Topo-Geometric Modeling of 3D Objects

Hamid Krim,

ECE Dept., NCSU,

Raleigh, NC.

Acknowledgements:

D. Aouada, S. Baloch, A. Ben Hamza, S. Feng

D. Dreisigmeyer, I. Kogan

*AFOSR and ONR

Outline

- 3D Shape Modeling
  - Topological modeling
  - Geometric modeling
- Geometric model matching
  - Integral Invariants
  - Differential Invariants
  - Riemannian metric
- Conclusion

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Motivation for 3D shapes

- Generalized framework for classification and recognition
- Biomedical imaging (surgery assistance…)
- Object compression for storage/retrieval
- CAD applications, Art archival, terrain modeling

Other 2D/3D Representations

- Shock graphs

[Kimia-Tannenbaum-Zucker, Siddiqi-Zucker….]

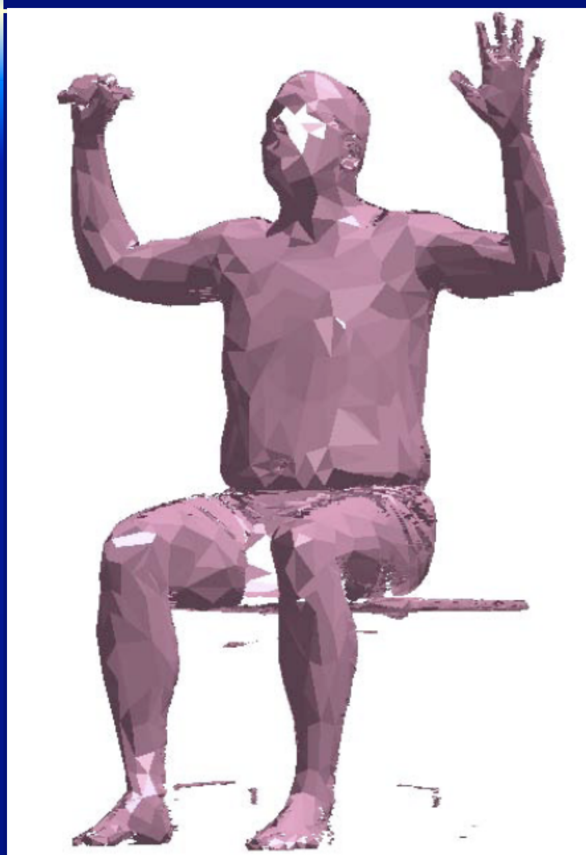
- Medial Axis

[Blum, Damon, Giblin, Kimia, Pizer, Siddiqi]

- M-Reps [Nackman-Pizer, Fletcher….]
- Morse-Theoretic

[Shinagawa, Kunii and Kergosien, Schroeder, Edelsbrunner, Schmidt *et. al*., Andres *et al*., Samir *et al*. …]

Coordinate-Free Representation



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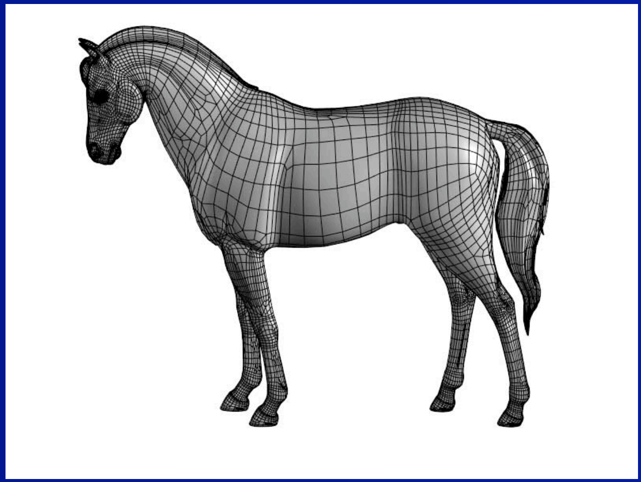
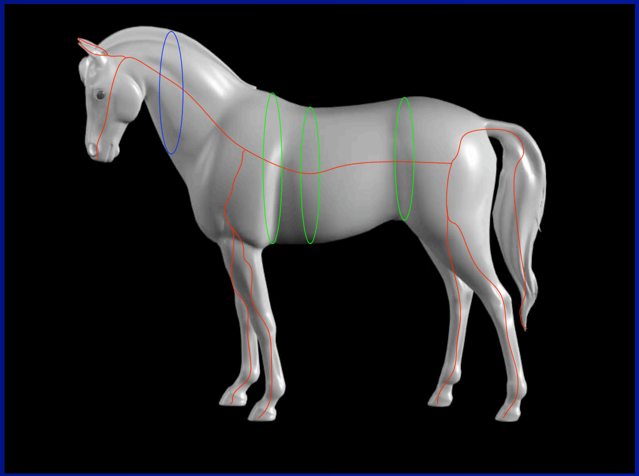
Pictorially…





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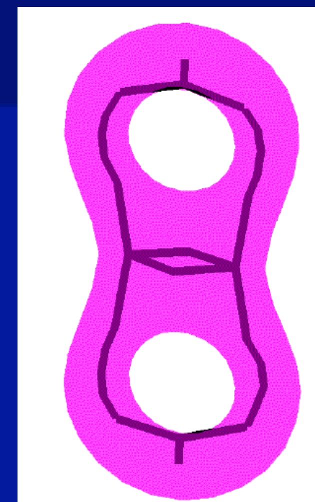
In words…

Coarse representation

- Fast and simple tool for surface comparison

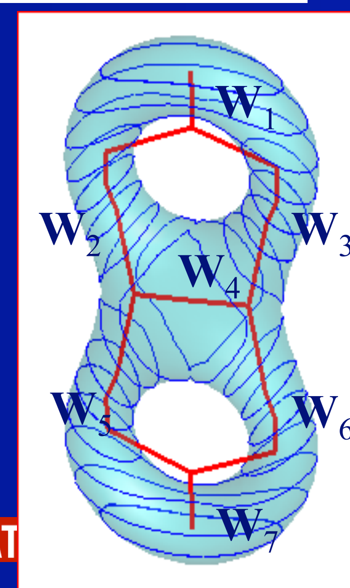
Classification ⇒ ***Topological*** or skeletal graph



Fine representation

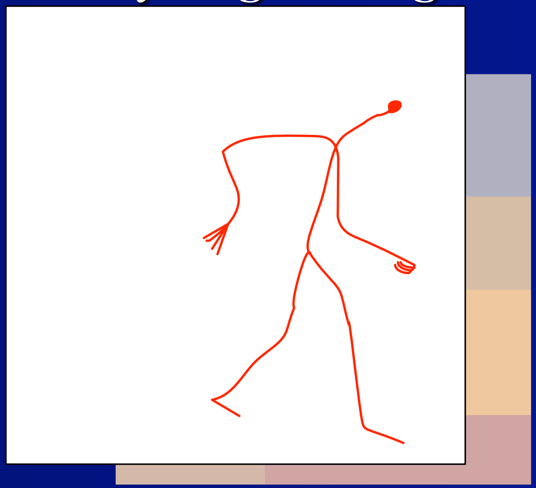
- Higher level of discrimination.

Recognition ⇒ ***Geometric*** **information**



Topology

- *Goal*: *Represent a surface/manifold in subparts which may be glued together*



- Information in Topology
- How to capture topology?
  - *Critical* points

Morse theory

- Study of smooth and compact surfaces by exploiting a defined Morse function

Definition:

A smooth function $h : \mathcal{S} \rightarrow \mathbb{R}$ on a smooth manifold S is called a *Morse* function if all of its critical points are non-degenerate.

➢ A critical point $p_0=(u_0,v_0)$ is degenerate if the Hessian of $h \circ \mathbf{x}$ is singular

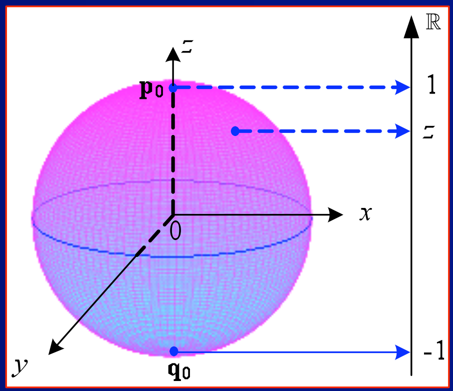
Properties:

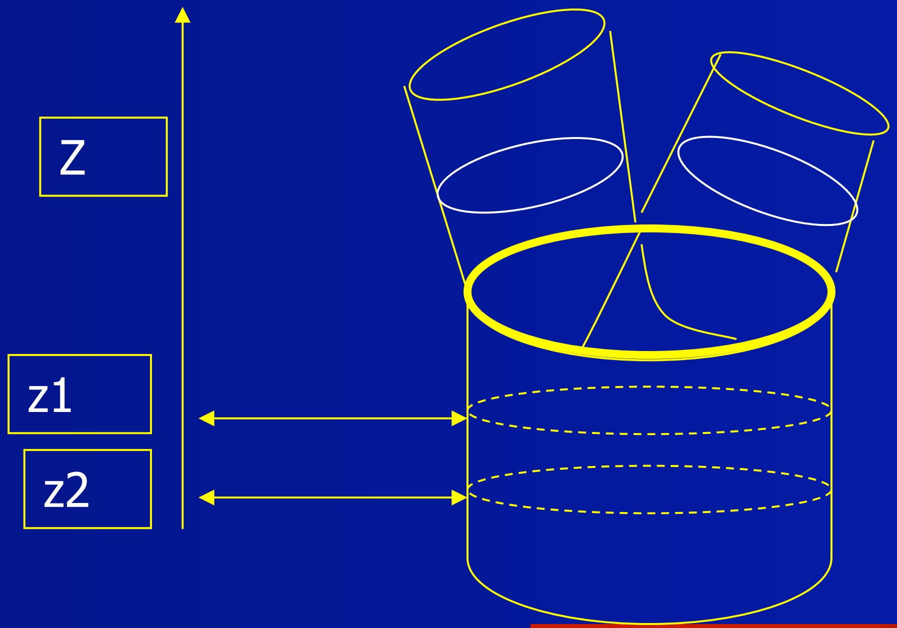
- Critical points of a Morse function are:
  - isolated
  - landmarks on the surface
- Saddle points determine topological changes of the surface

Height Function

- A height function $h : \mathcal{M} \rightarrow \mathbb{R}$ on smooth manifold is a real valued function such that

$$h(x, y, z) = z, \forall (x, y, z) \in \mathcal{M}$$





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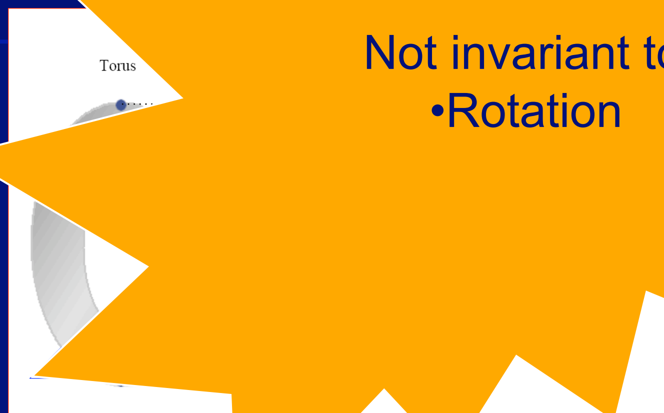
Reeb g



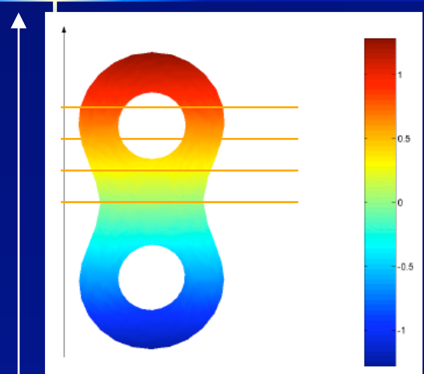
Not invaluant to
•Rotation

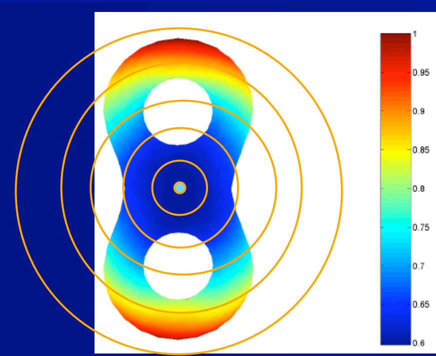
– Reeb graph may ernatively be described as a **quotient space** $\mathcal{M}/\sim$ where the **equivalence relation** $\sim$ is defined as:

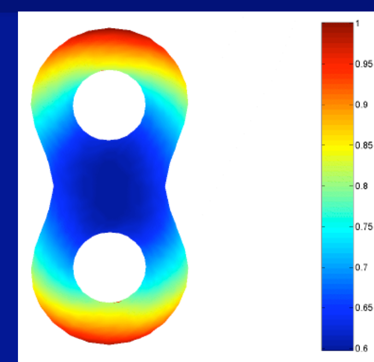
– $\mathbf{p} \sim \mathbf{q}$ iff
  - $h(\mathbf{p}) = h(\mathbf{q})$ $\mathbf{p} \in \text{ConnComp}(\text{Levelset}(\mathbf{q})) = h^{-1}(h(\mathbf{q}))$

$\mathcal{M}/\sim := \{[\mathbf{p}] : \mathbf{p} \in \mathcal{M}\}$ $[\mathbf{p}] = \{\mathbf{q} \in \mathcal{M} : \mathbf{q} \sim \mathbf{p}\}$

Choice of a Morse function







Height Function:

Not invariant to rotation.(***Reeb, Shinagawa et al., Edelsbrunner et al., Ben Hamza et al.***)

Spherical sampling of a surface:

Requires a reference point.

