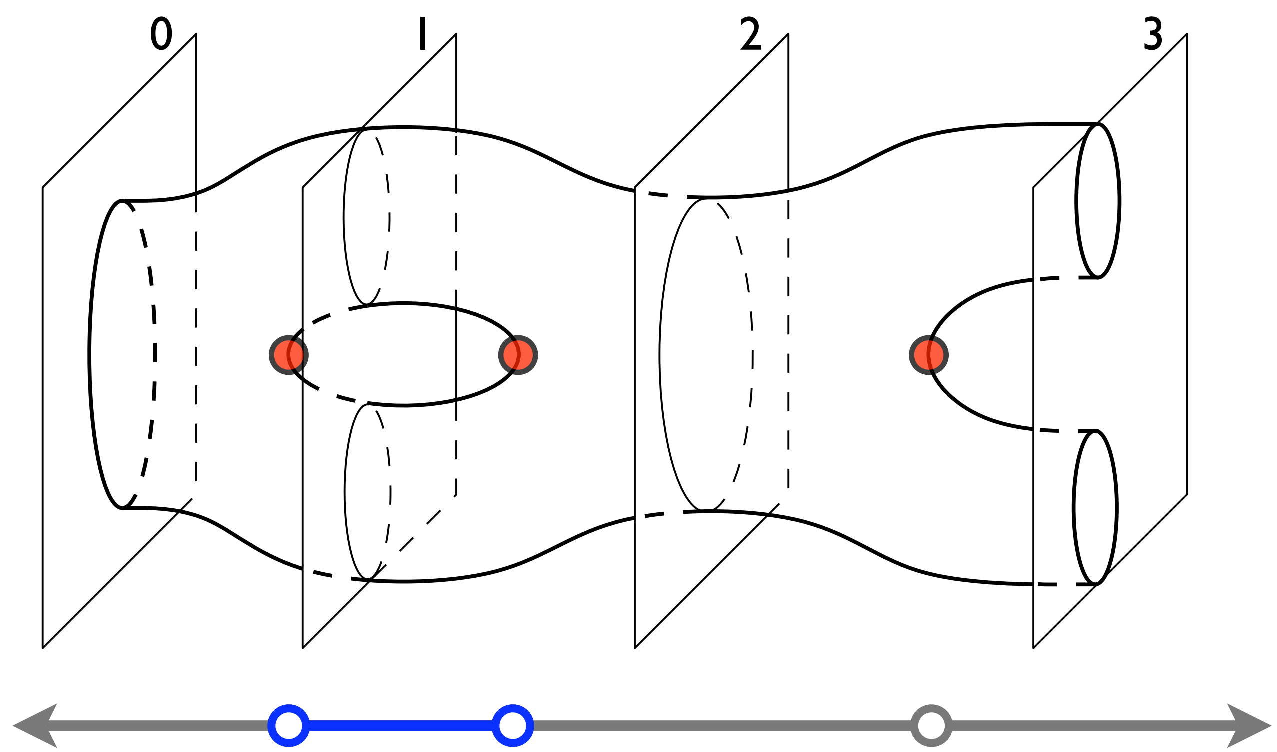
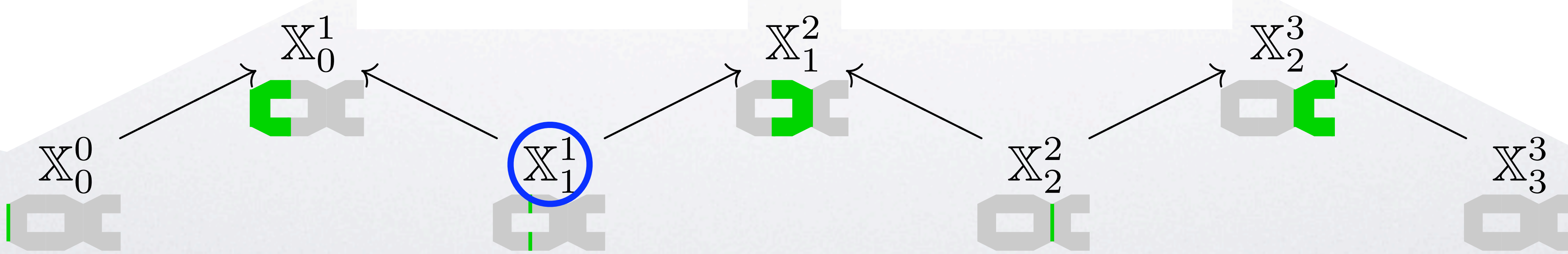
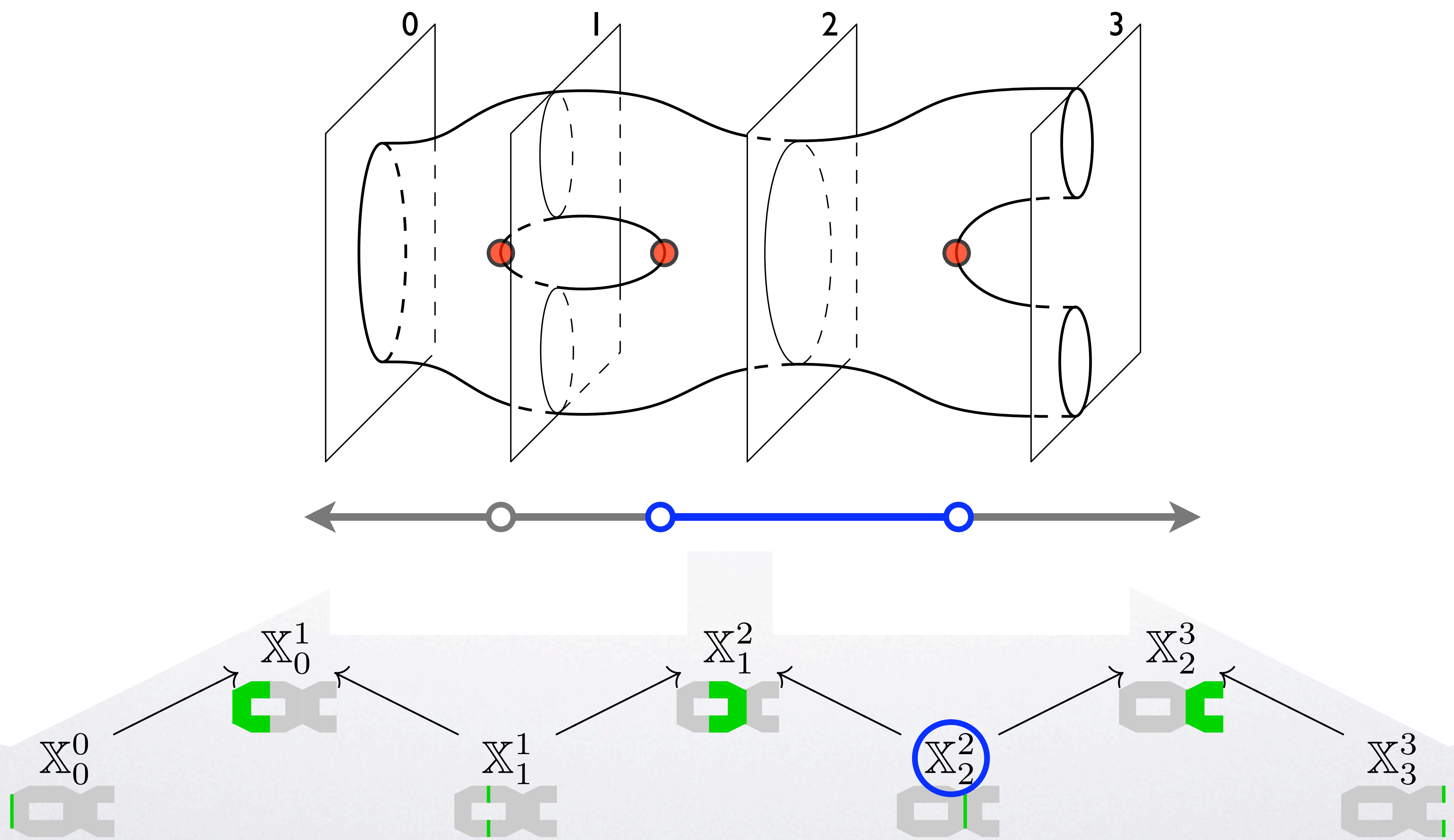




Levelset zigzag persistence



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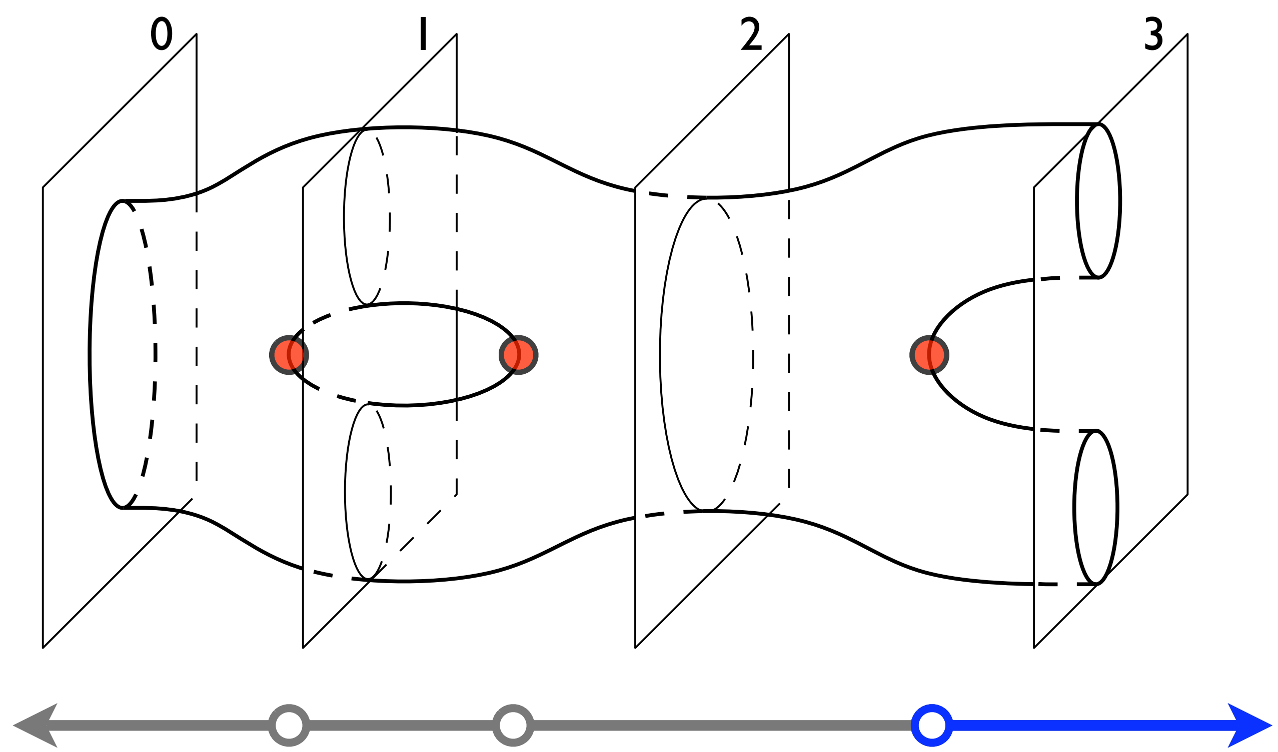
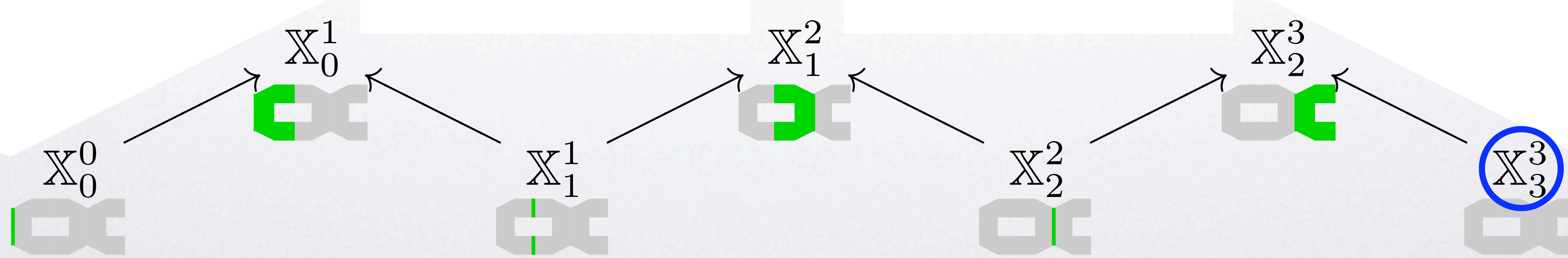
Levelset zigzag persistence





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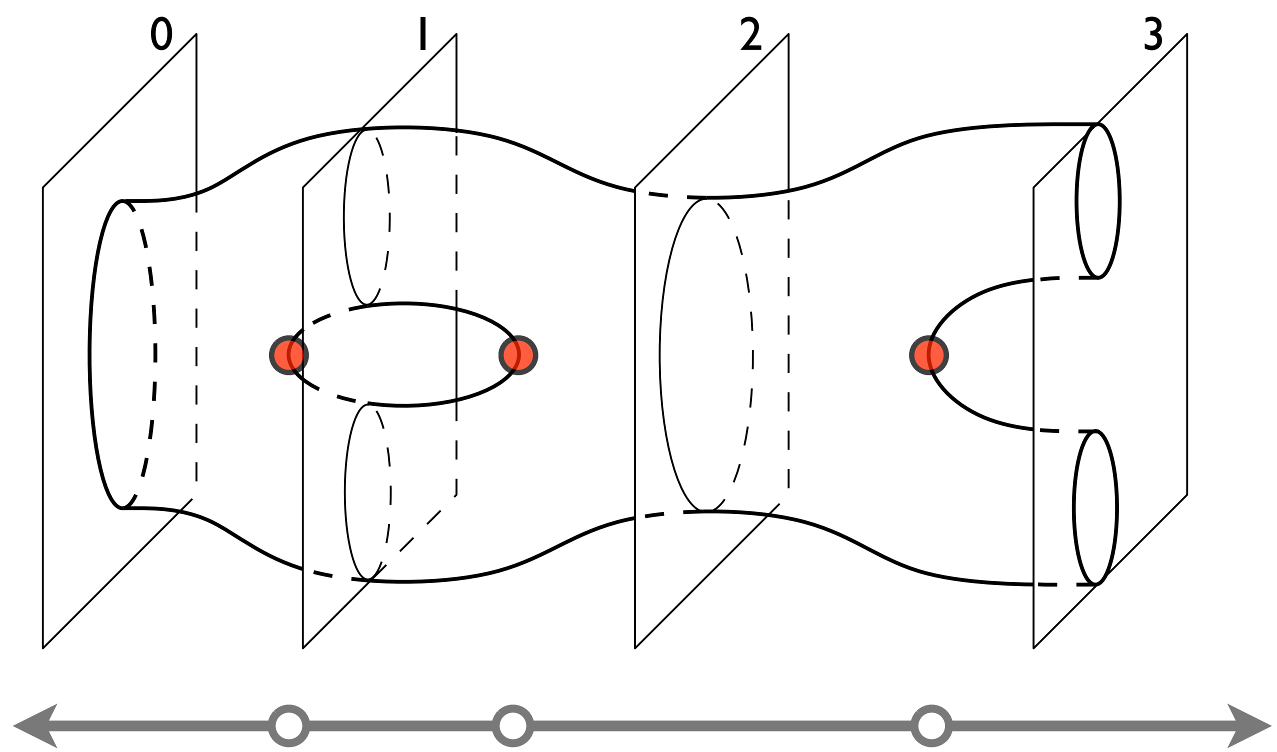

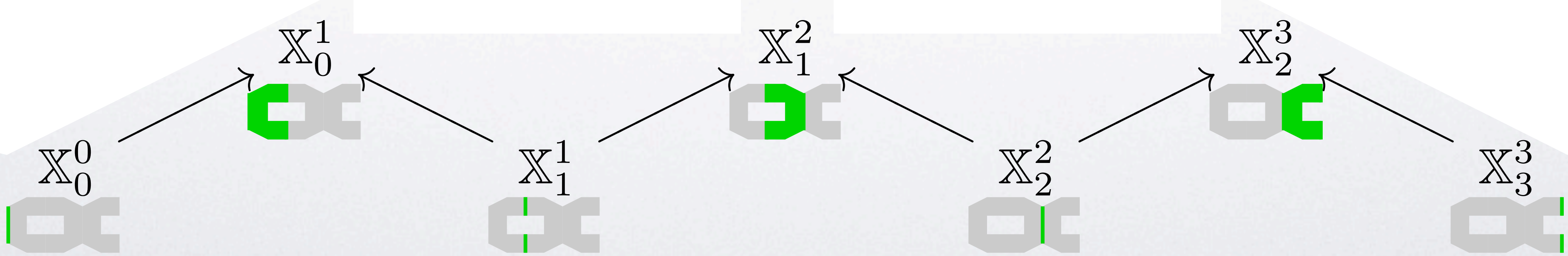

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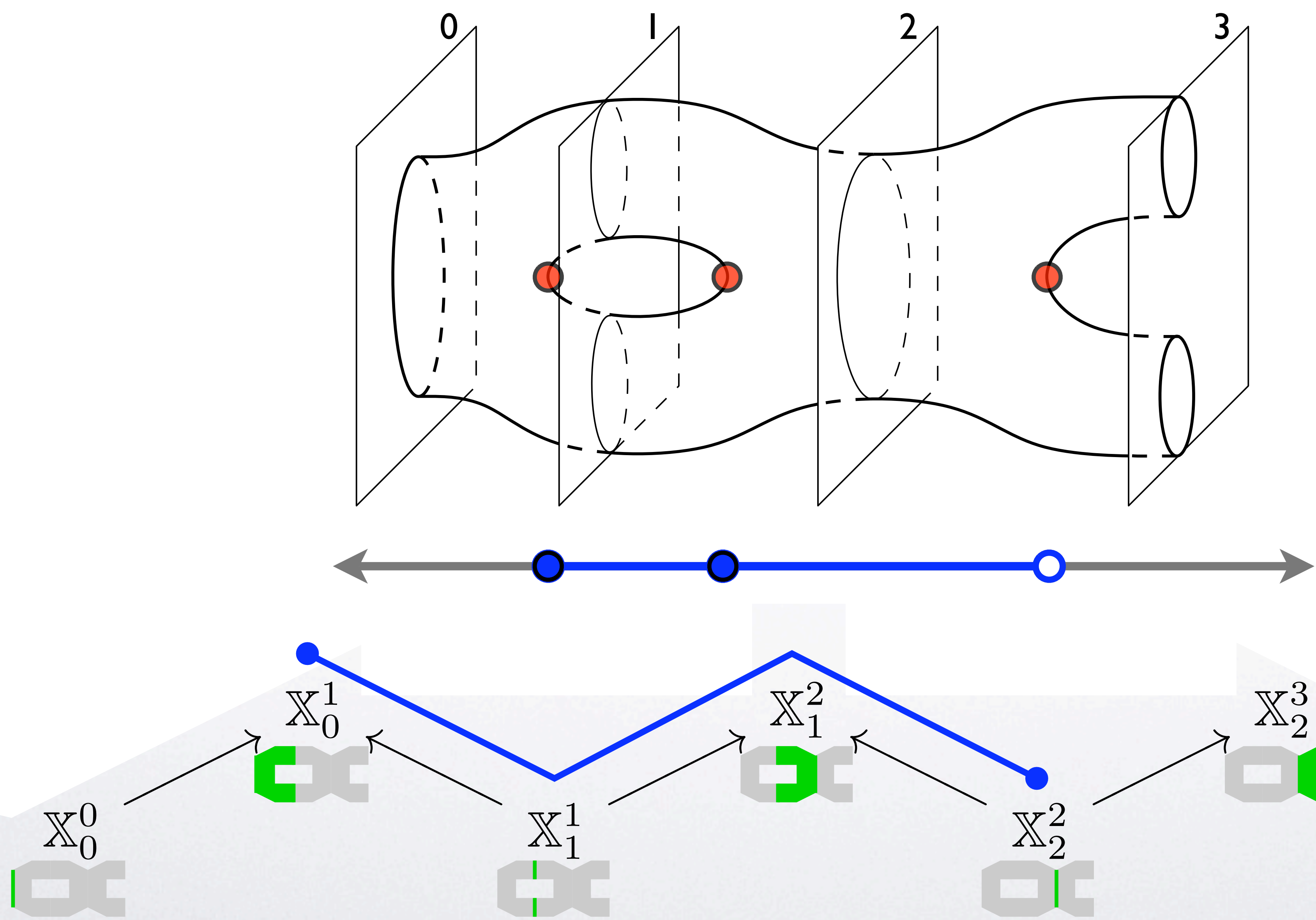

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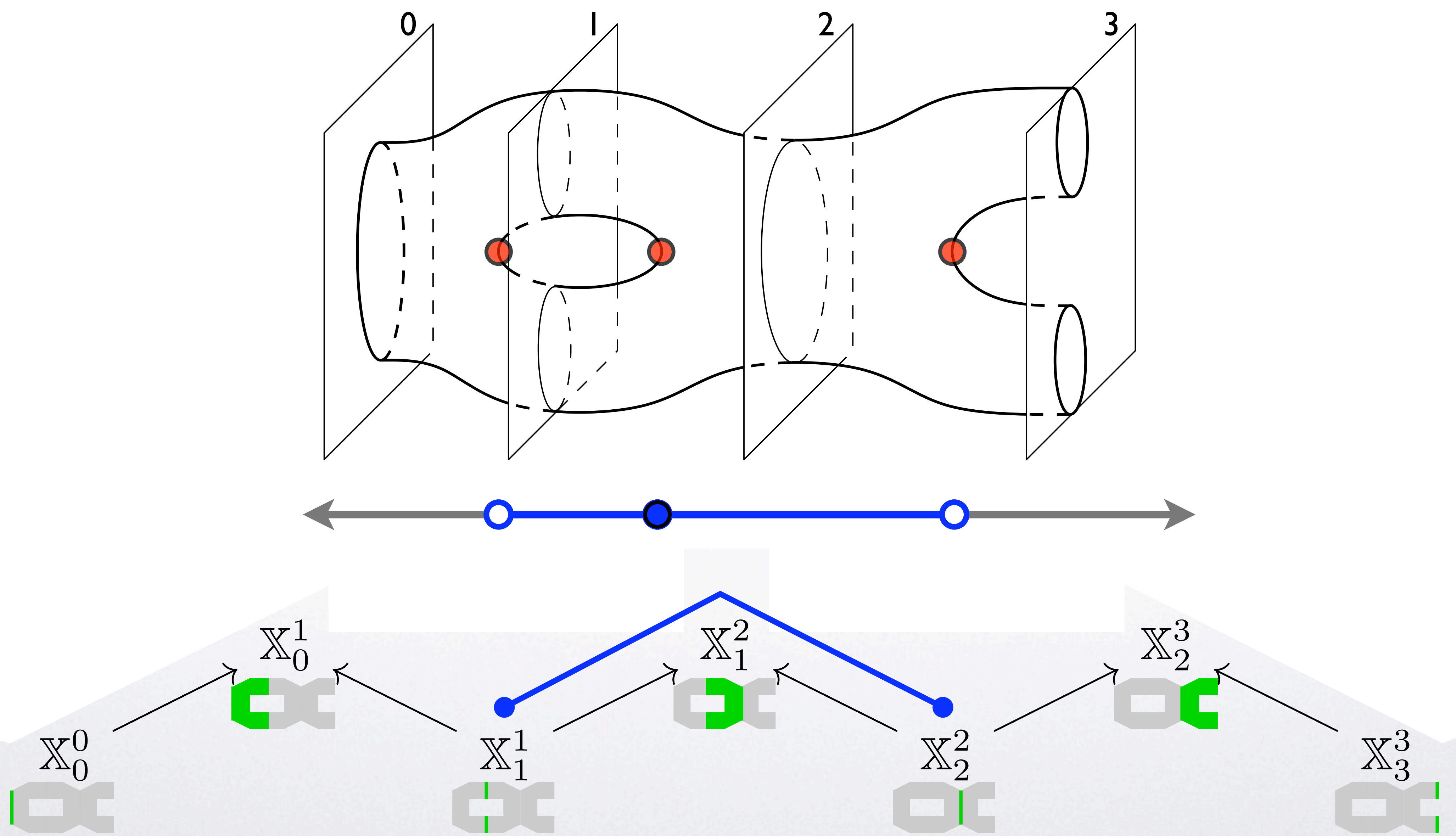
Levelset zigzag persistence



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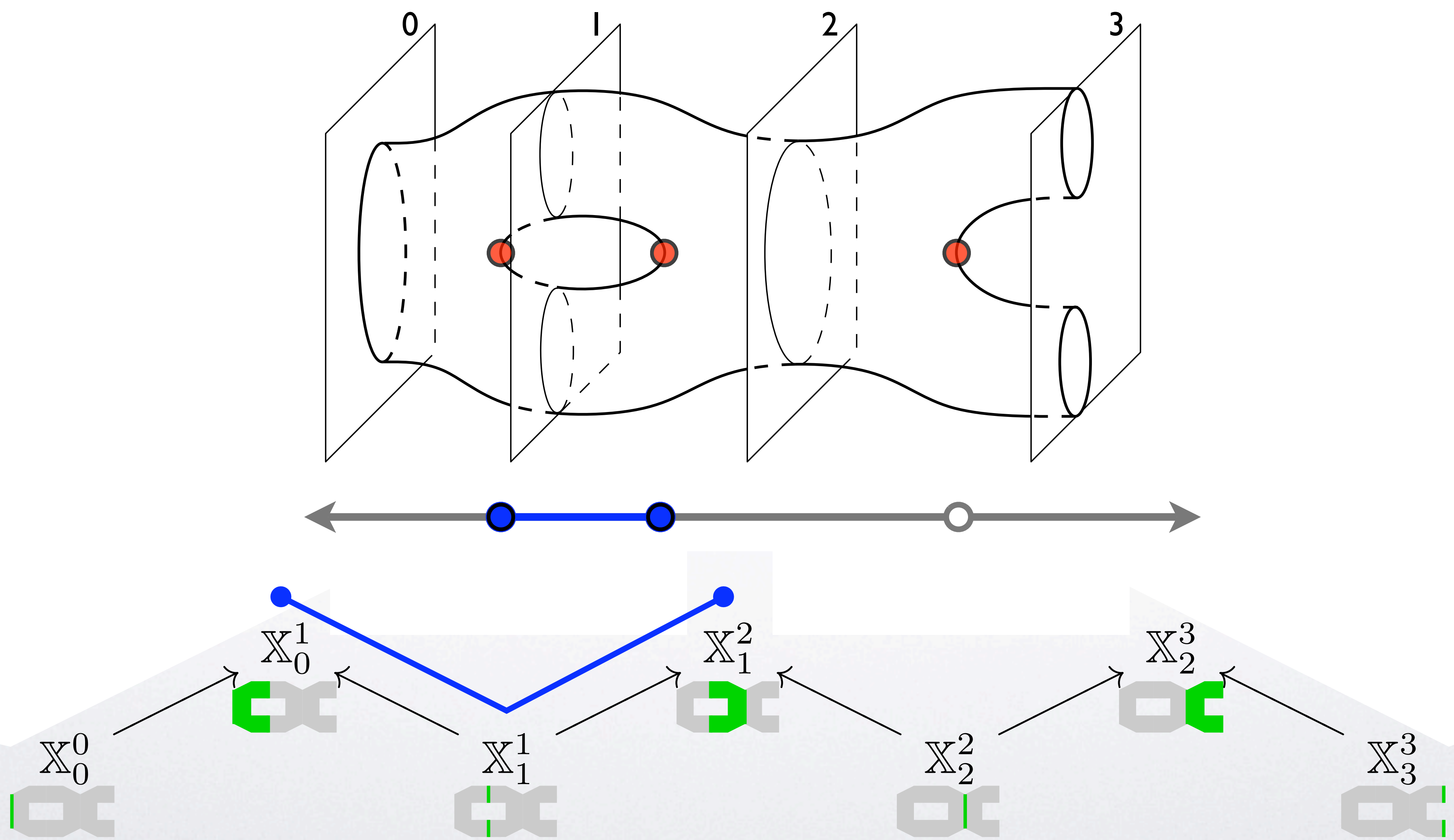