(***S. Baloch and K.)***

Geodesic Function

Completely intrinsic to a surface.

(***Hilaga et al.)***

Global Geodesic Function

- Integrated geodesic distance

$$f(\mathbf{v}) = \int_{\mathbf{p}\in\mathbf{S}} \mathbf{d}(\mathbf{v},\mathbf{p})\mathbf{d}\mathcal{S}.$$

- Given a 2D surface as a 3D mesh represented by $p_{\{i=1,\cdots m\}}$ vertices
- $d(.)$ being the geodesic distance between two vertices, *GGF* $g(.)$ is defined as:

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Global Geodesic Function *GGF* (2)

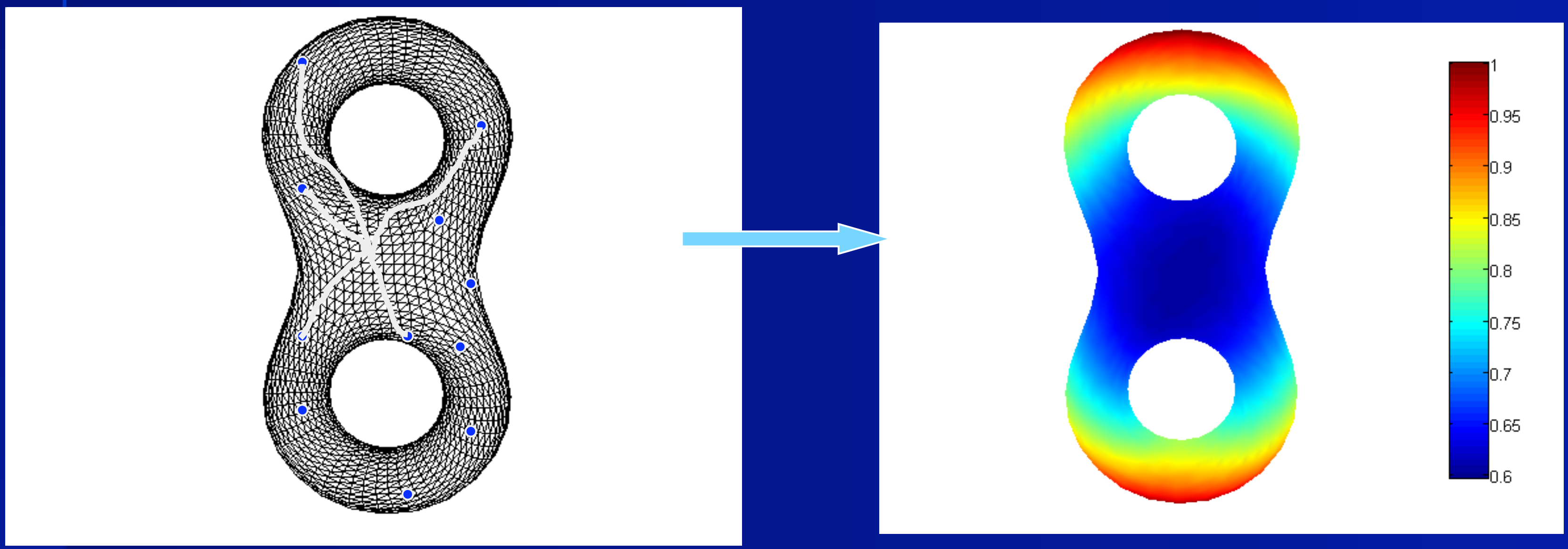

Illustration of the GGF

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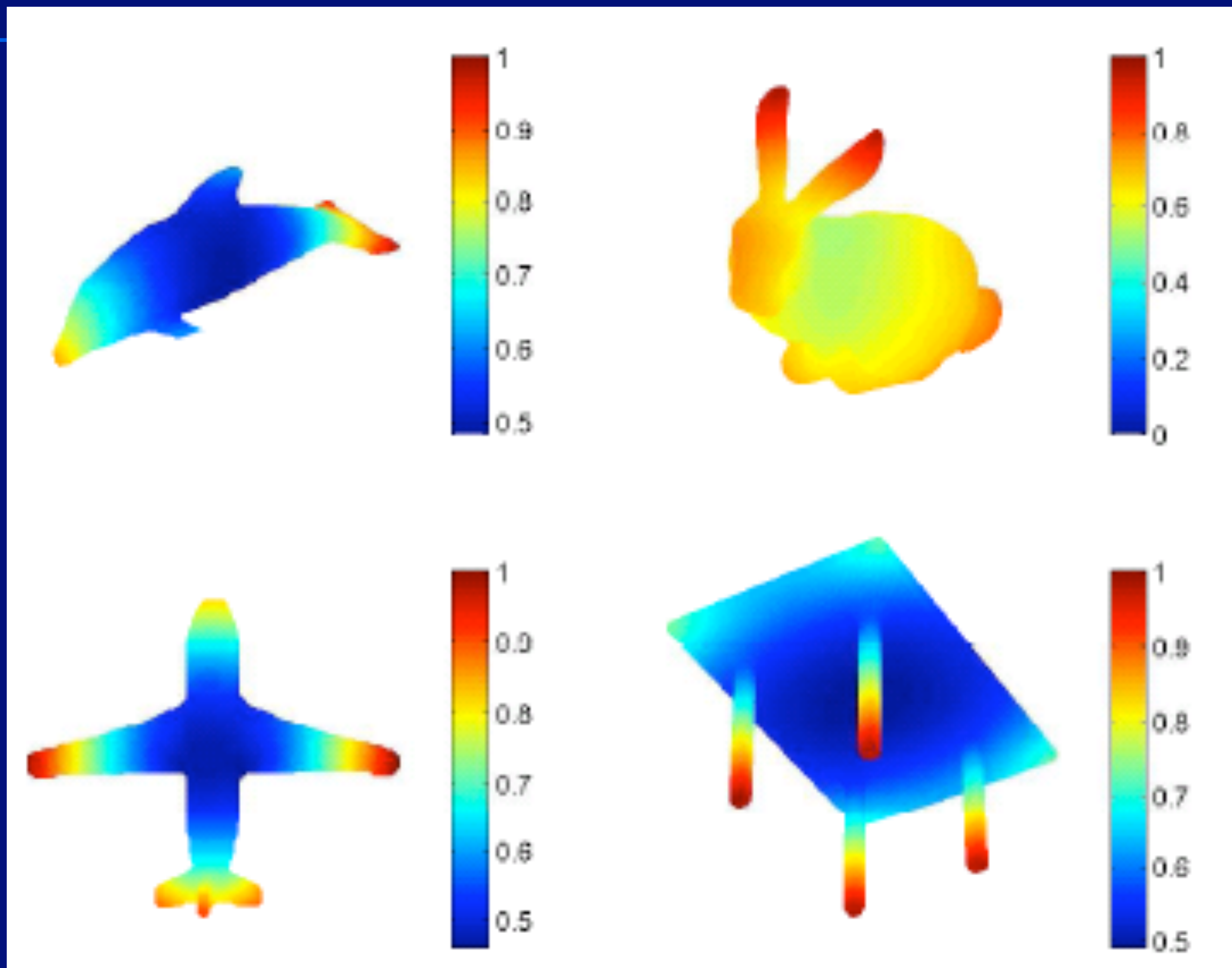
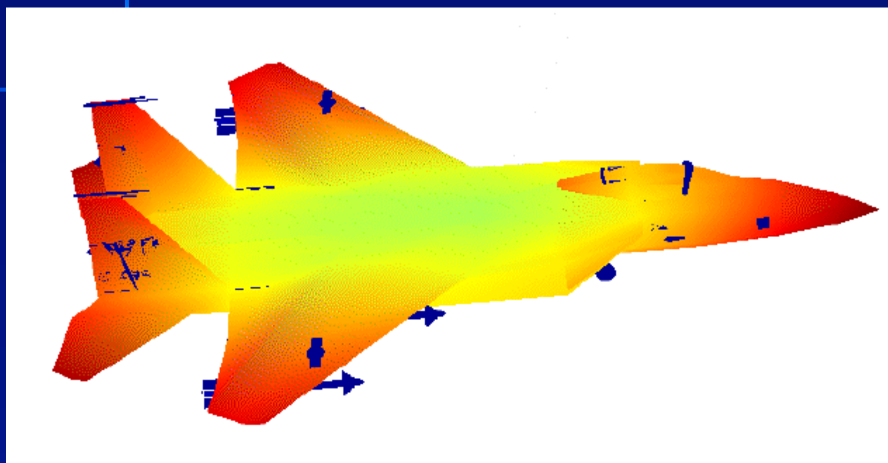
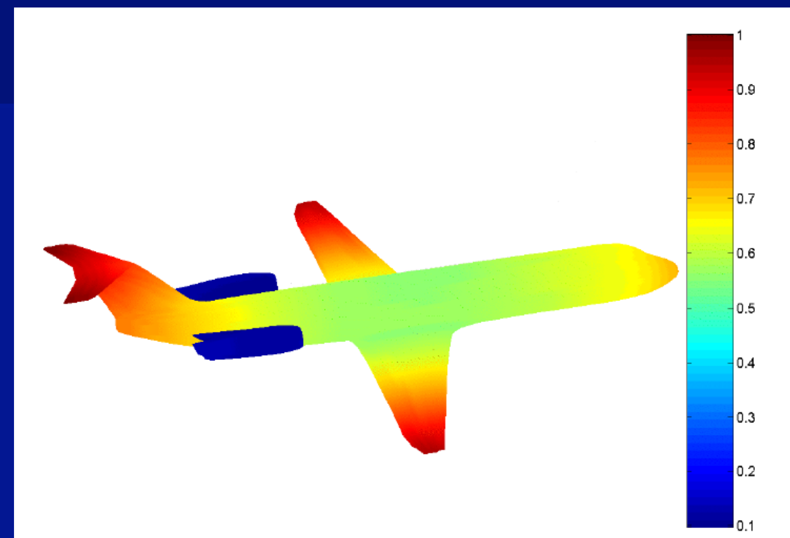
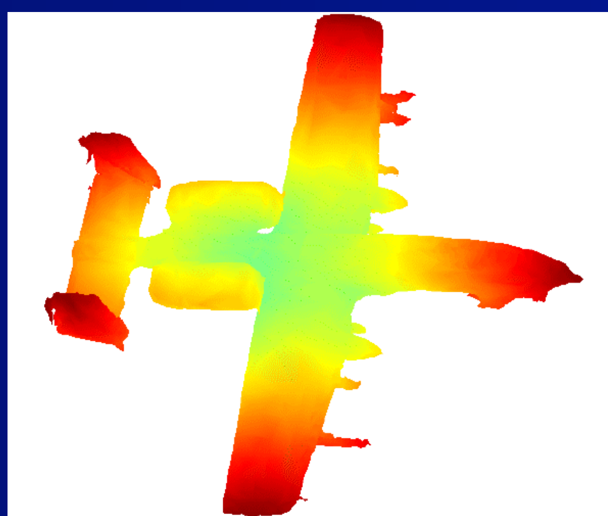
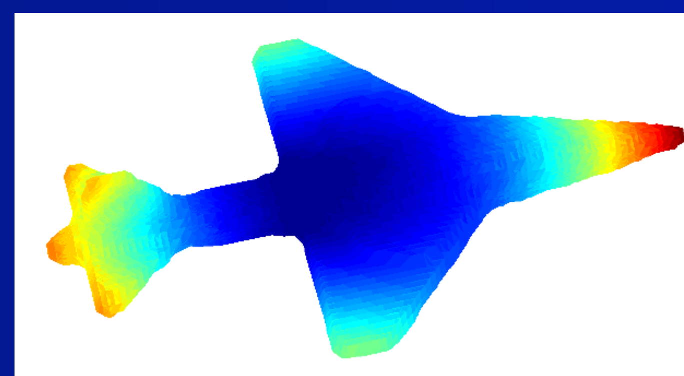
Global Geodesic Function GGF











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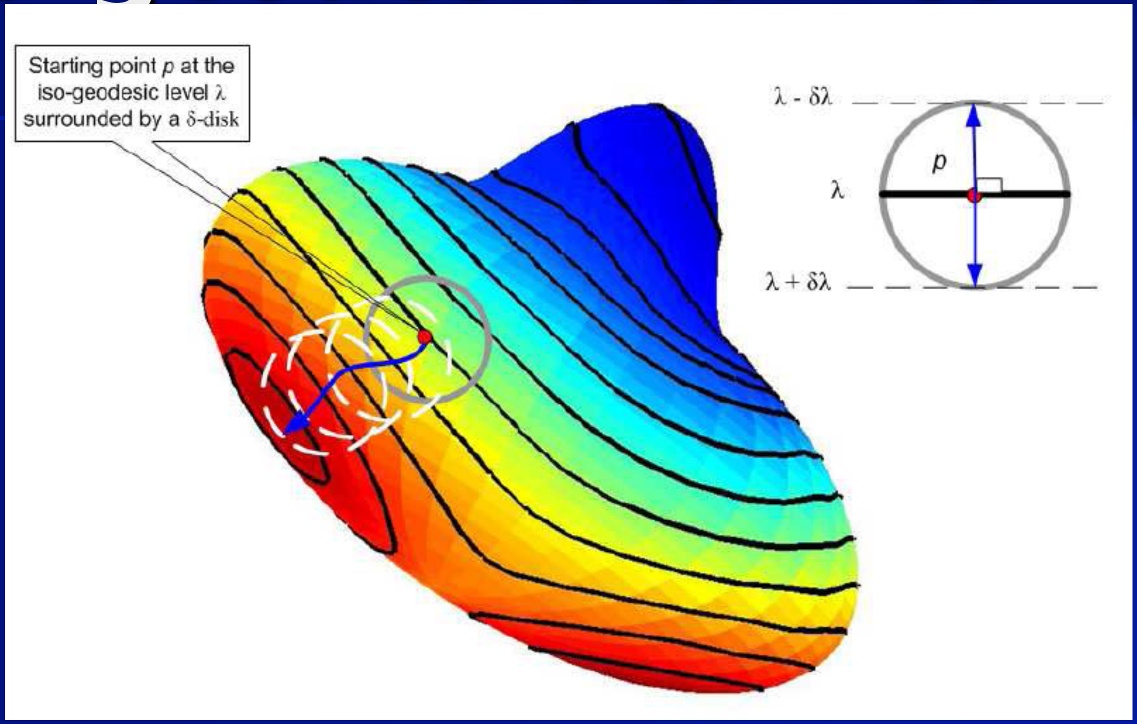
TG Characterization

- Morse framework provides a powerful means for gleaning topological and geometric information,
- For a surface F(x,y,z) and a height function h(f(.),.), we seek level sets

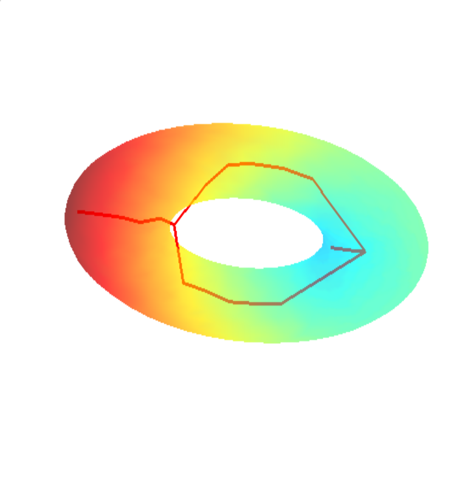
$$F(x,y,z).(h(x,y,f(x,y)) - C_t) = 0$$

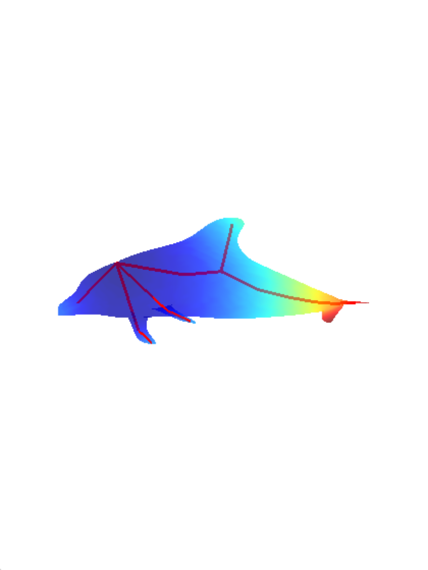
Iso-geodesic curves

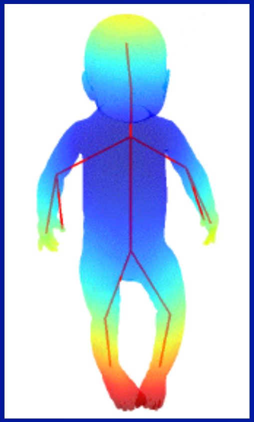


- Iso-geodesic curves are smoothly interpolated.
- The adjacency of nodes is verified through path connectedness. (with D. Aouada)

Examples of Reeb-like graphs







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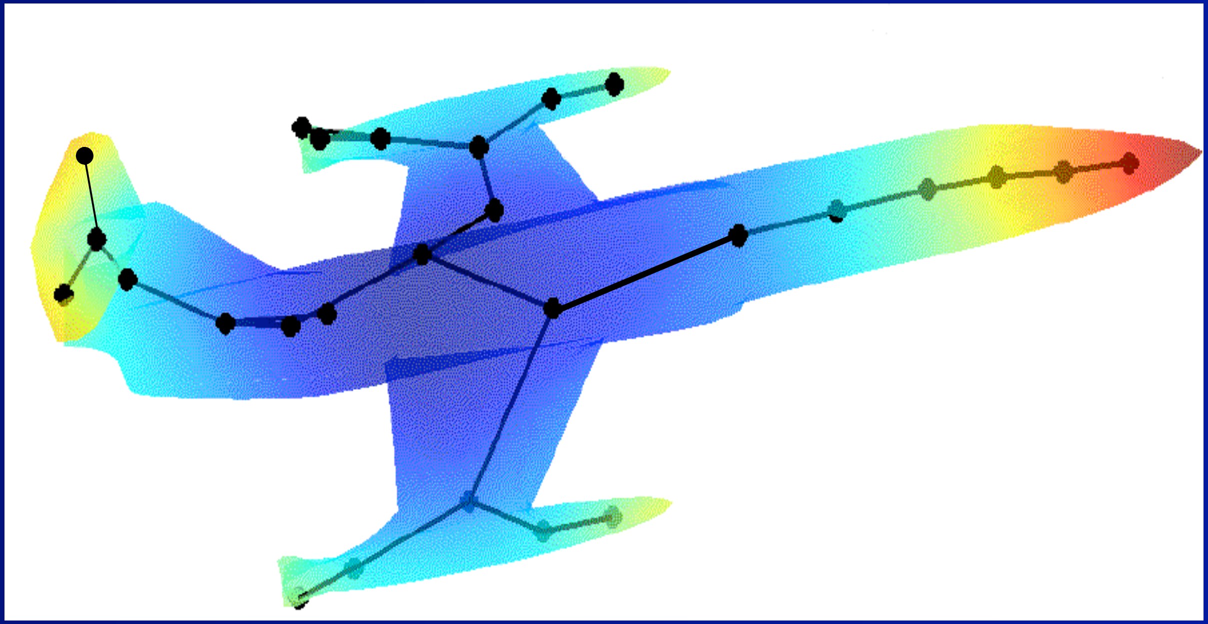
Topological Characterization