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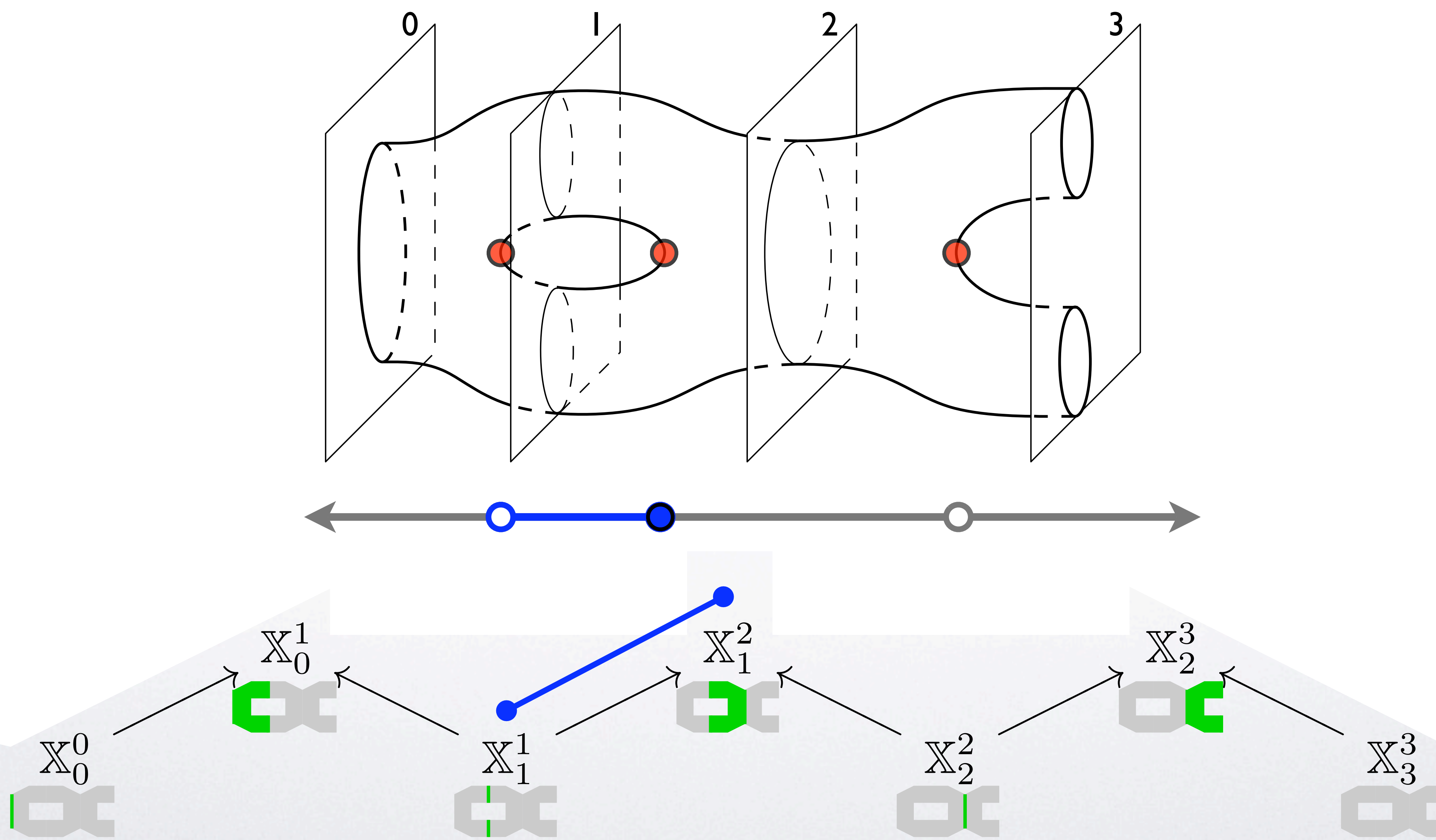
Levelset zigzag persistence



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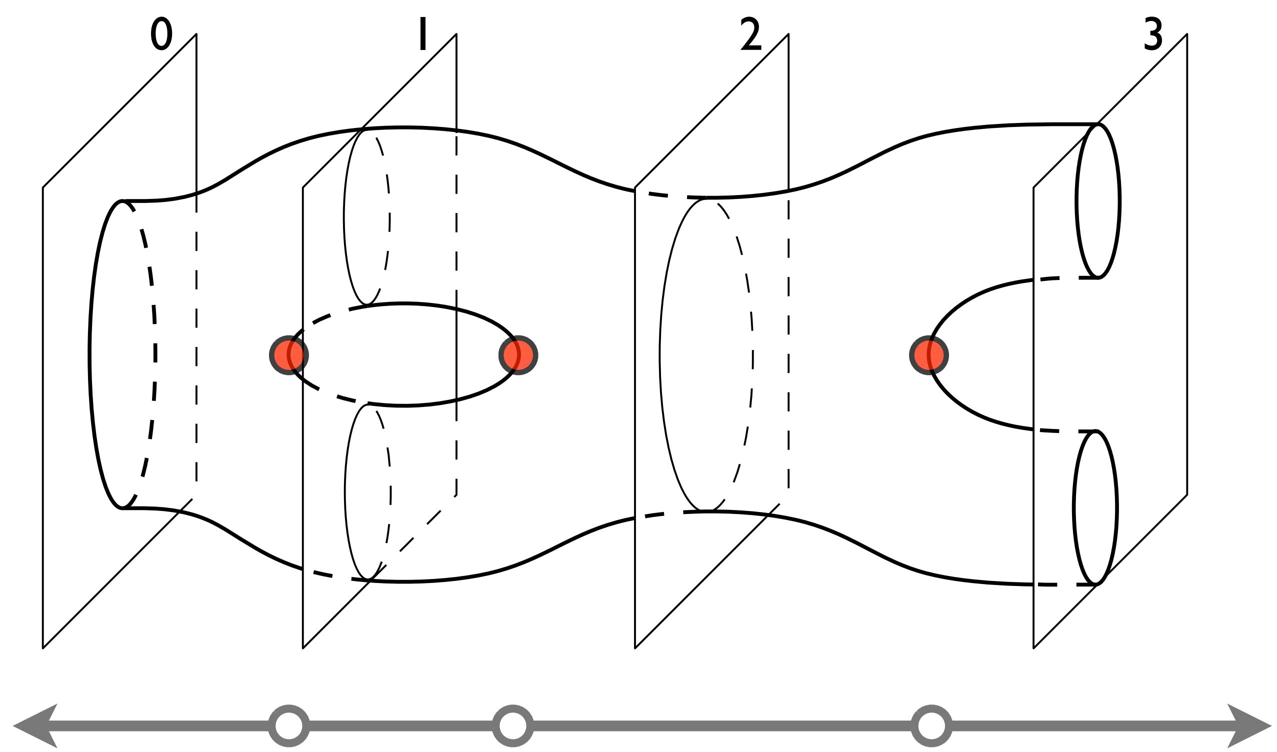

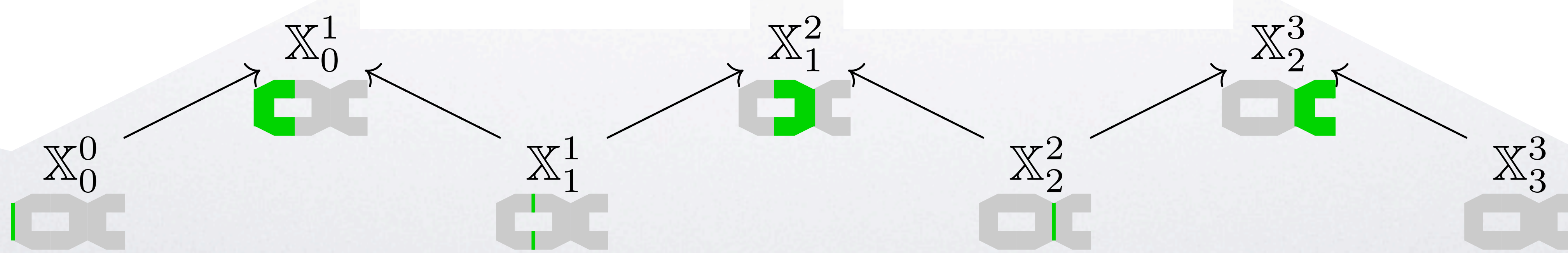

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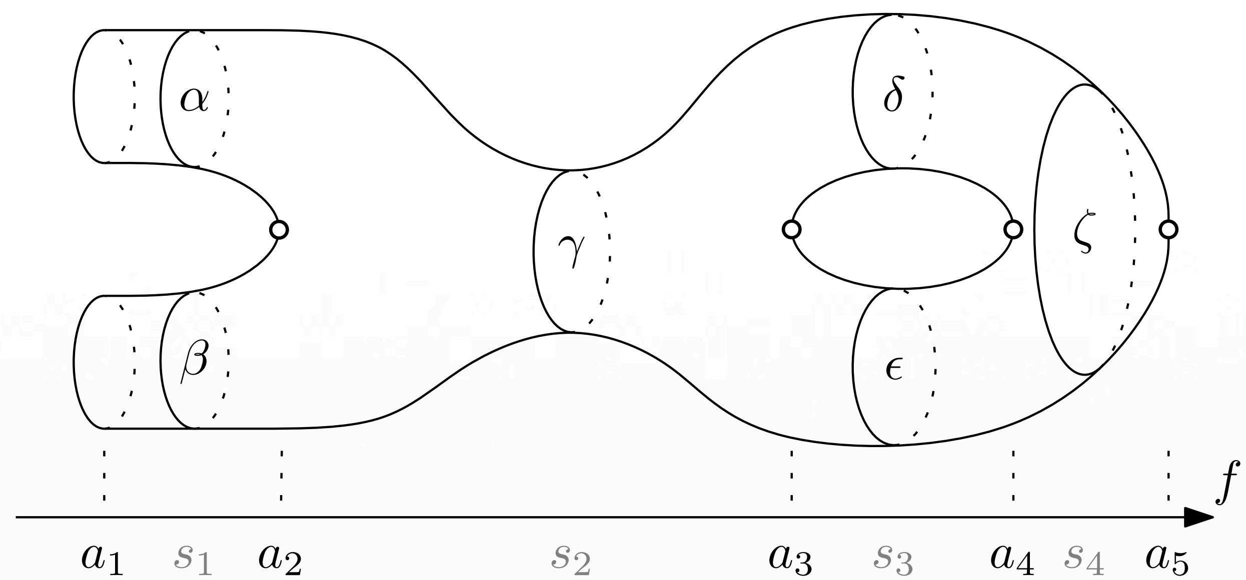
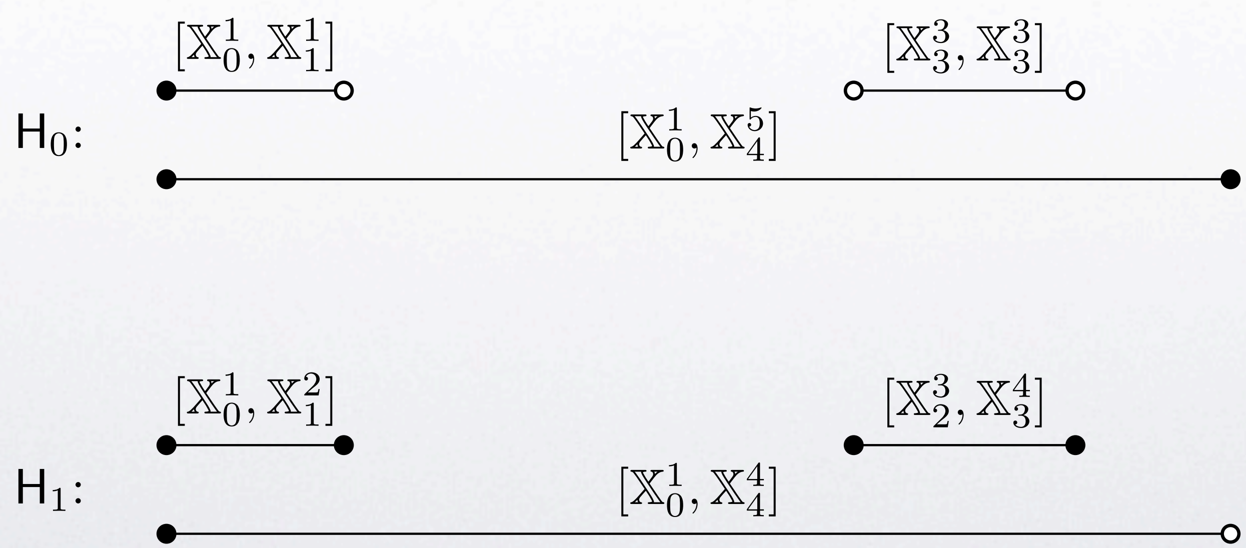
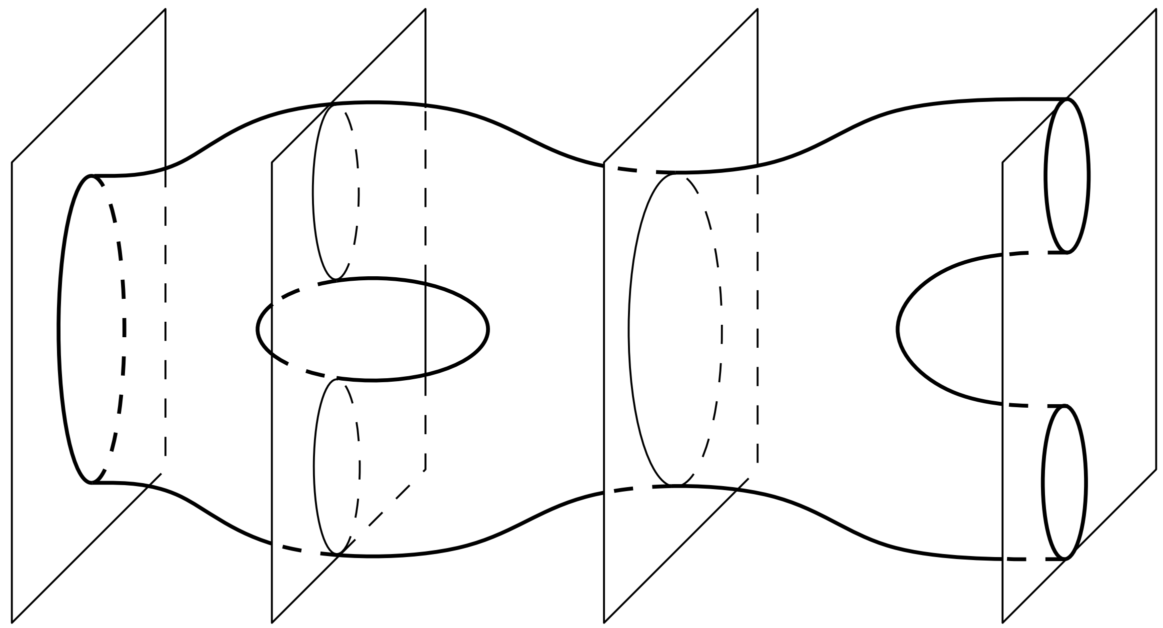
Levelset zigzag persistence