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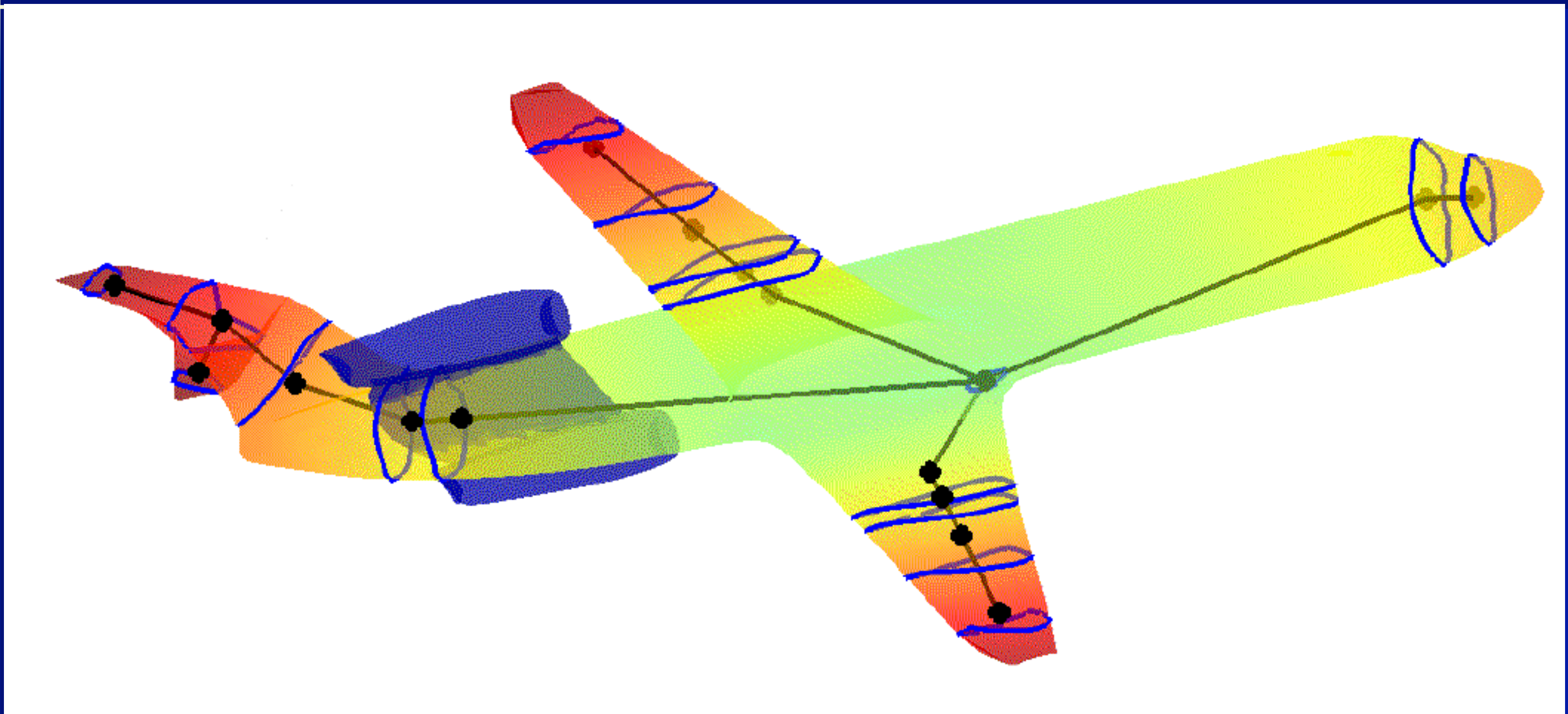
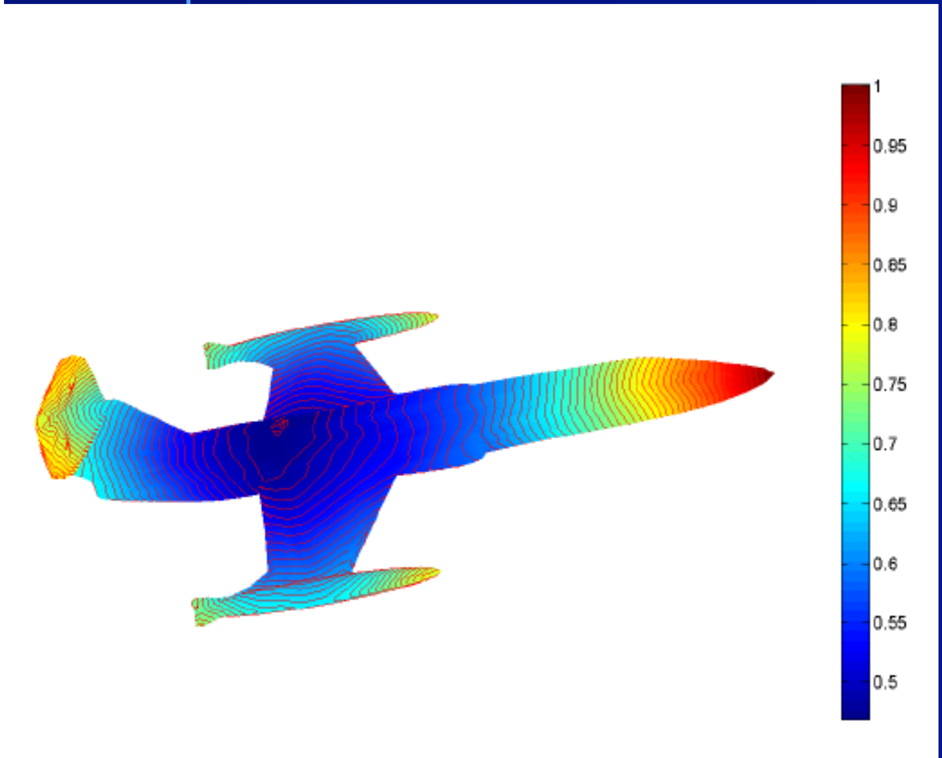
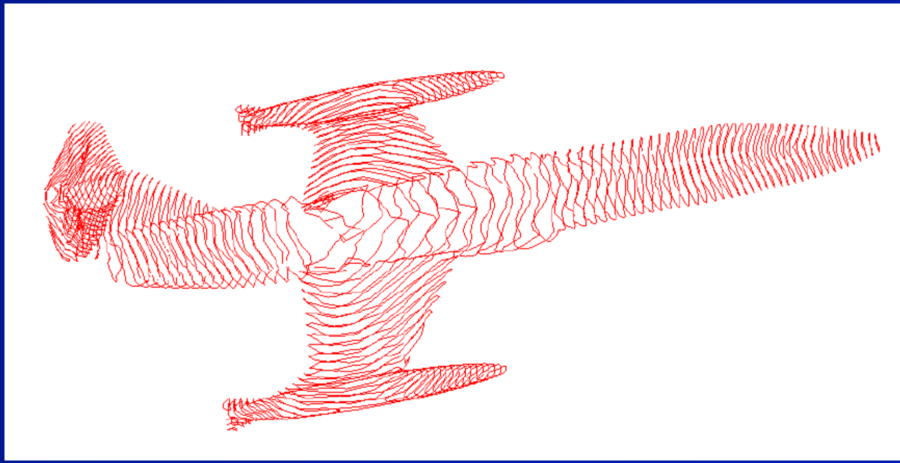
Topological Characterization (2)



- The adjacency of nodes is verified through path connectedness.

Geometric characterization





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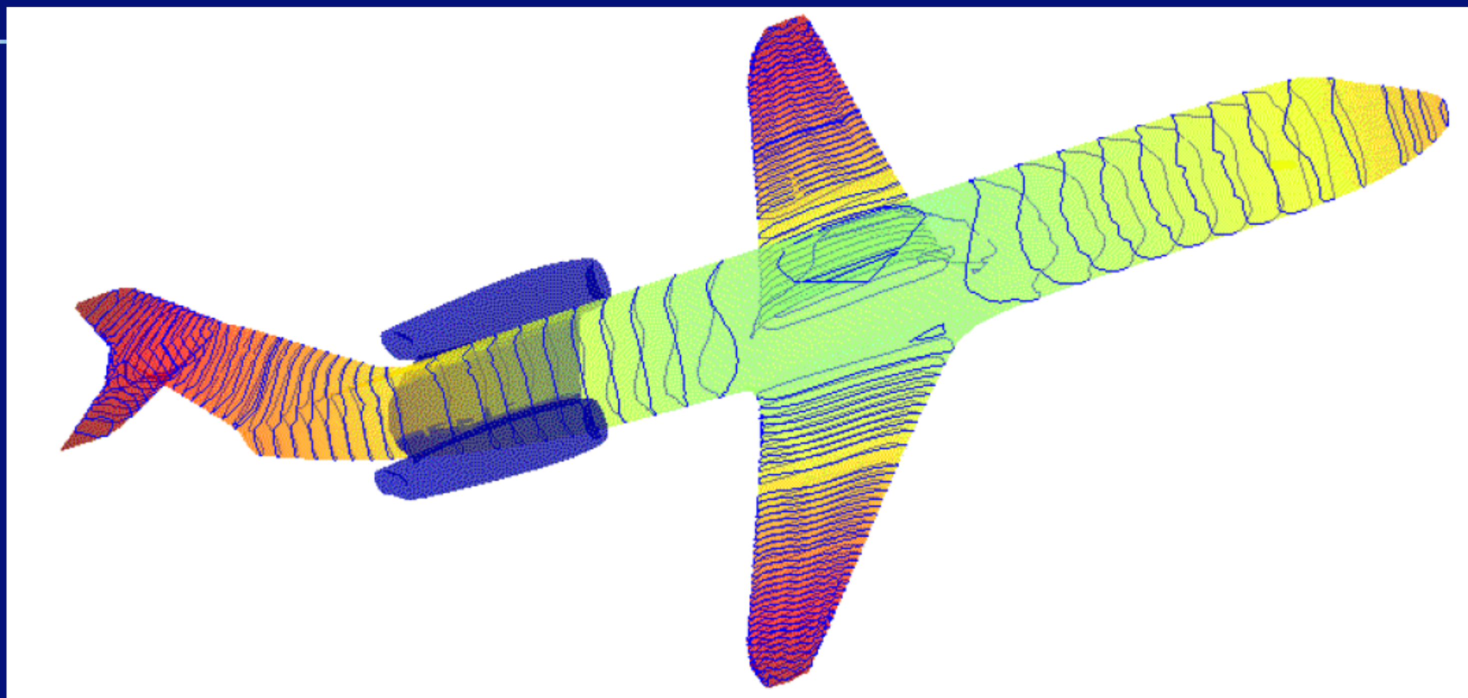
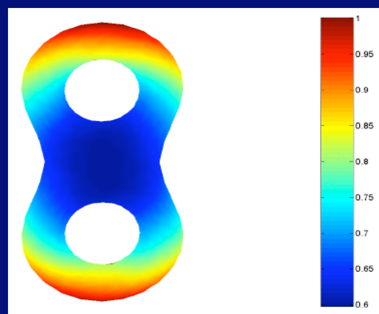
Geometric characterization (2)

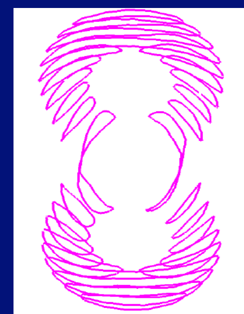


- Surface sampling.
- Iso-geodesic curves are smoothly interpolated.

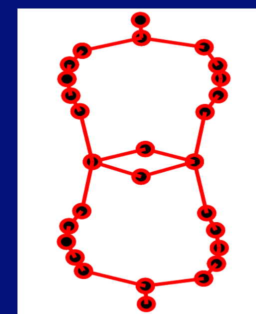
Tag modeling



GGF of the surface



Surface sampling.



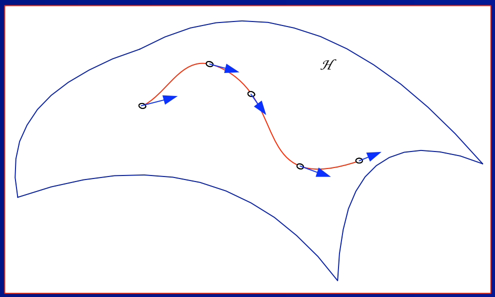
Topological graph

Assign a geometric weight to each edge

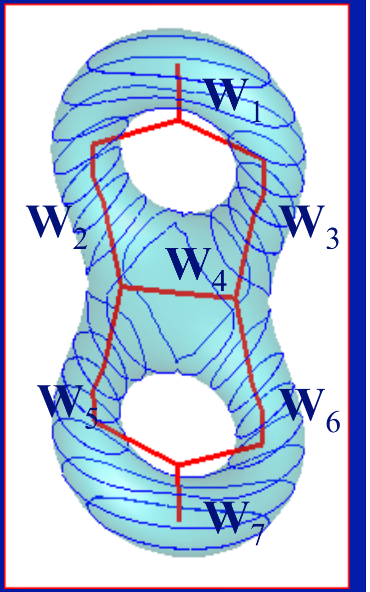
Iso-geodesic curves bear geometric features: Find a model to simply represent a set of these curves along each edge.

Geometric modeling (1)

- Capturing geometry

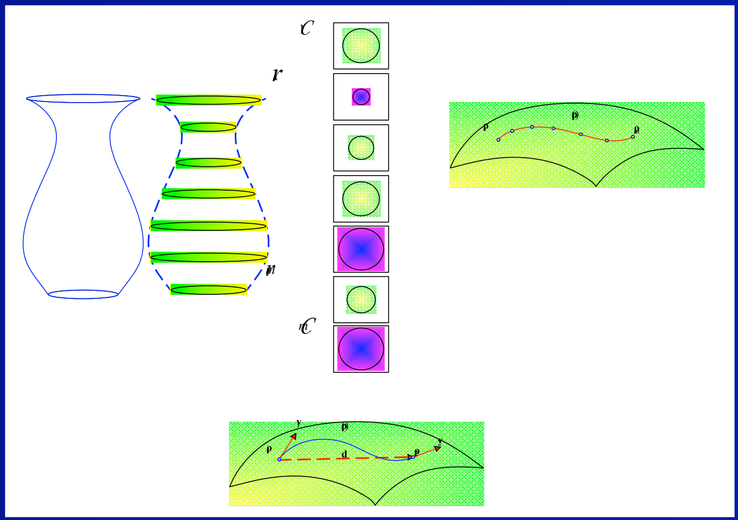


- TaG weighted skeletal graphs
  - Unique, compact and "complete" representation of 3D shape

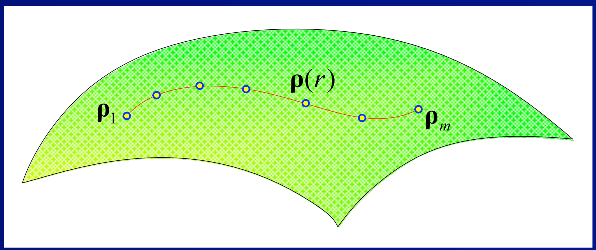


Geometric Modeling (2)

- Given m curves $C_1,\ldots,C_m$, at levels $r_1,\ldots,r_m$

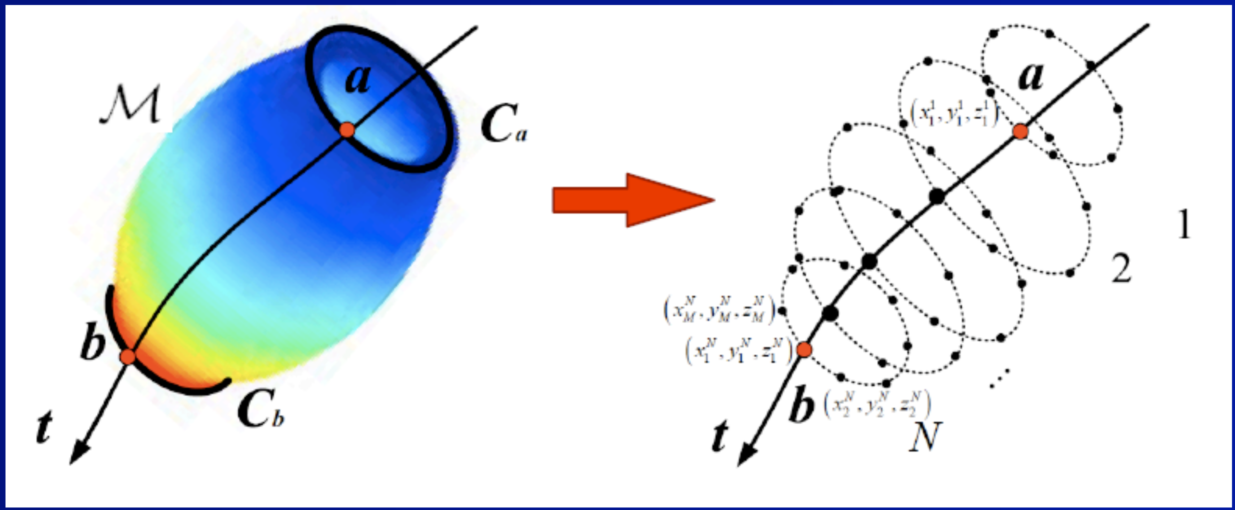


- Find optimal trajectory $\alpha: \mathrm{I} \rightarrow \mathrm{R}^n$



S.H. Baloch, H.K., W. Mio, and A. Srivastava, "3D curve interpolation and object reconstruction", *IEEE ICIP*, 2005.

In Summary



$$\mathcal{M} = \bigcup_{t \in [a,b]} C(t), \quad \text{with} \quad g_0 \leq a < b \leq 1,$$

$$\text{and} \quad C(t_1) \cap C(t_2) = \varnothing \quad \text{if} \quad t_1 \neq t_2,$$

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Geometric modeling: Whitney embedding-Dimension Reduction

Theorem:

Let ***m*** be a compact *Hausdorff* *n*-dimensional manifold, $2 \leq r \leq \infty$, then there is a embedding of *m* in $\mathbb{R}^{2n+1}$

(Constructive proof due to Broomhead and Kirby, SIAM DS 2000)

- In our problem, we are to embed ***a*** in $\mathbb{R}^3$

[***Aouada, et al. ICIP 08, IEEE Trans. On IP 09***]

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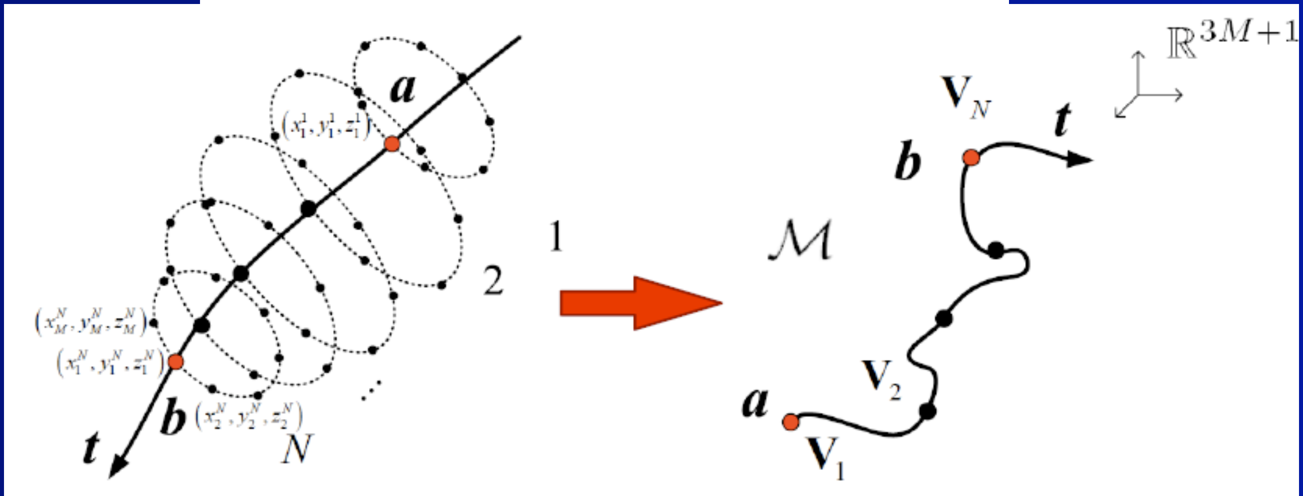
Curve Modeling

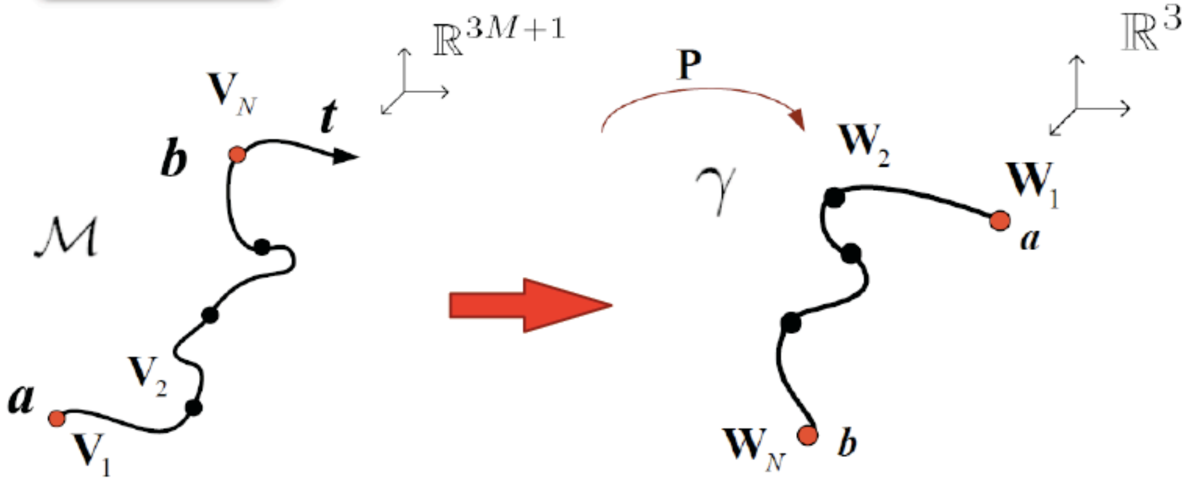
$$\mathbf{V}_i = \left[\left[\mathbf{V}_1^i\right]^T \cdots \left[\mathbf{V}_M^i\right]^T\right]^T \text{th } \mathbf{V}_j^i = \left[x_j^i, y_j^i, z_j^i\right]^T$$



$$\Psi = \left\{\frac{\mathbf{V}_i - \mathbf{V}_j}{\|\mathbf{V}_i - \mathbf{V}_j\|},\ (i,j) \in \{1,\ldots,N\}^2 \text{ and } i \neq j\right\}$$
$$= \{\Psi_i, \quad i = 1, \ldots, L\},$$

Whitney Embedding

$$\mathbf{W} = \mathbf{P}^T\mathbf{V},$$



$$\hat{\mathbf{P}} = \arg\min\left[-\min_{i=1,\dots,L} \| \mathbf{P}\Psi_i \|_2\right]$$

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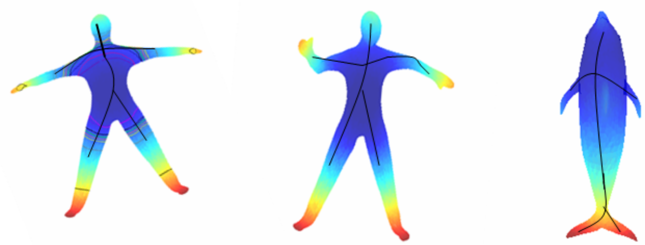
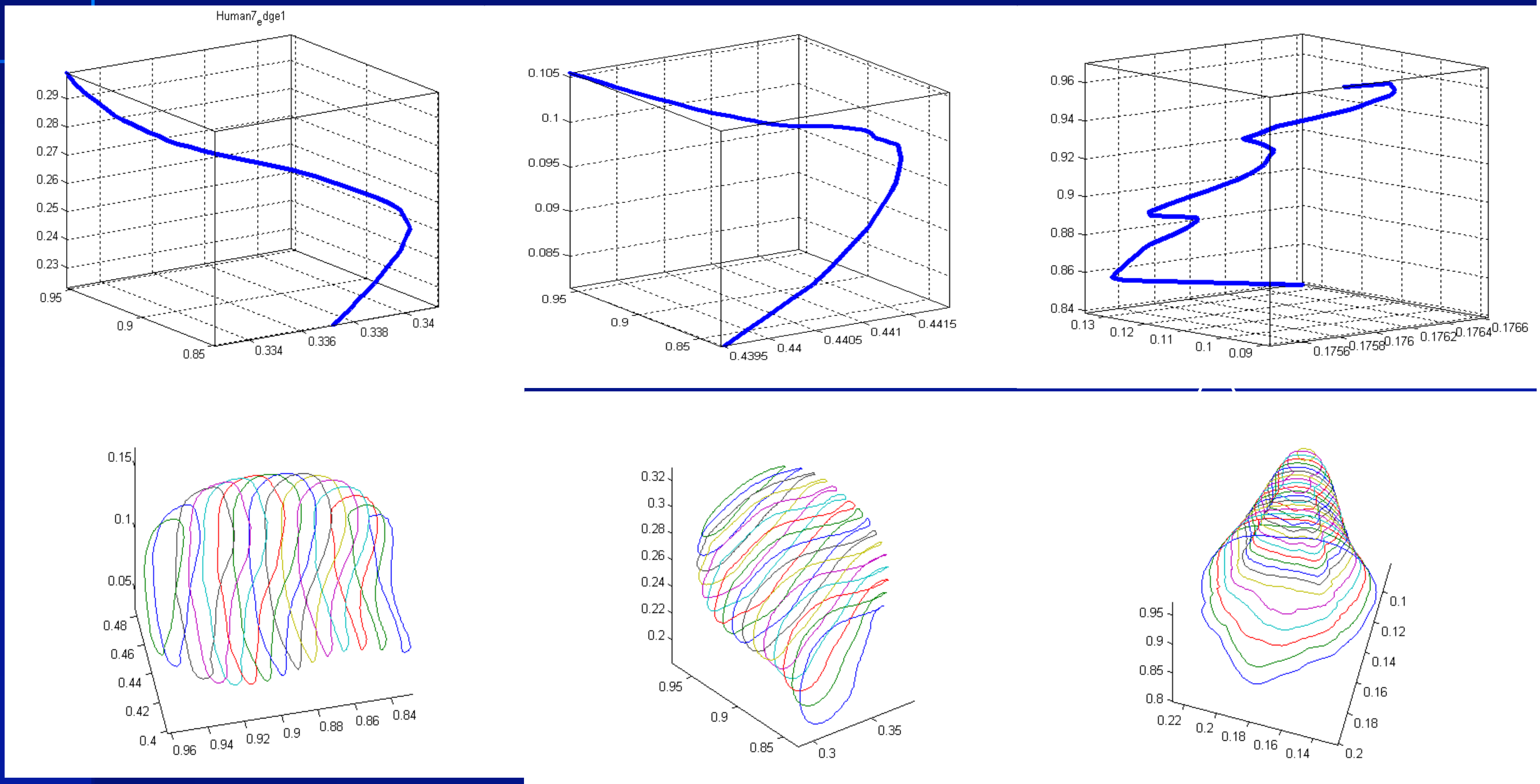
Example illustrating the problem

- Distinguishing three objects with the same graphical representation.
- Find 3D curves to assign to each edge.