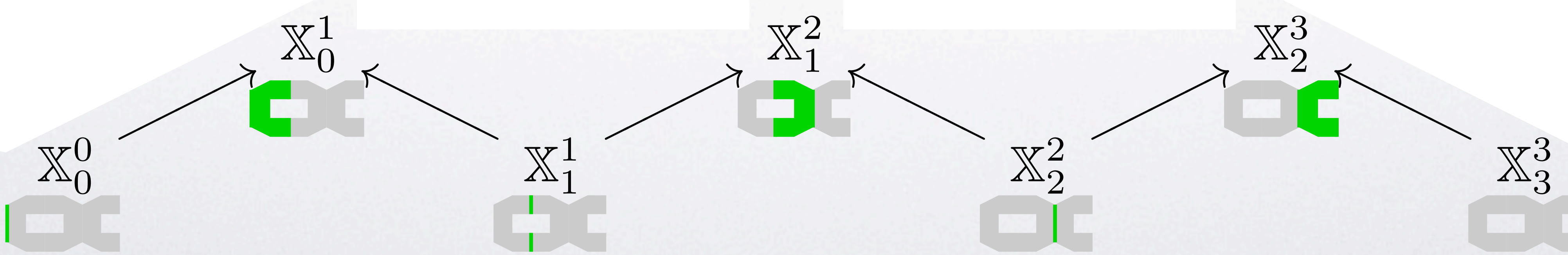




The pyramid theorem

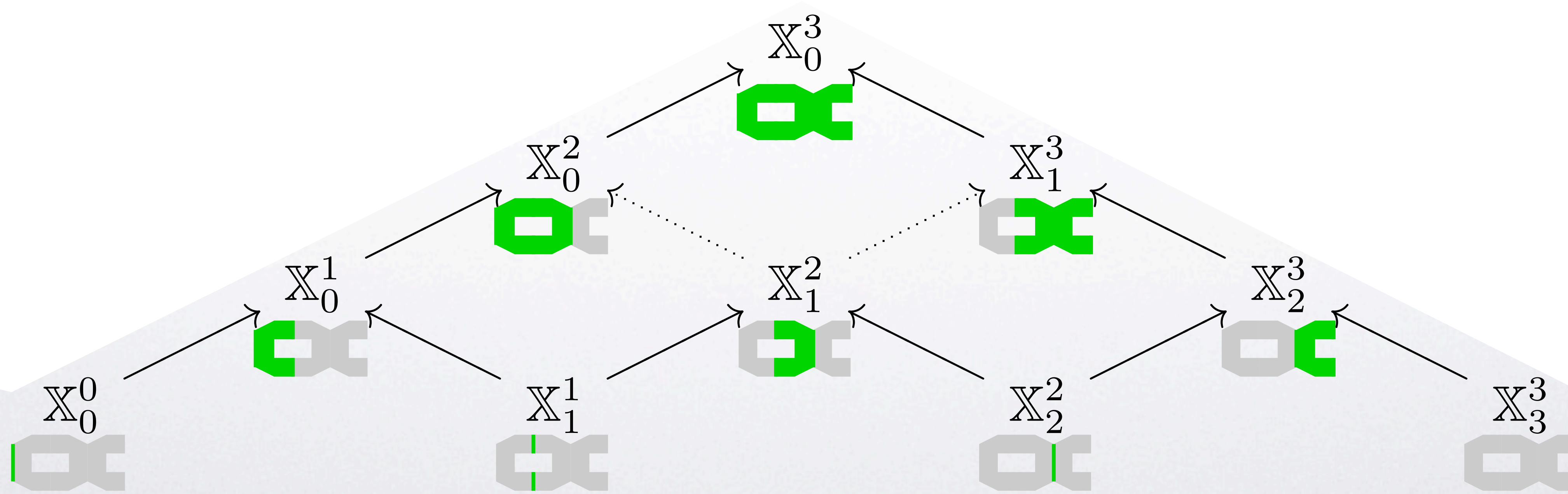
A vast commutative diagram





$\mathbb{X}_i^j = f^{-1}[i, j]$

A vast commutative diagram



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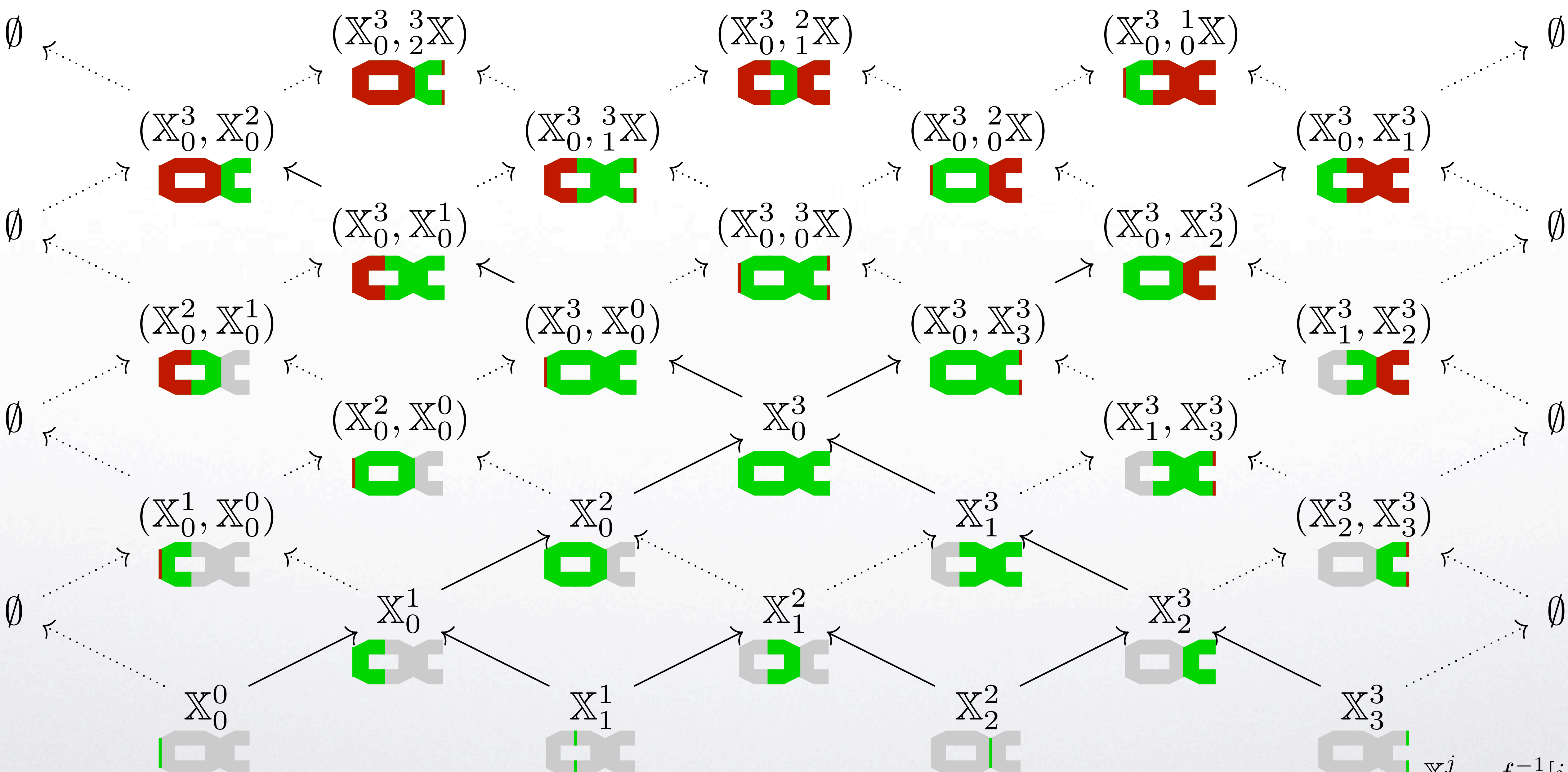
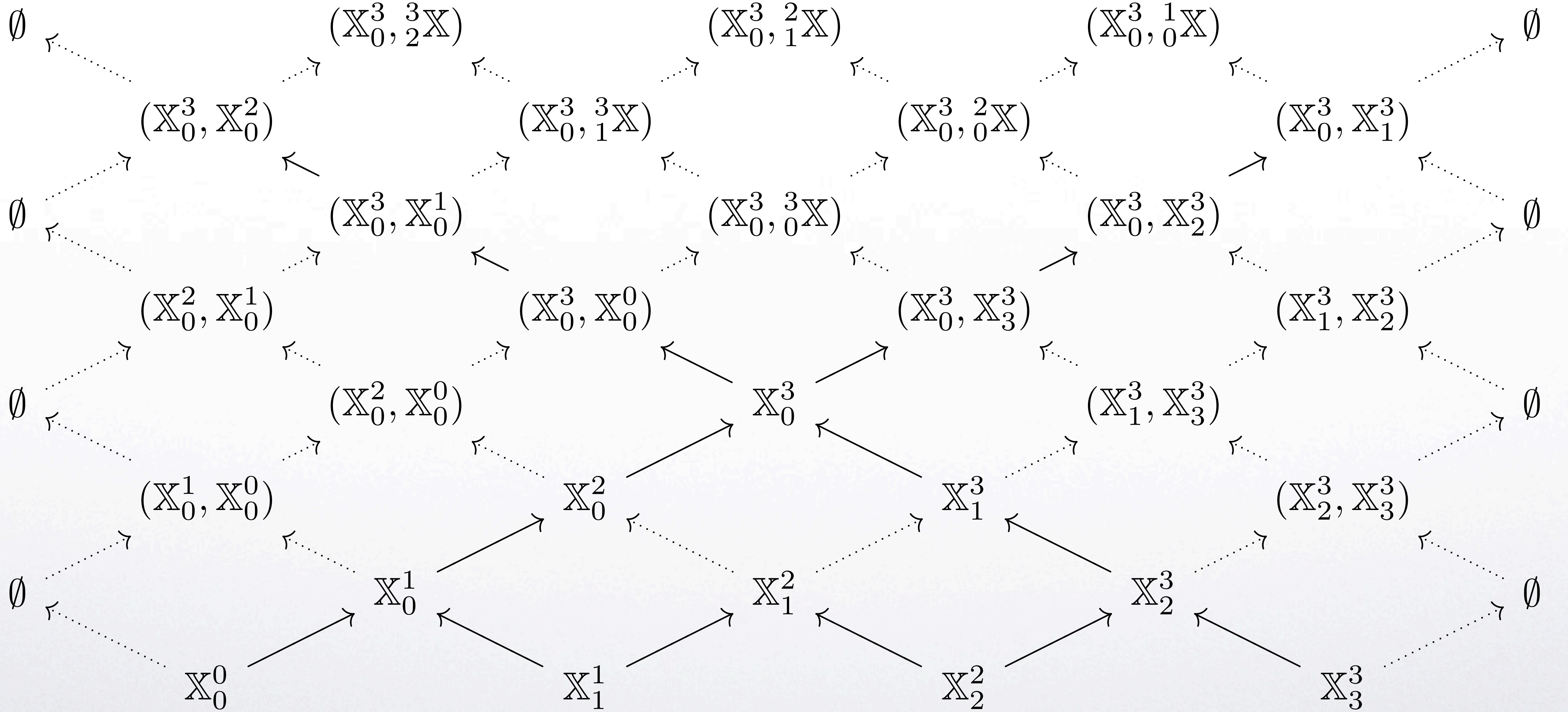


$$\mathbb{X}_i^j = f^{-1}[i, j]$$

$${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^3$$

A vast commutative diagram



$\mathbb{X}_i^j = f^{-1}[i, j]$

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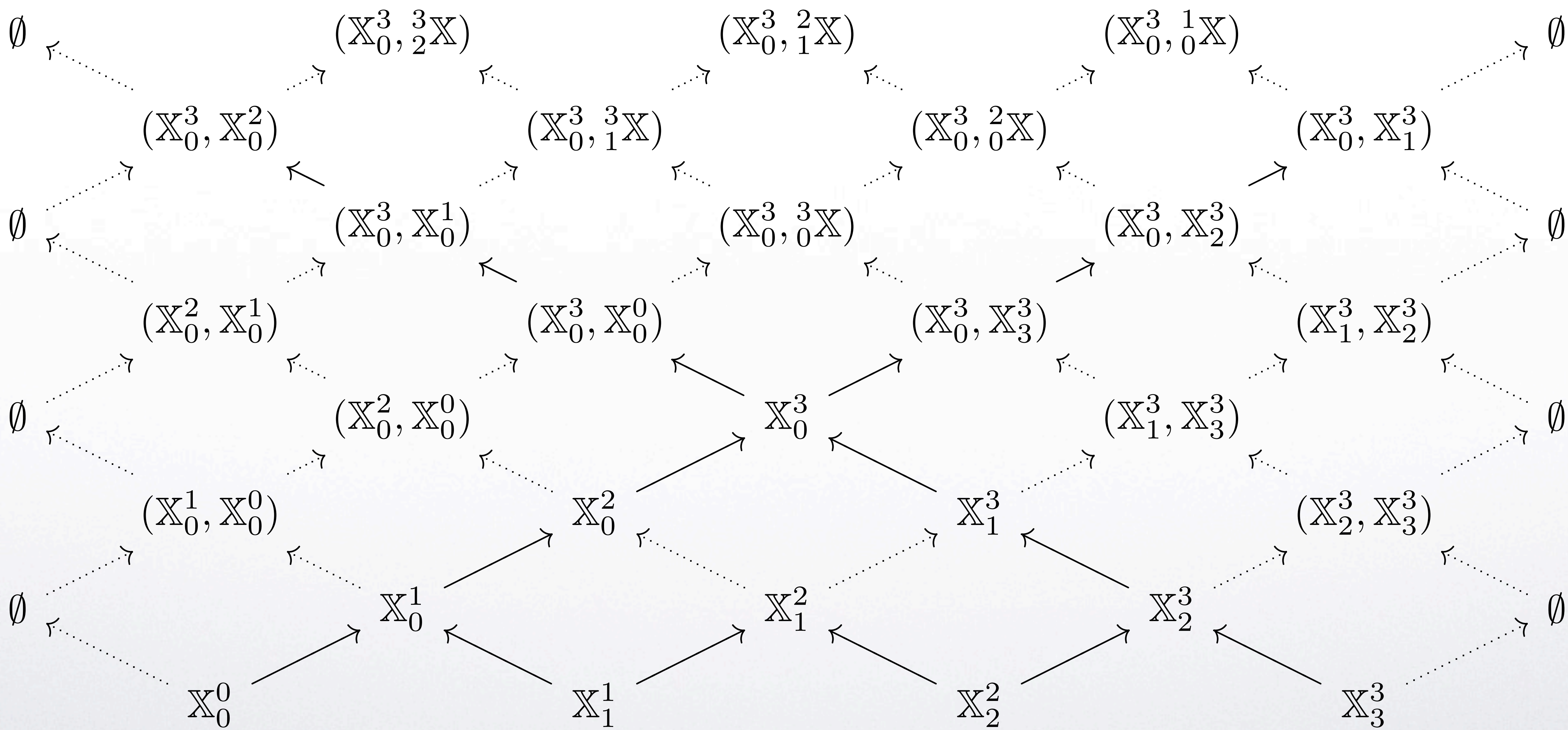

(In homology, there are connecting maps from the top edge to the bottom edge, turning this into a commutative Möbius band.)

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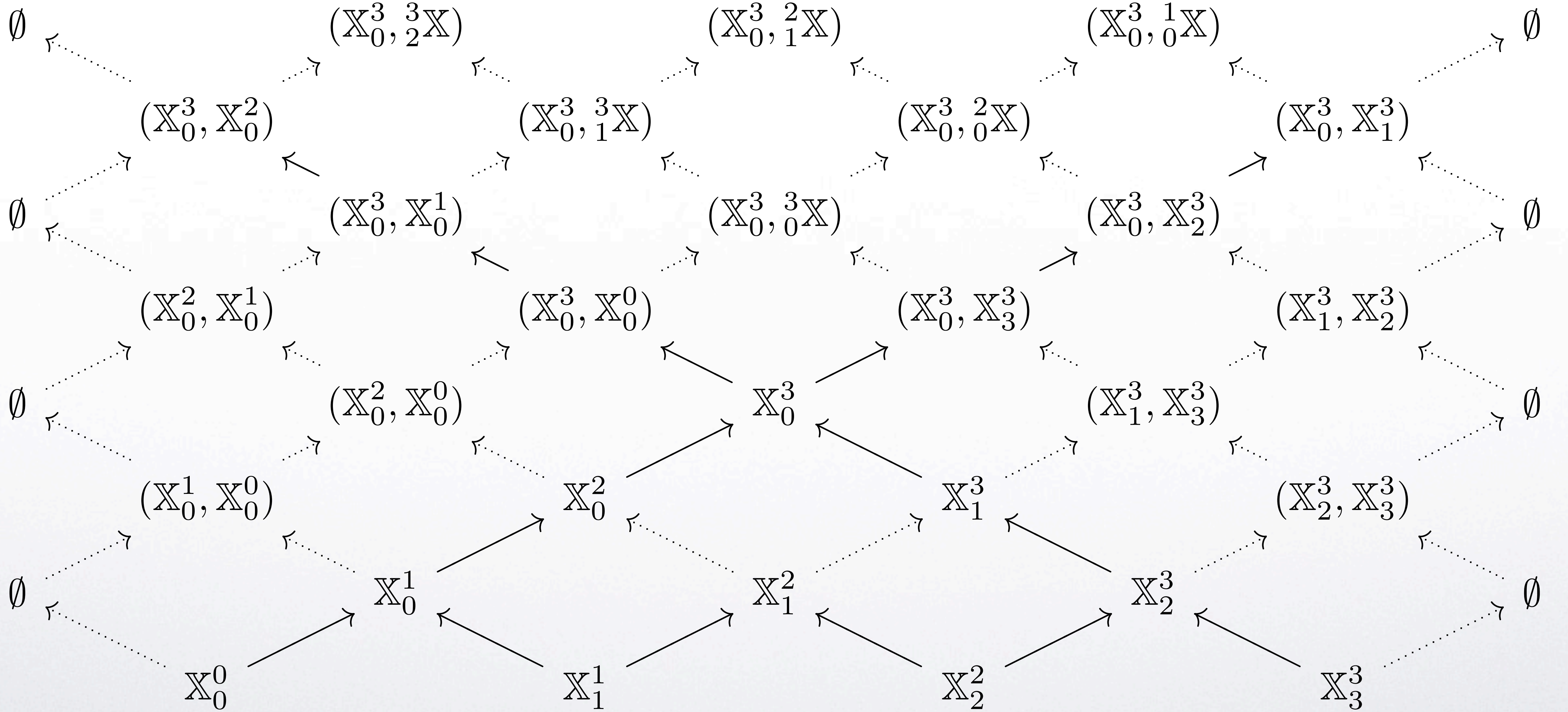

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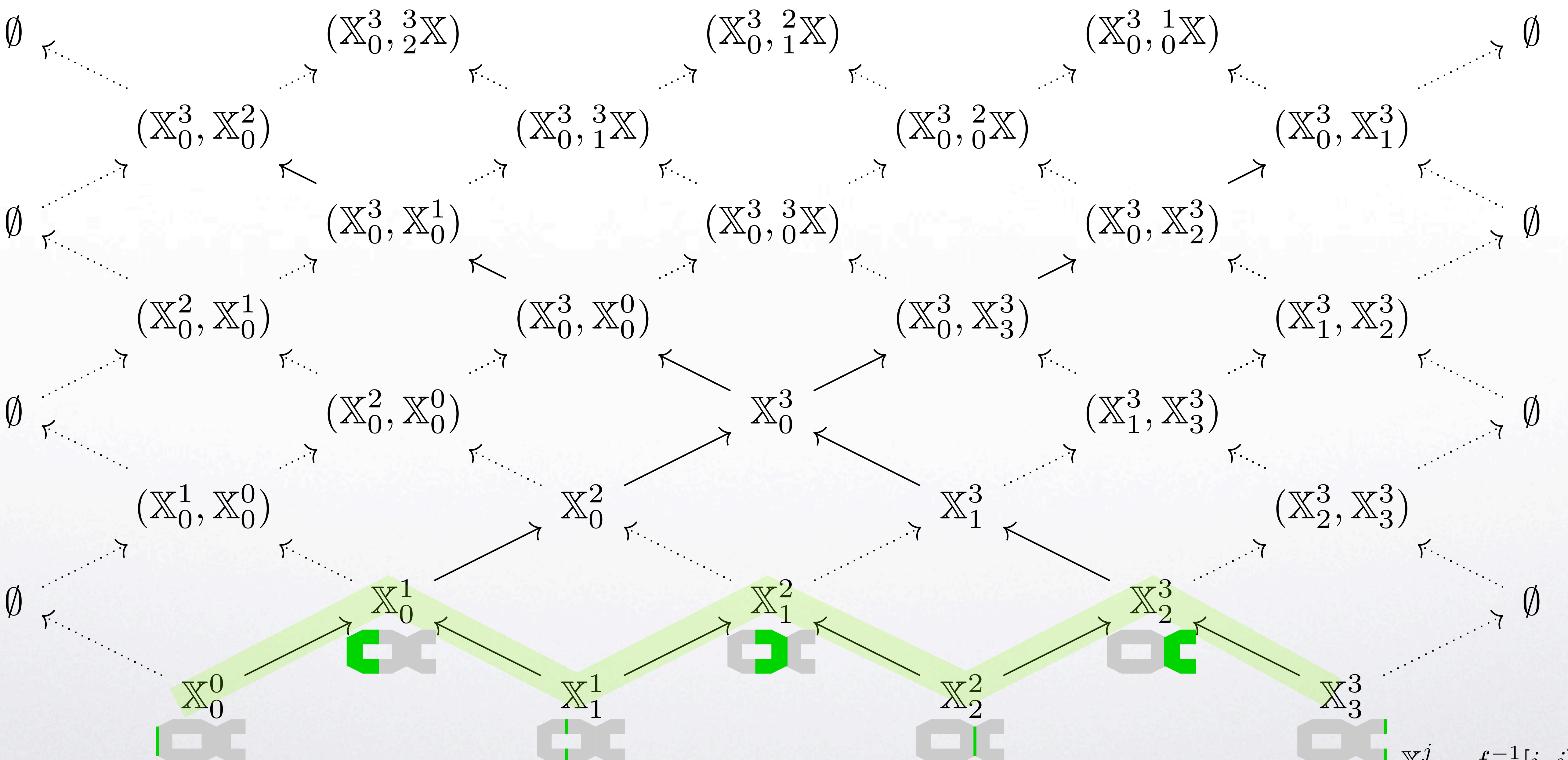

Levelset zigzag persistence

$\mathbb{X}_i^j = f^{-1}[i, j]$

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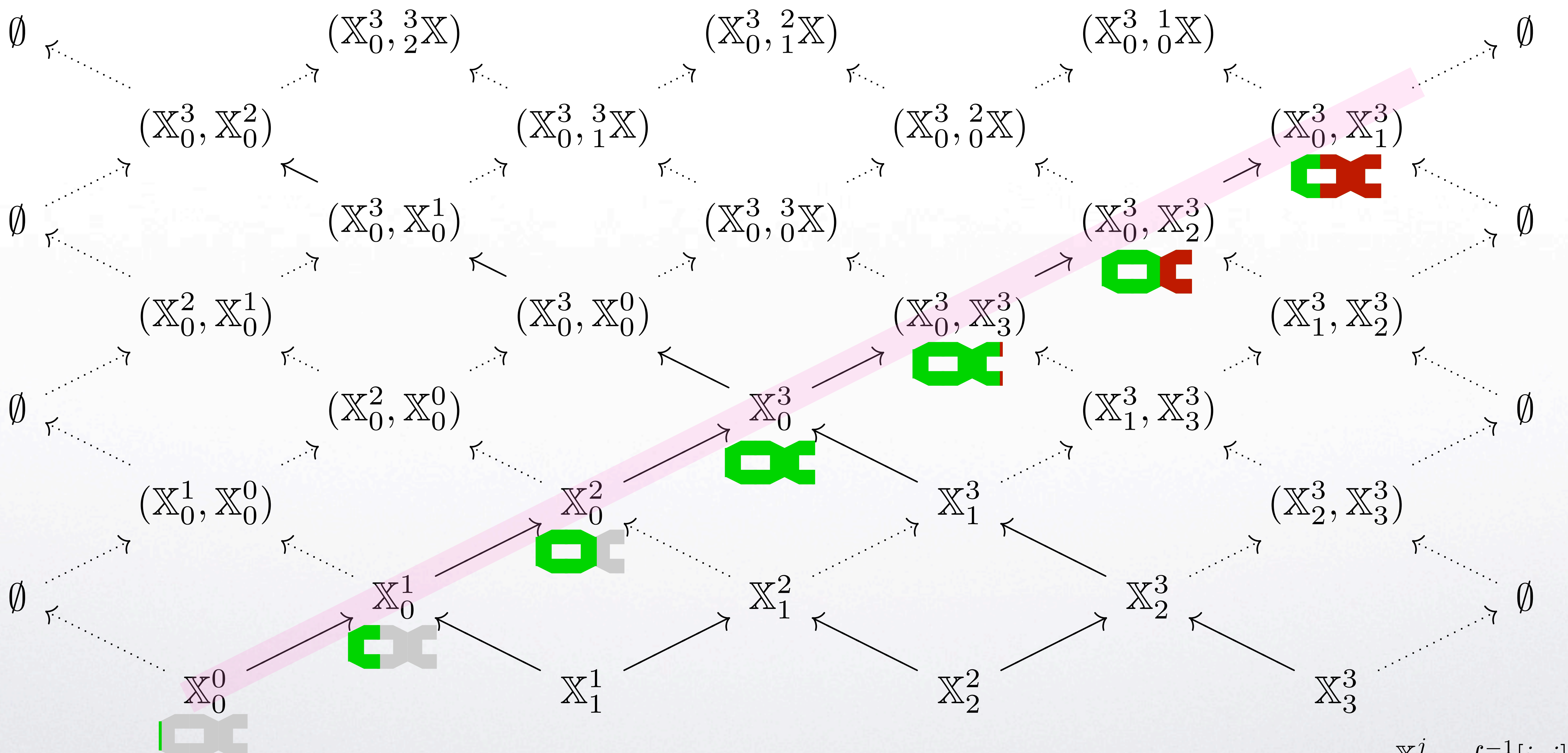

$\mathbb{X}_i^j = f^{-1}[i,j]$

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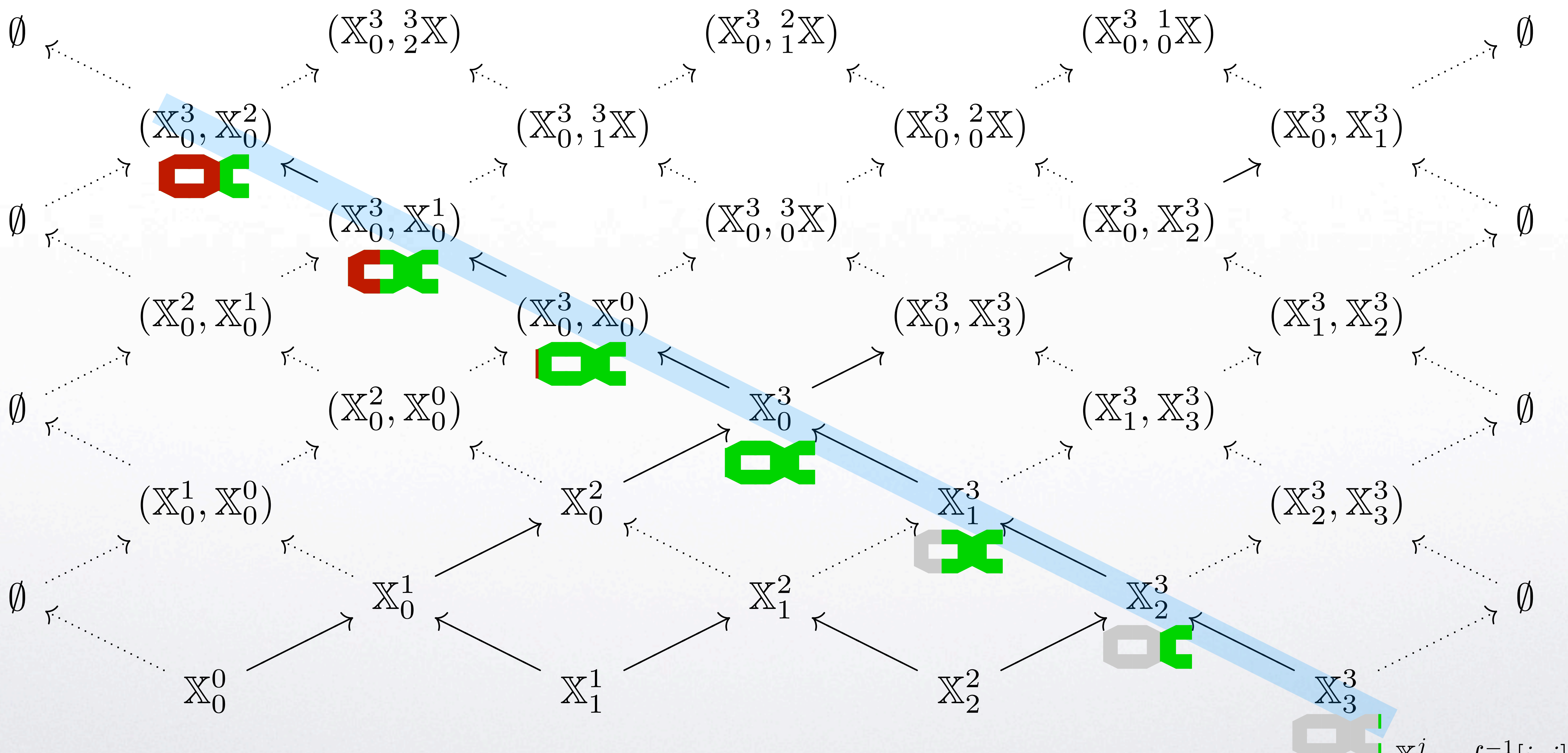
Cohen-Steiner, Edelsbrunner, Harer: Extended persistence of (X,f)