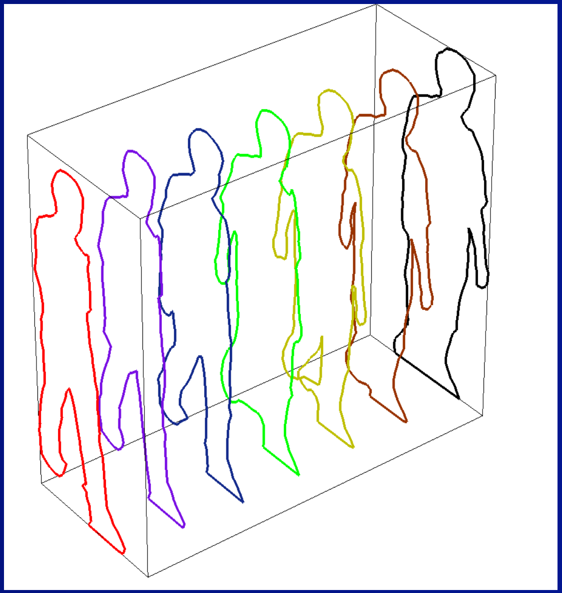


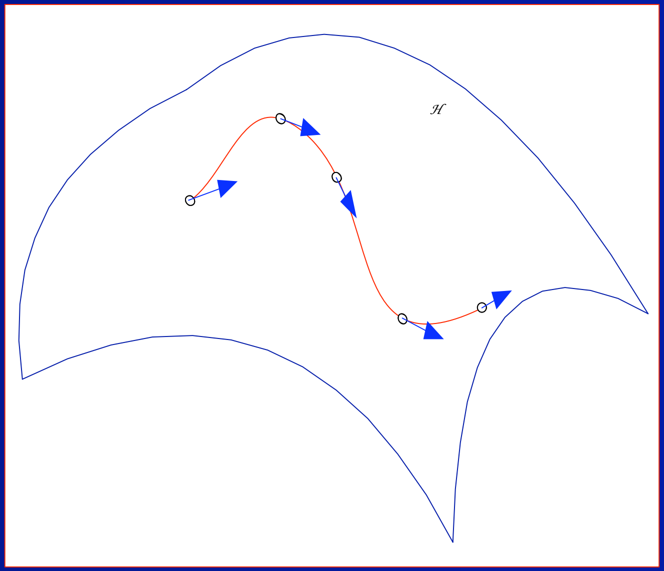
(a) (b) (c)

“Modeling of heads”



Other Applications





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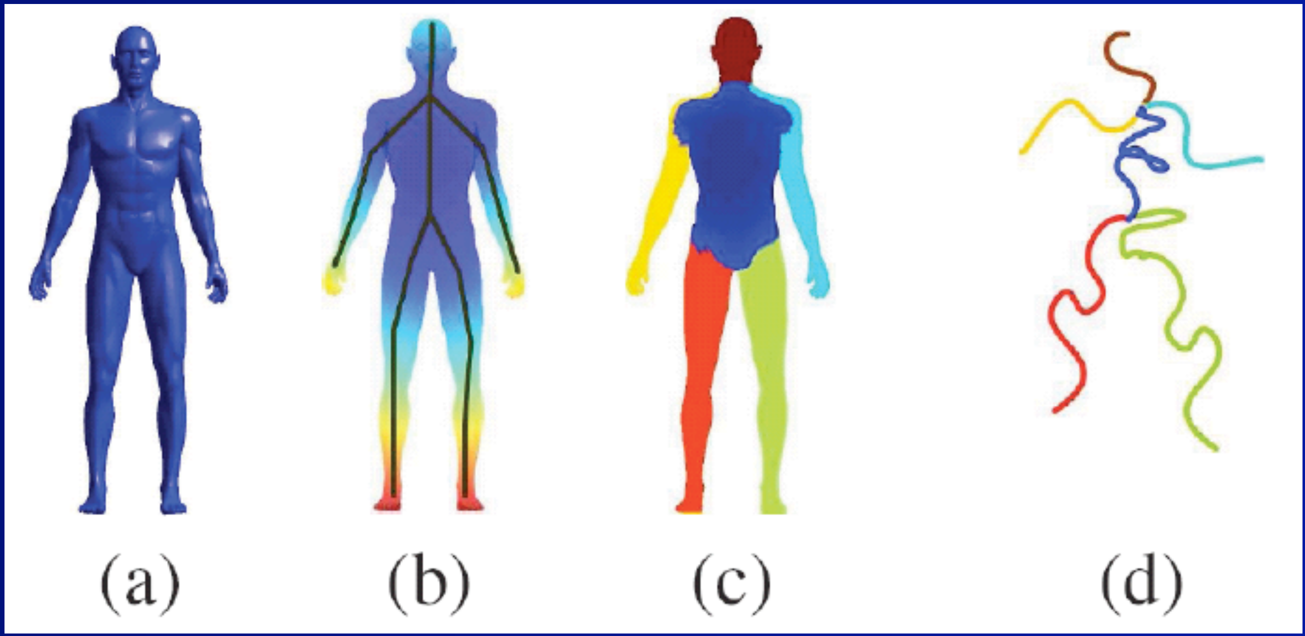
Squigraph Model



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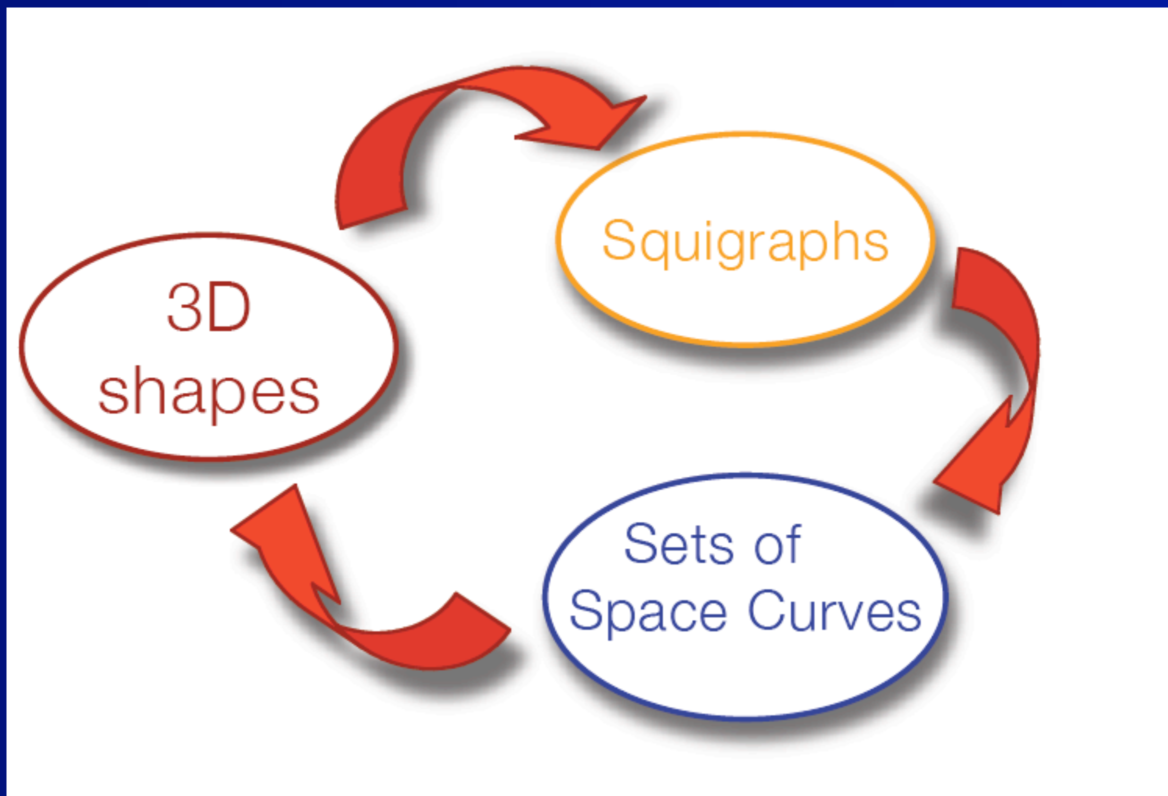
Representation



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Object comparison/ classification

Registration

- Complicated
- Time consuming

Invariant

- Easy to compute
- Inexpensive to store
- From curve comparison to invariant comparison

Invariants offer a simple alternative !

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Geometric Model Matching

- Curves are fundamental blocks in vision and imaging
- Curves undergo geometrical transformations (e.g. biological organs)
- Matching tantamount to coping with variability of characteristic curves

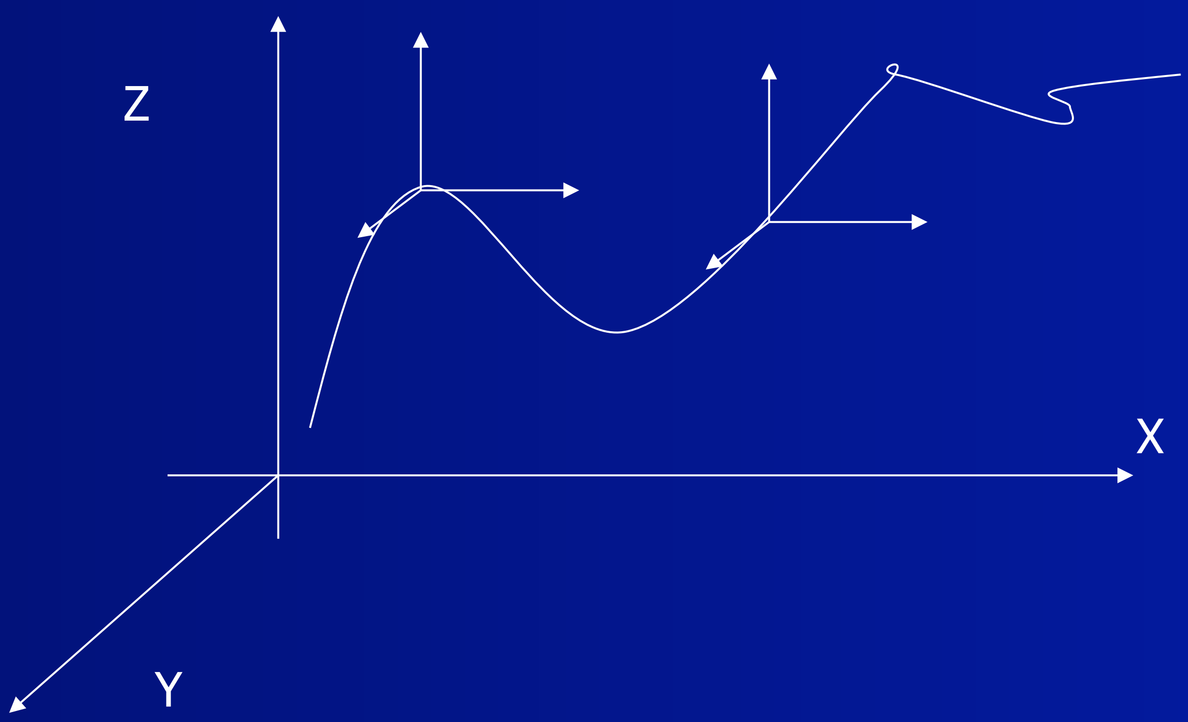
↕

- Invariants (Boutin, Fels-Olver, Kogan, Hickman *et al.*, Manay *et al.*)

Invariants

- Integral Invariants
  – Robust to noise
  – Too costly to make independent of parameterization
  – Hard to compare
- Differential Invariants
  – Easy to compute and to apply
  – Sensitive to noise

Intuitively...invariant



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Integral Jet Space

- Special affine action on spatial curves can be prolonged to the action on the integral variables of order l=i+j ***(with S. Feng, I. Kogan)***

$$X_{ijk} = \int_{t_0}^{t} x^i y^j z^k dx, \quad j+k \neq 0 \quad Y_{ijk} = \int_{t_0}^{t} x^i y^j z^k dy \quad i+k \neq 0$$

$$Z_{ijk} = \int_{t_0}^{t} x^i y^j z^k dz, \quad i+j \neq 0$$

- The jet space is defined as:

$$(X, Y, Z, \mathrm{Z}_{100}, \mathrm{Z}_{010}, \mathrm{Y}_{100}, \mathrm{Z}_{011}, \mathrm{Z}_{020}, \mathrm{Z}_{101}, \mathrm{Z}_{110}, \mathrm{Y}_{101}, \mathrm{X}_{110}, \mathrm{X}_{101}, \mathrm{X}_{020})$$