A vast commutative diagram



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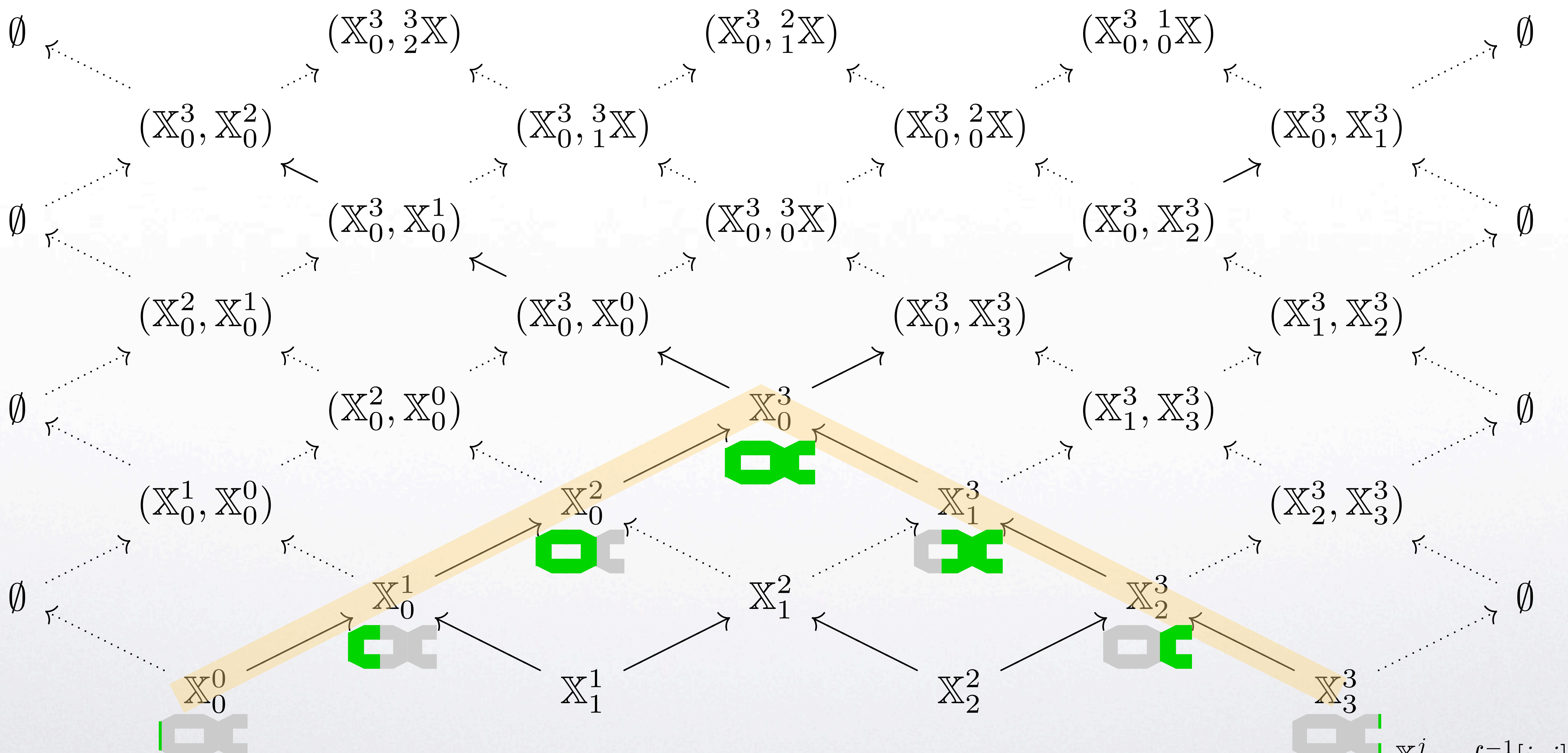
Cohen-Steiner, Edelsbrunner, Harer: **Extended persistence of (X,-f)**

A vast commutative diagram



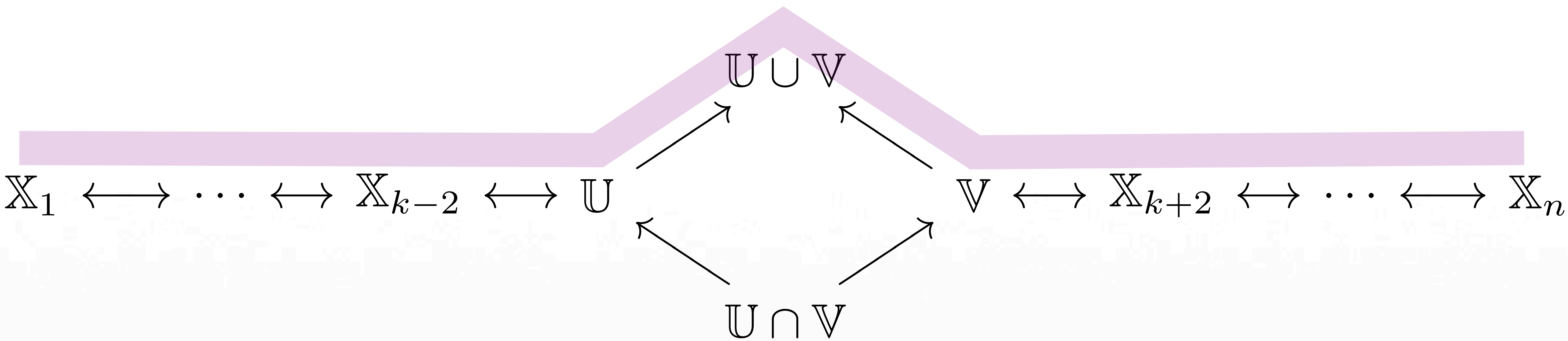
$\mathbb{X}_i^j = f^{-1}[i,j]$

${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^3$

Mayer–Vietoris Diamond Principle

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathbb{U} \cup \mathbb{V} & & & & & & \\
 & & & & & \nearrow & & \nwarrow & & & & & \\
\mathbb{X}_1 \longleftrightarrow & \cdots & \longleftrightarrow & \mathbb{X}_{k-2} & \longleftrightarrow \mathbb{U} & & & & \mathbb{V} \longleftrightarrow & \mathbb{X}_{k+2} & \longleftrightarrow & \cdots & \longleftrightarrow \mathbb{X}_n \\
 & & & & & \nwarrow & & \nearrow & & & & & \\
 & & & & & & \mathbb{U} \cap \mathbb{V} & & & & & &
\end{array}
$$

Mayer–Vietoris Diamond Principle



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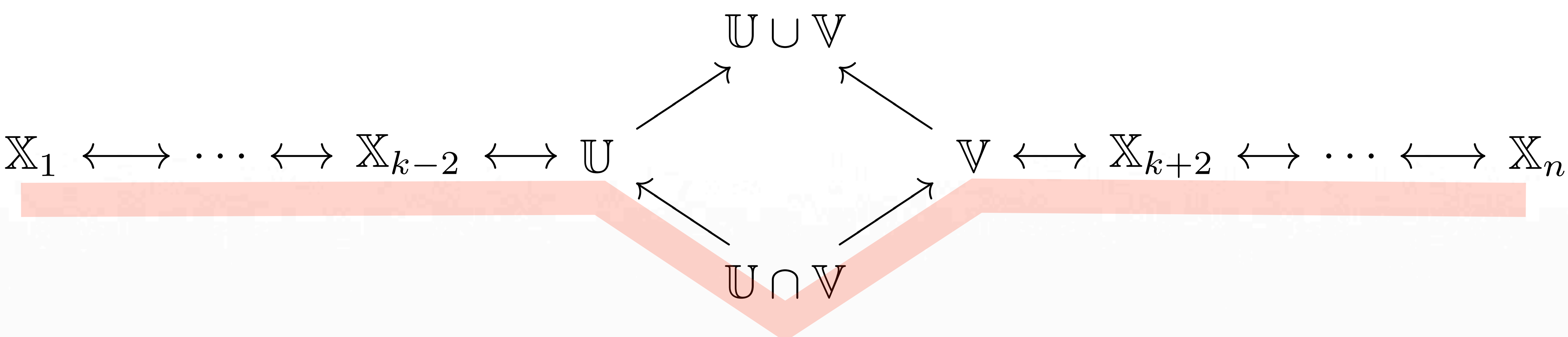

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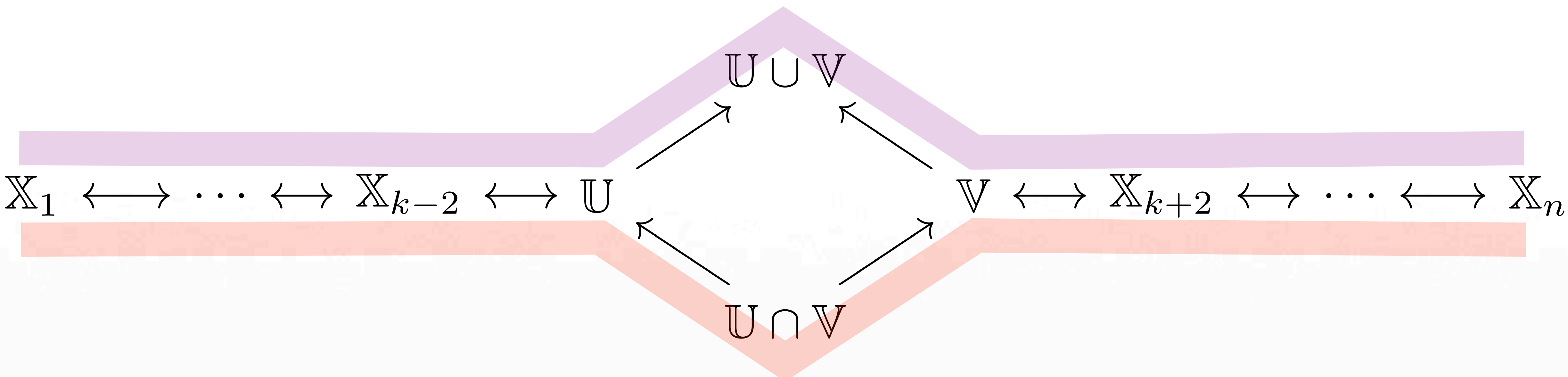
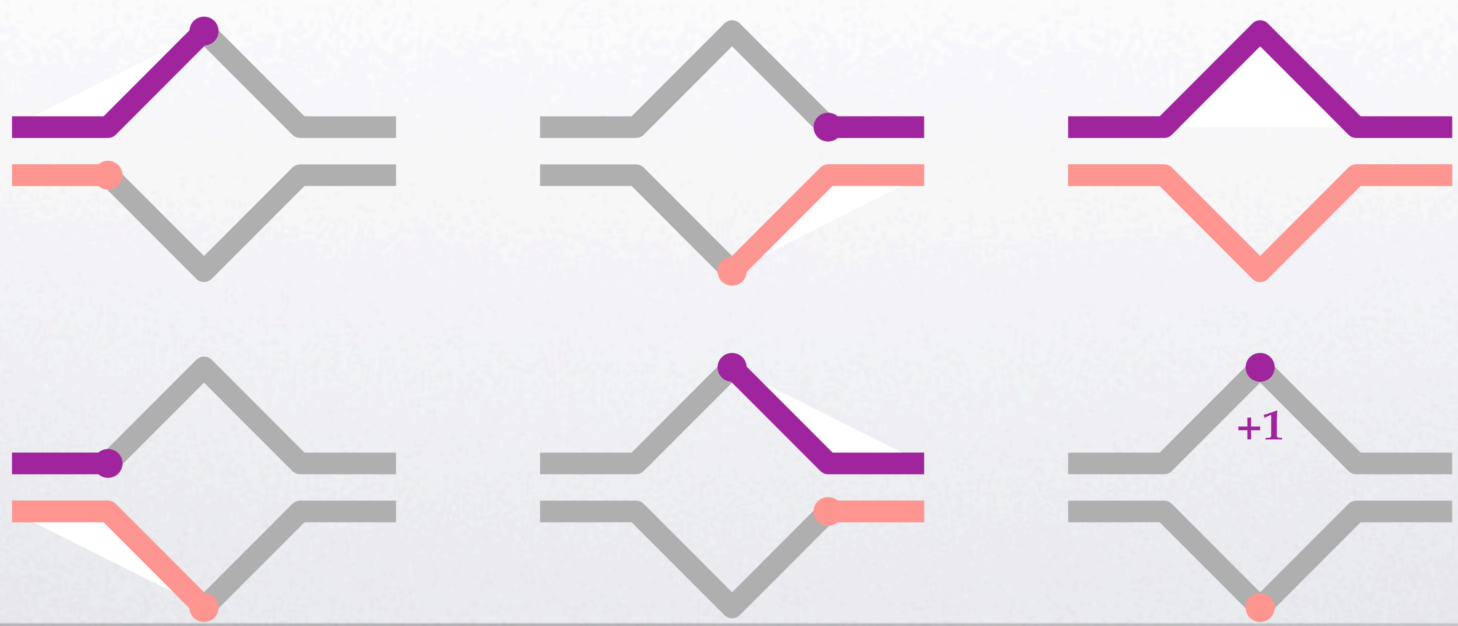


Both zigzags encode the same information in persistent homology:

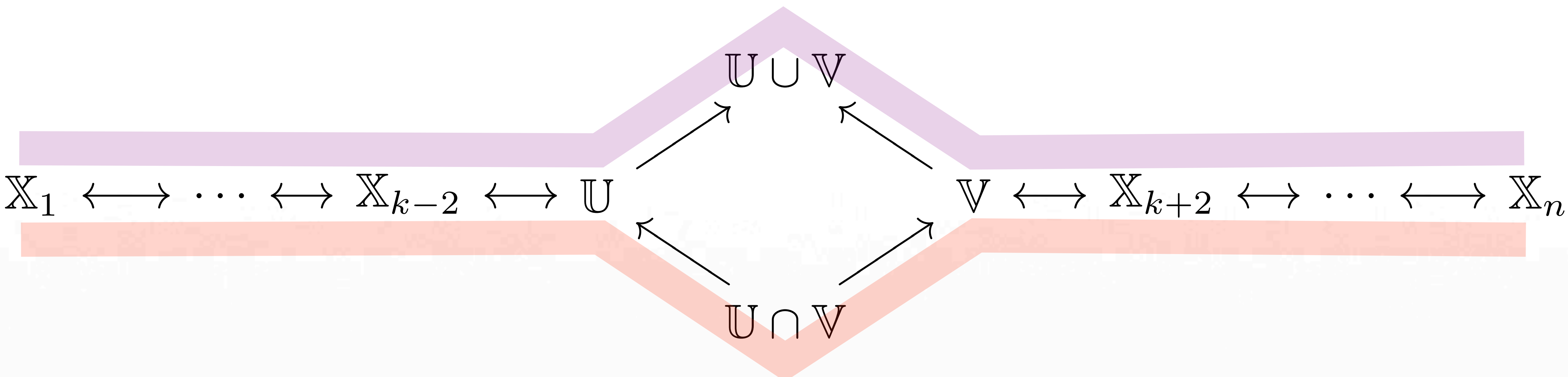
Mayer–Vietoris Diamond Principle

$$\mathbb{X}_1 \longleftrightarrow \cdots \longleftrightarrow \mathbb{X}_{k-2} \longleftrightarrow \mathbb{U} \quad \begin{matrix} \mathbb{U}\cup\mathbb{V} \\ \nearrow \quad \nwarrow \\ \\ \nwarrow \quad \nearrow \\ \mathbb{U}\cap\mathbb{V} \end{matrix} \quad \mathbb{V} \longleftrightarrow \mathbb{X}_{k+2} \longleftrightarrow \cdots \longleftrightarrow \mathbb{X}_n$$

Both zigzags encode the same information in persistent homology:



Mayer–Vietoris Diamond Principle



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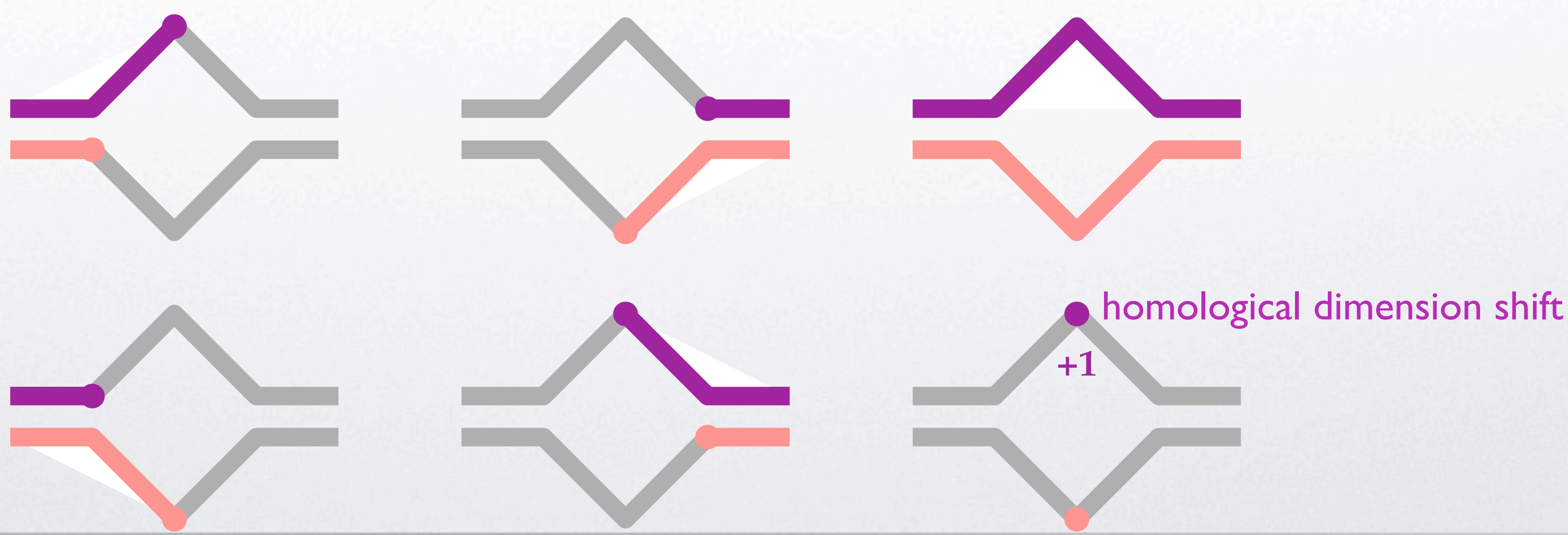

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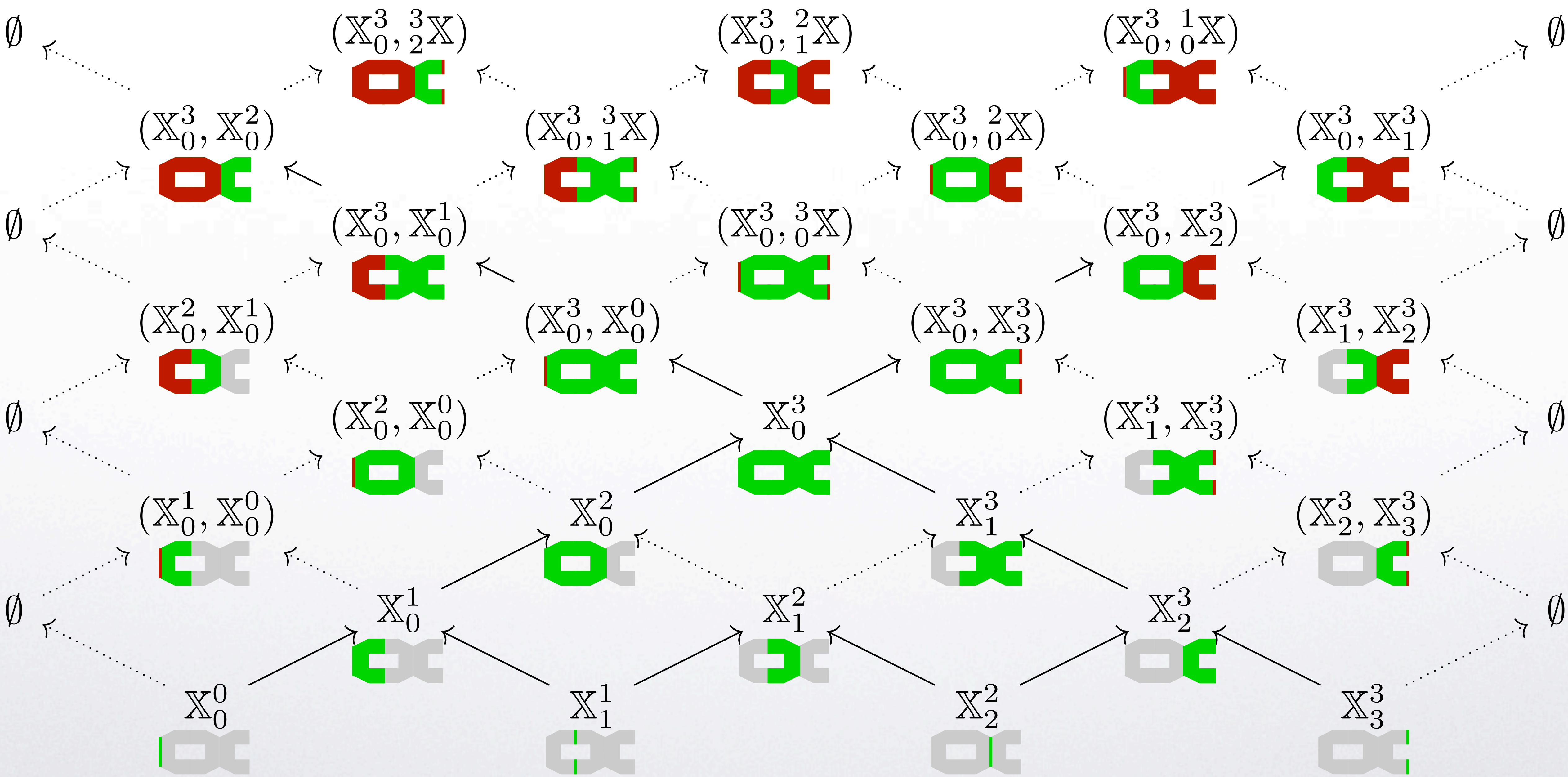
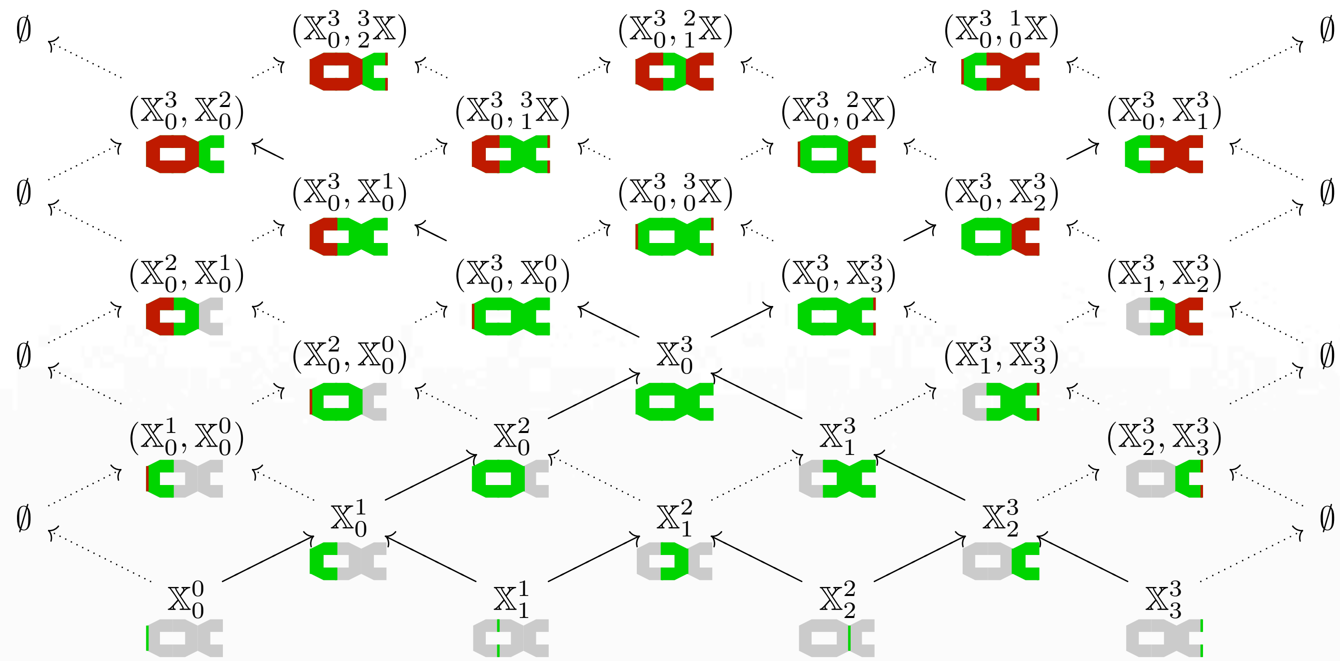
The pyramid theorem



The pyramid theorem



- Every diamond is Mayer–Vietoris.
- Thus all monotone paths from left to right carry the same zigzag persistent information (rearranged, with dimension shifts).
- In particular, the following carry equivalent information:

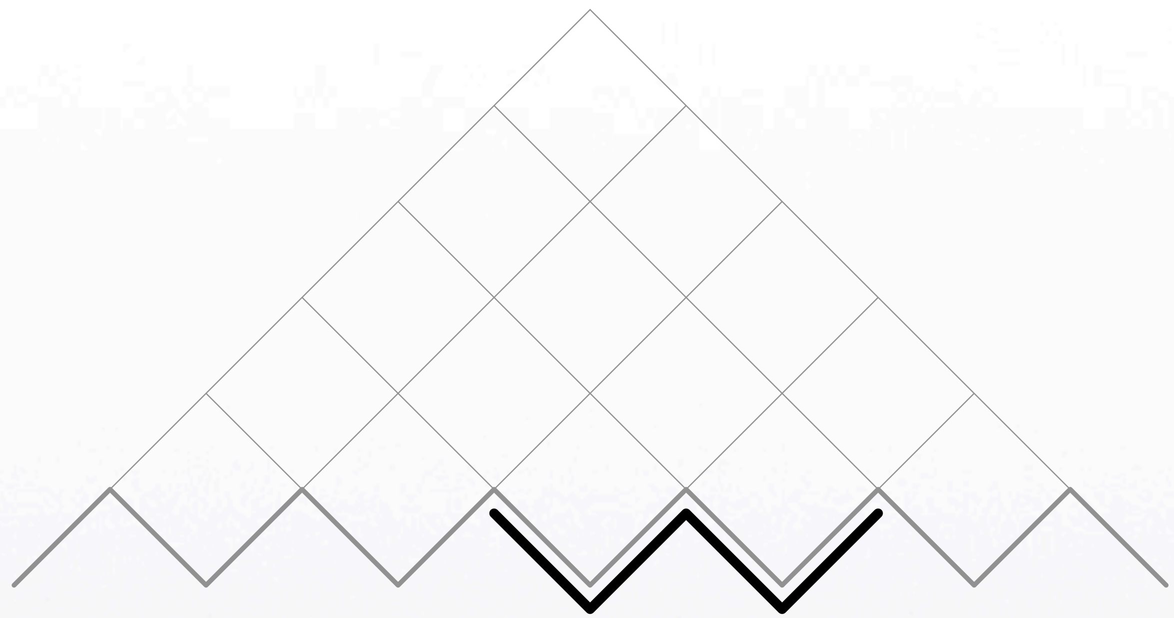
Extended persistence of (X,f)	Extended persistence of (X,-f)
Levelset zigzag persistence of (X,f)	Up-down persistence of (X,f)

The pyramid theorem

	Type I	Type II	Type III	Type IV
	$i < j$	$i < j$	$i \leq j$	$i < j$
LZZ	$[\mathbb{X}_{i-1}^{i}, \mathbb{X}_{j-1}^{j-1}]$ $[a_i, a_j)$	$[\mathbb{X}_{i}^{i}, \mathbb{X}_{j-1}^{j}]$ $(a_i, a_j]$	$[\mathbb{X}_{i-1}^{i}, \mathbb{X}_{j-1}^{j}]$ $[a_i, a_j]$	$[\mathbb{X}_{i}^{i}, \mathbb{X}_{j-1}^{j-1}]$ (a_i, a_j)
UD	$[\mathbb{X}_{0}^{i}, \mathbb{X}_{0}^{j-1}]$ $[a_i, a_j)$	$[\mathbb{X}_{i}^{n}, \mathbb{X}_{j-1}^{n}]$ $(\bar{a}_i, \bar{a}_j]$	$[\mathbb{X}_{0}^{i}, \mathbb{X}_{j-1}^{n}]$ $[a_i, \bar{a}_j]$	$[\mathbb{X}_{0}^{j}, \mathbb{X}_{i-1}^{n}]^{+}$ $[a_j, \bar{a}_i]^{+}$
EP(f)	$[\mathbb{X}_{0}^{i}, \mathbb{X}_{0}^{j-1}]$ $[a_i, a_j)$	$[(\mathbb{X}_{0}^{n}, \mathbb{X}_{j-1}^{n}), (\mathbb{X}_{0}^{n}, \mathbb{X}_{i}^{n})]^{+}$ $[\bar{a}_j, \bar{a}_i)^{+}$	$[\mathbb{X}_{0}^{i}, (\mathbb{X}_{0}^{n}, \mathbb{X}_{j}^{n})]$ $[a_i, \bar{a}_j)$	$[\mathbb{X}_{0}^{j}, (\mathbb{X}_{0}^{n}, \mathbb{X}_{i}^{n})]^{+}$ $[a_j, \bar{a}_i)^{+}$
EP$(-f)$	$[(\mathbb{X}_{0}^{n}, \mathbb{X}_{0}^{j-1}), (\mathbb{X}_{0}^{n}, \mathbb{X}_{0}^{i})]^{+}$ $(\bar{a}_j, \bar{a}_i]^{+}$	$[\mathbb{X}_{i}^{n}, \mathbb{X}_{j-1}^{n}]$ $(a_i, a_j]$	$[(\mathbb{X}_{0}^{n}, \mathbb{X}_{0}^{i-1}), \mathbb{X}_{j-1}^{n}]$ $(\bar{a}_i, a_j]$	$[(\mathbb{X}_{0}^{n}, \mathbb{X}_{0}^{j-1}), \mathbb{X}_{i-1}^{n}]^{+}$ $(\bar{a}_j, a_i]^{+}$

- In particular, the following carry equivalent information:
  - Extended persistence of (X,f)
  - Extended persistence of (X,-f)
  - Levelset zigzag persistence of (X,f)
  - Up-down persistence of (X,f)