3D Integral Invariants

- We achieve two special affine integral invariants:

$$I_1 = n_1 X + n_2 Z - n_3 Y$$

$$I_2 = 2n_1(XYZ^2 - 3Z_{011}X + 3YZ_{101} - ZZ_{110} - 2ZY_{101}) + n_2(2XY^2Z + 3XZ_{020} - 6ZX_{020} - 4YZ_{110} - 2YY_{101}) - 2n_3(3YX_{101} - 3ZX_{110} + XZ_{110} - XY_{101})$$

$$n_1 = YZ - 2Z_{010} \quad n_2 = XY - 2Y_{100} \quad n_3 = XZ - 2Z_{100}$$

where

- The full affine integral invariant is easily to derived as [Feng et al., 09]

$$I = \frac{I_2}{I_1^2}$$

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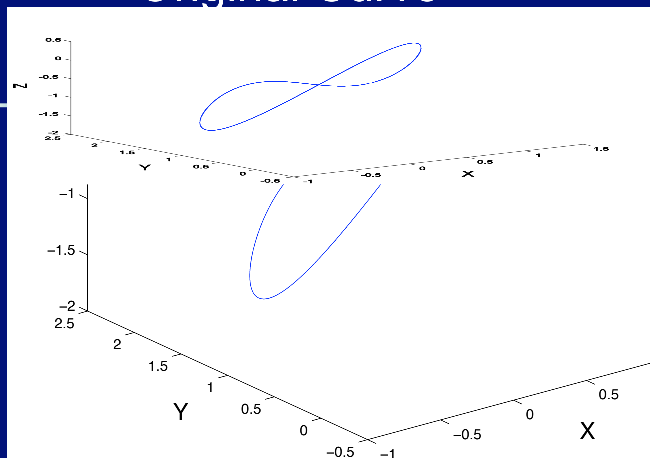
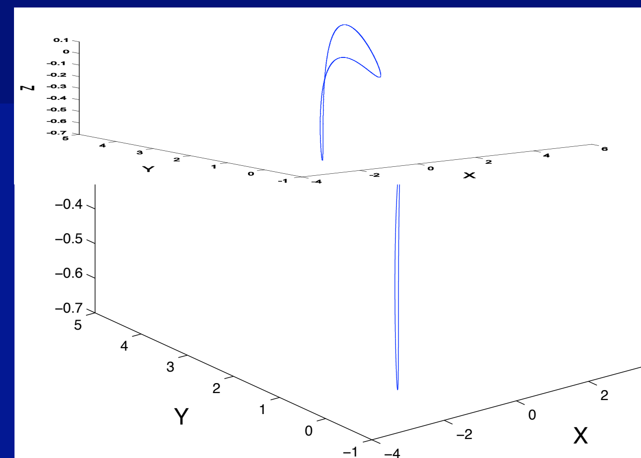
Examples of invariants

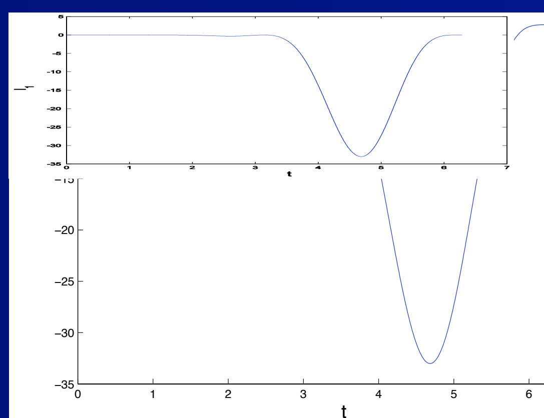
Original Curve



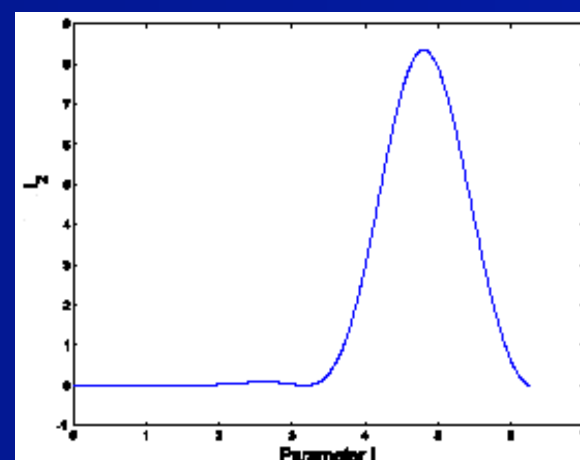
Transformed Curve



Invariant I1 for both of the curves above



Invariant I2 for both of the curves above



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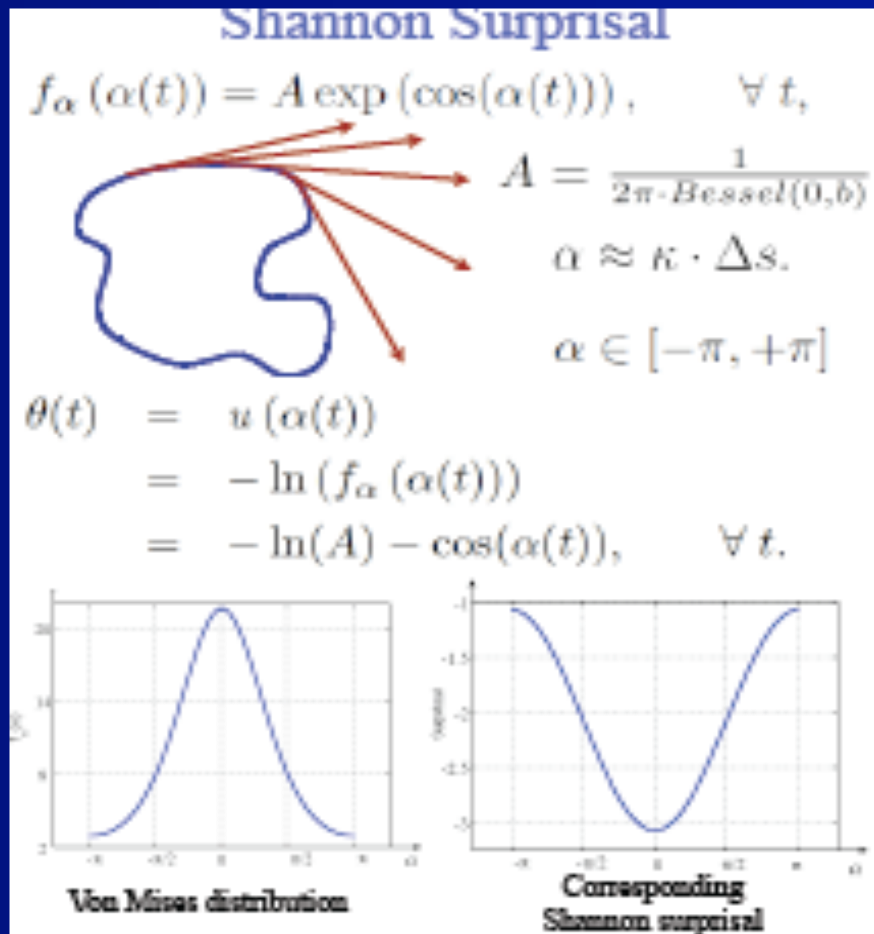
Invariants: Shannon Surprisal



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Space Curves

Using Frenet frame to generalize:

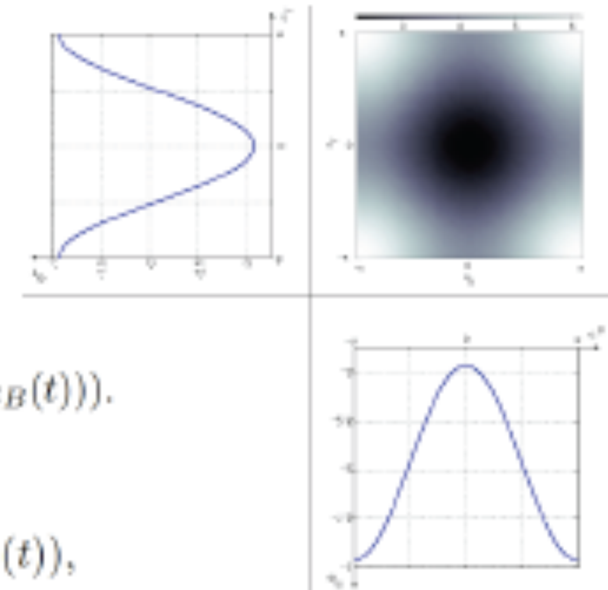
From planar to space curves

$$\frac{d\mathbf{T}}{dt} = \kappa\mathbf{N},$$
$$\frac{d\mathbf{N}}{dt} = -\kappa\mathbf{T} + \tau\mathbf{B},$$
$$\frac{d\mathbf{B}}{dt} = -\tau\mathbf{N},$$

$$\alpha_T(t) \approx \kappa(t)\cdot\Delta t \quad \text{and} \quad \alpha_B(t) \approx \tau(t)\cdot\Delta t.$$

$$f_\alpha(\alpha_T(t), \alpha_B(t)) = A^2 \exp(\cos(\alpha_T(t)) + \cos(\alpha_B(t))).$$

$$\begin{aligned}\theta(t) &= -\ln\left(f_\alpha(\alpha_T, \alpha_B)\right) \\ &= -2\ln(A) - \cos(\alpha_T(t)) - \cos(\alpha_B(t)),\end{aligned}$$



Intrinsic Riemannian Metric

- For two given curves γ_1, γ_2 , define an oriented curve $\lambda_\Delta = \lambda_1 - \lambda_2$
- First take form F,

$$\vec{\mathrm{F}} : ([-\pi,\pi])^2 \to \mathbb{R}^2$$

$$(\alpha_T, \alpha_B) \to -\ln(f_\alpha(\alpha_T)\,\vec{i} - \ln(f_\alpha(\alpha_B)\,\vec{j}$$

One then defines a projection on $[2\pi]^2$

$$\lambda_\Delta^*(\phi) = \int_{\lambda_\Delta^*} \phi(\vec{F}(t))dt$$

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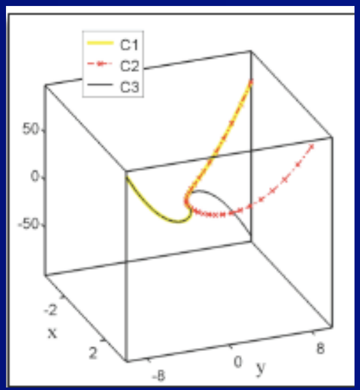
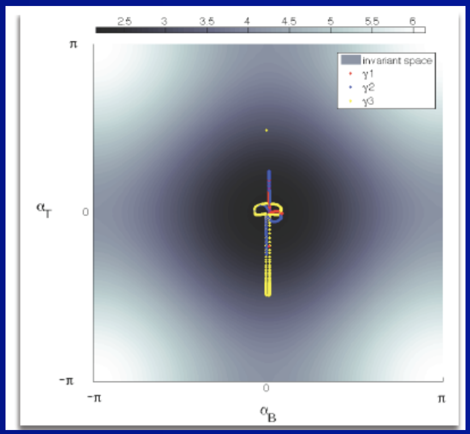
Riemannian Metric

- Define a Flat norm (Vaillant, Glaunes 07)

$$D(\gamma_1,\gamma_2) = \mathfrak{F}(\lambda_1^* - \lambda_2^*) \triangleq \sup\left\{ \lambda_\Delta^*(\phi) : \|d\phi\| \le 1, \forall \|\phi\| \le 1 \right\}$$