Example: LZZ to UD



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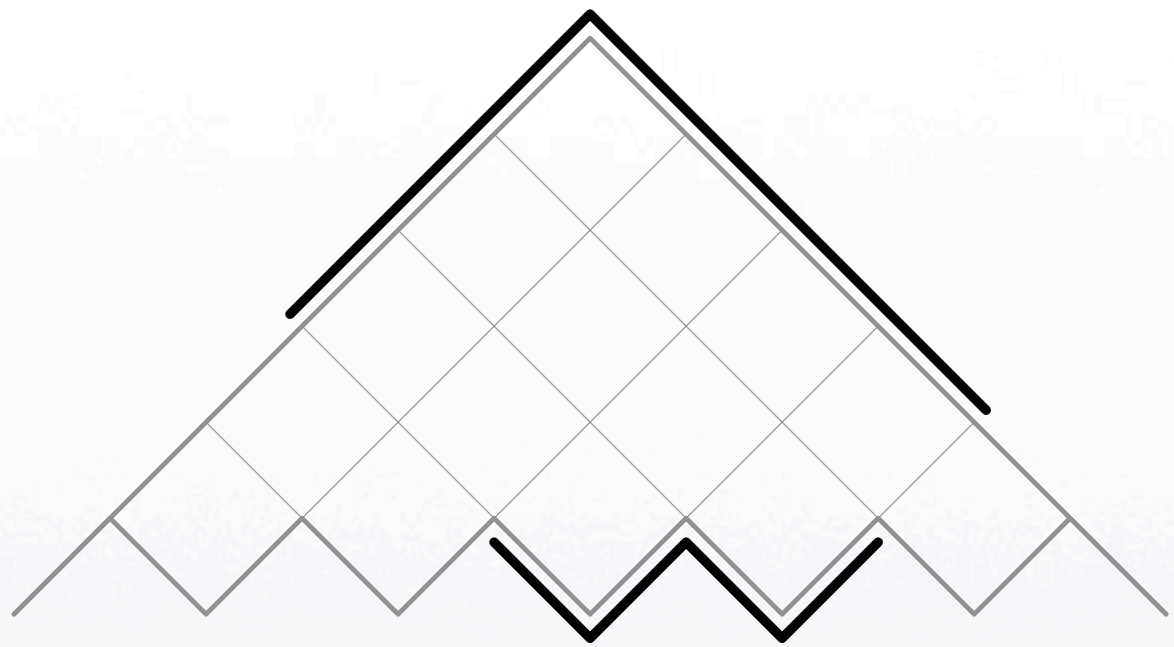

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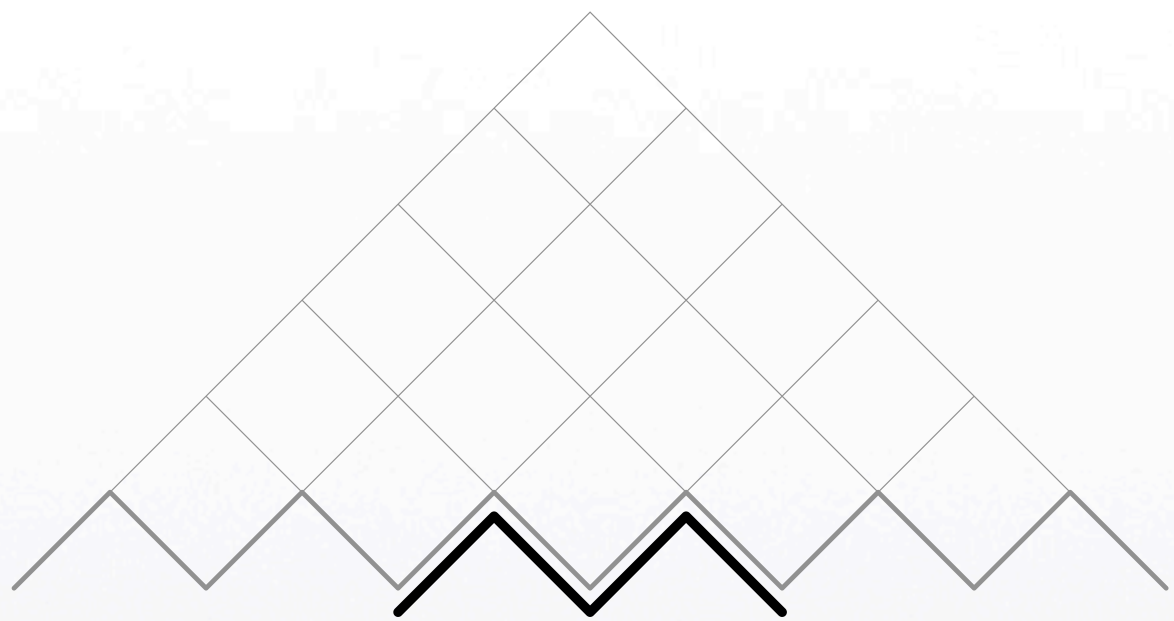

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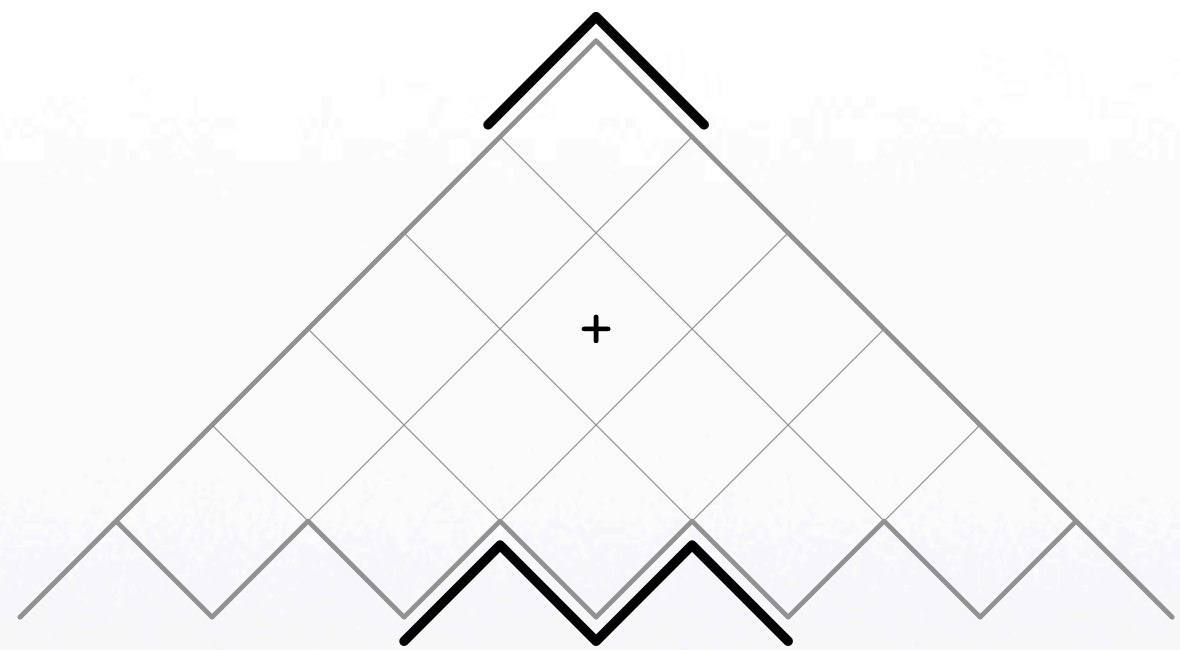

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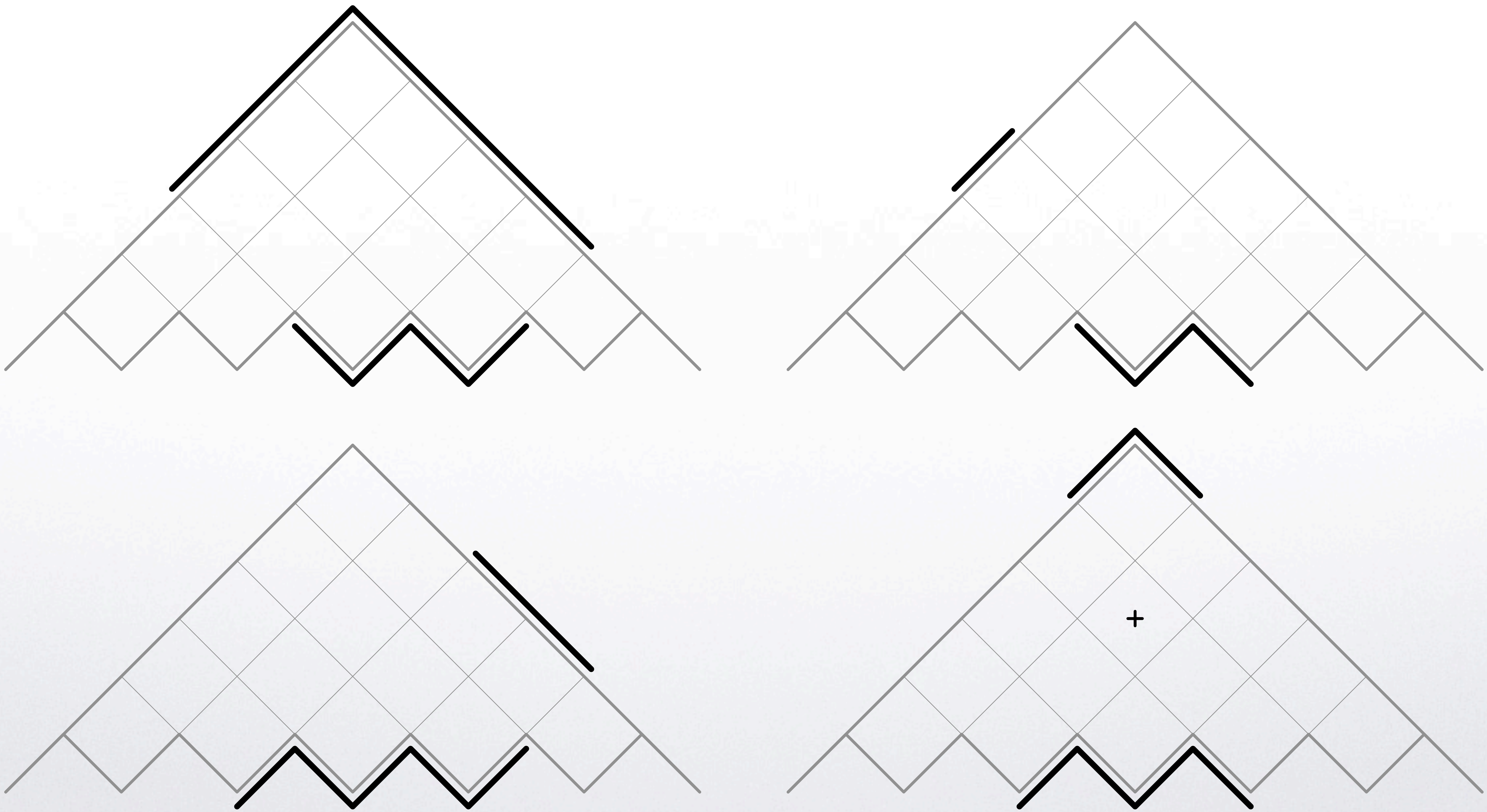
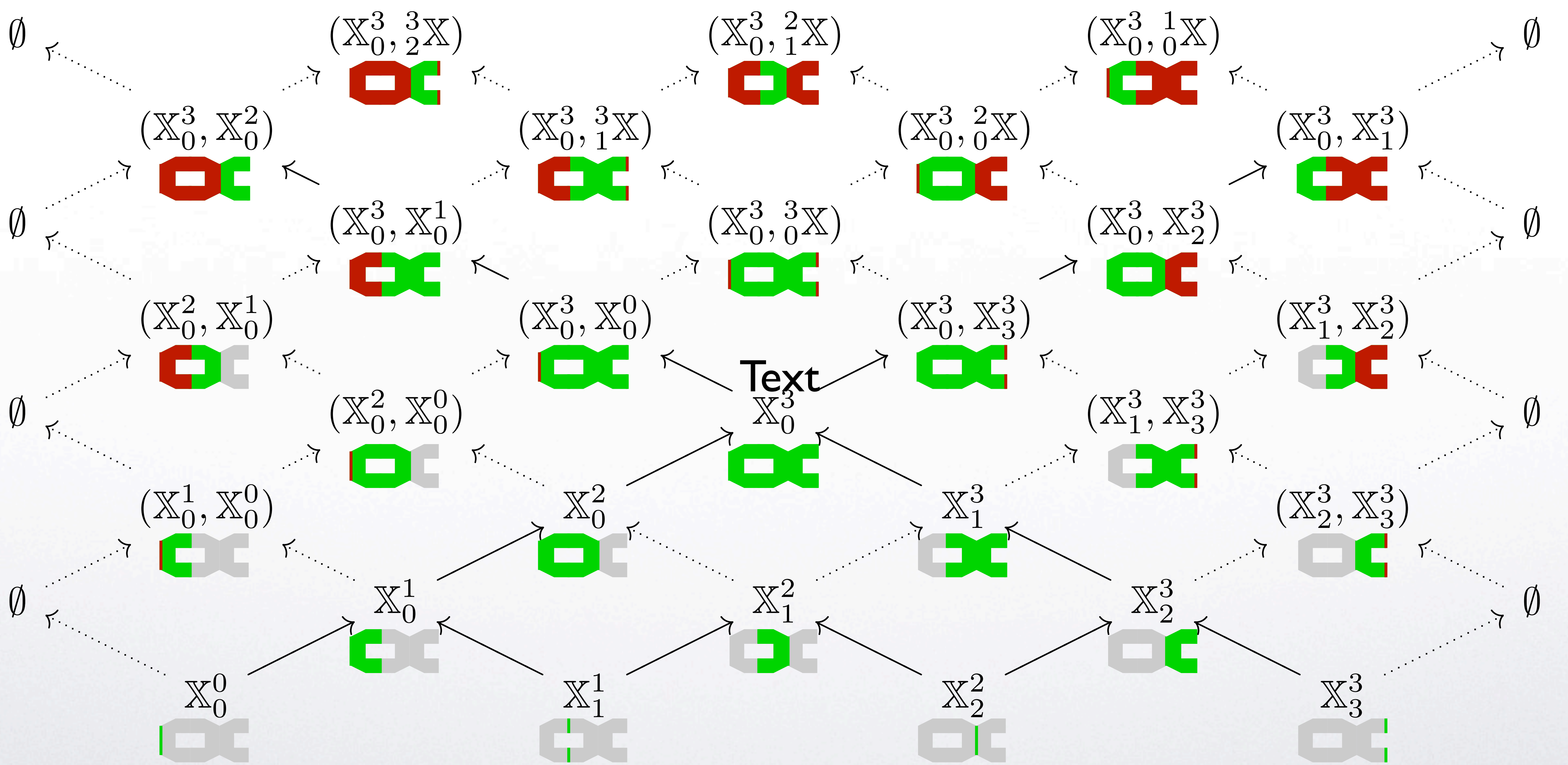


Corollaries

- Stability of levelset zigzag persistence
  - follows from stability of extended persistence
- Symmetry theorem for extended persistence
  - extended persistence for (X,f) and (X,-f) record same information as LZZ
- Explanation of Interval Persistence (Dey, Wenger '07)
  - interval persistence of (X,f) recovers intervals of types [b,d) and (b,d) in LZZ
  - interval persistence of (X,-f) recovers intervals of types (b,d] and (b,d) in LZZ
  - zigzag persistence algorithm computes interval persistence efficiently
- Space efficient computation of extended persistence
  - at most, calculations involve the band between two adjacent levelsets (as opposed to the whole space so far)

Thank you



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