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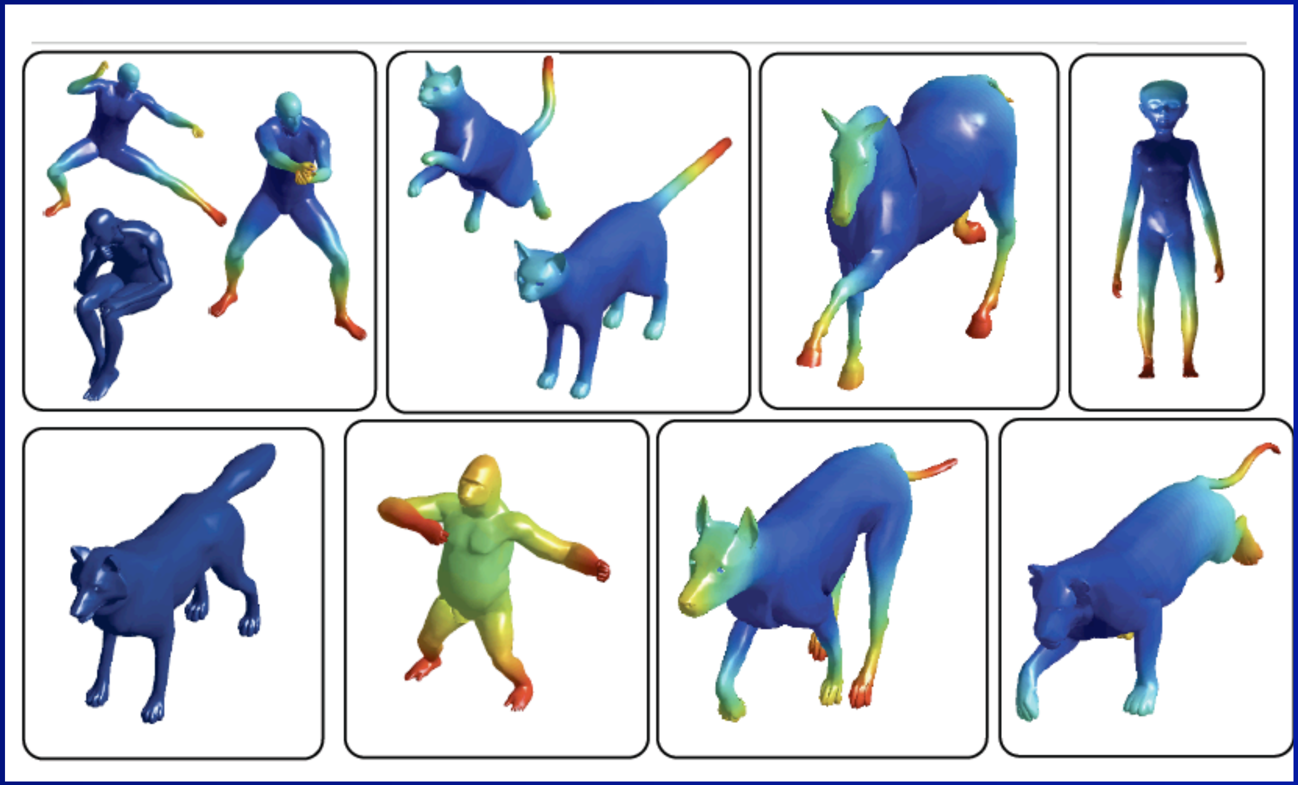
Comparison of complex shapes



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Performance Evaluation

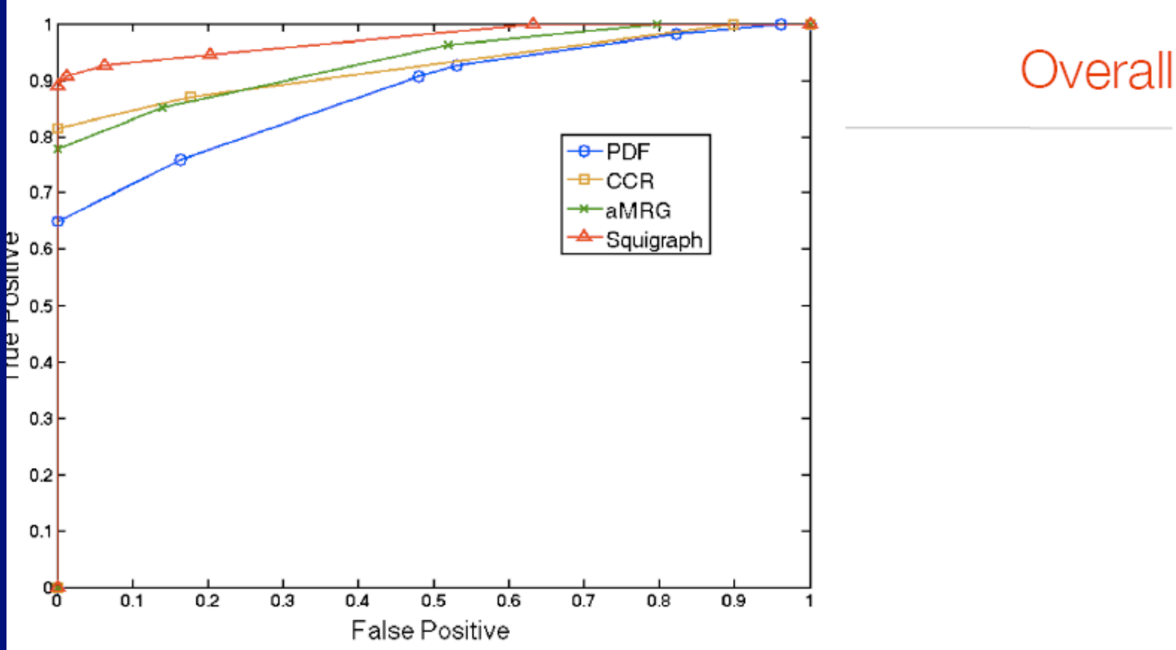

Table 1. Overall performance summary.

Method	PDF	CCR	aMRG	Squigraph
AUC	87.95%	92.53%	93.389%	97.35%

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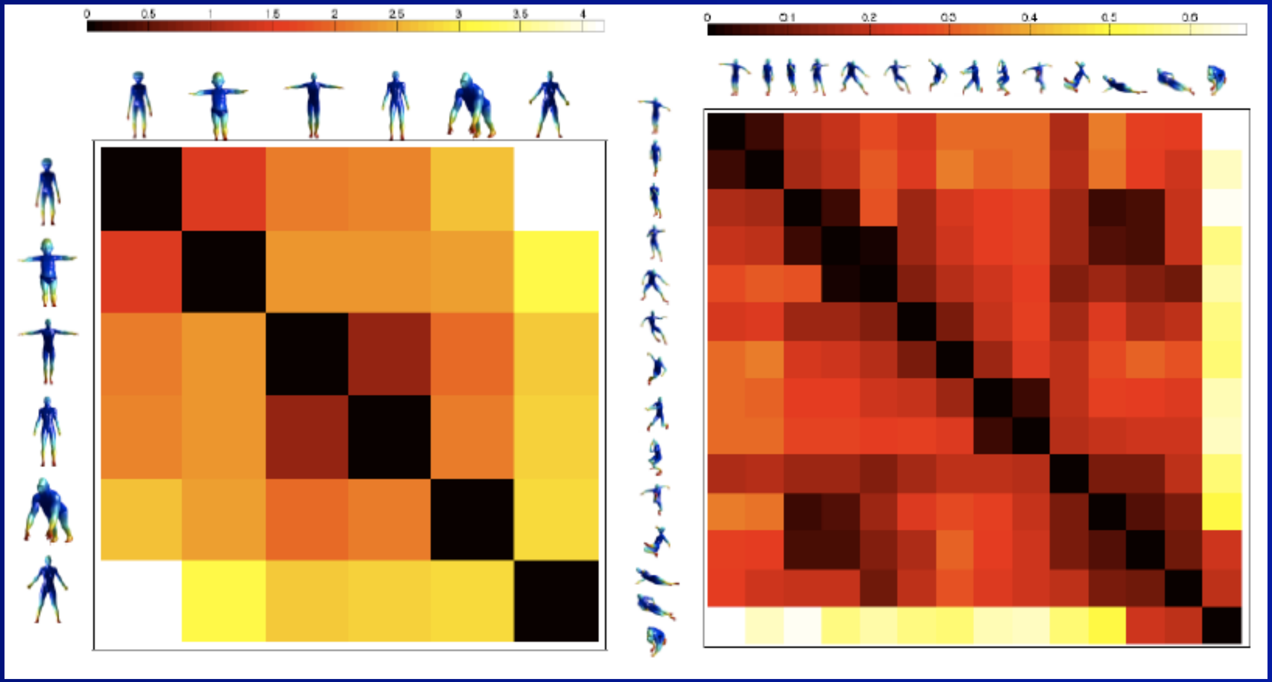
Bipedal Shapes



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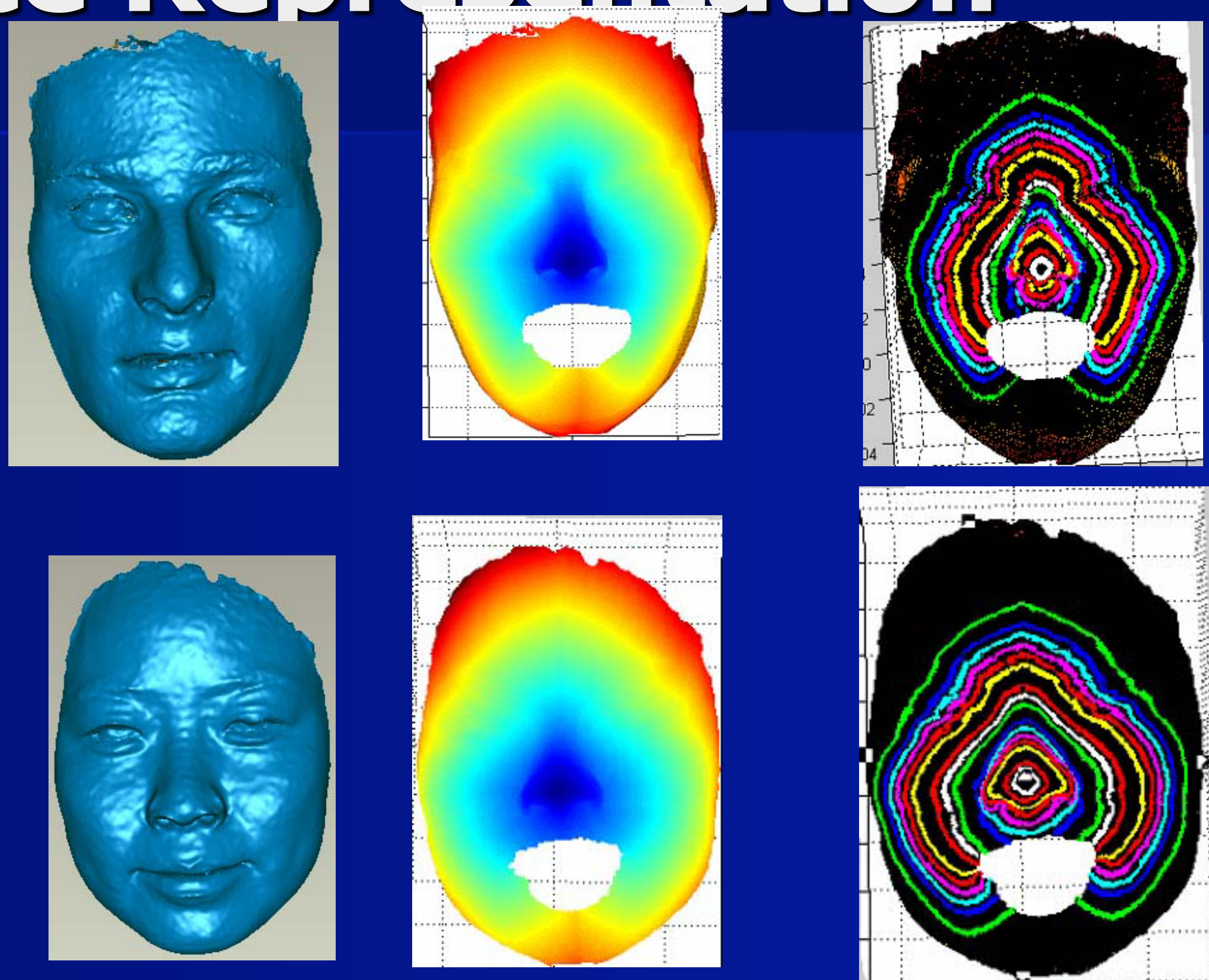
Face Representation



Conclusion/Ongoing Work

- Framework for 3D shape modeling
- Other theoretical issues
  - Robustness issues
  - Sampling in 3D
- Other application avenues
- Experiment with real data

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